On quasi pseudo-GP-injective rings and modules

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Abstract

In 2010, Sanh et al. introduced a class of pseudo-M-gp-injective modules, following this, a right R-module N is called pseudo-M-gp-injective if for any homomorphism $0 \neq \alpha \in End(M)$, there exists $n \in \mathbb{N}$ such that $\alpha^n \neq 0$ and every monomorphism from $\alpha^n(M)$ to N can be extended to a homomorphism from M to N. In this paper, we give more properties of pseudo-gp-injective modules.

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1 Introduction

Throughout the paper, R is an associative ring with identity $1 \neq 0$ and all modules are unitary R-modules. We write M_R (resp., ${}_RM$) to indicate that M is a right (resp., left) R-module. Let J (resp., Z_r, S_r) be the Jacobson radical (resp. the right singular ideal, the right socle) of R and $E(M_R)$ the injective hull of M_R. If X is a subset of R, the right (resp. left) annihilator of X in R is denoted by $r_R(X)$ (resp., $l_R(X)$) or simply $r(X)$ (resp. $l(X)$). If N is a submodule of M (resp., proper submodule) we write $N \leq M$ (resp. $N < M$). Moreover, we write $N \leq^e M$, $N \ll M$, $N \leq^\oplus M$ and $N \leq^{max} M$ to indicate that N is an essential submodule, a small submodule, a direct summand and a maximal submodule of M, respectively. A module M is called uniform if $M \neq 0$ and every non-zero submodule of M is essential in M. A module M is *finite dimensional* (or has *finite rank*) if $E(M)$ is a finite direct sum of indecomposable submodules. A right R-module N is called M-generated if there exists an epimorphism $M^{(I)} \to N$ for some index set I. If the set I is finite, then N is called finitely M-generated. In particular, N is called M-cyclic if it is isomorphic to M/L for some submodule L of M. Hence, any M-cyclic submodule X of M can be considered as the image of an endomorphism of M.

Following Nicholson, Yousif (see [15]), a ring R is called right *P-injective* if every R-homomorphism from a principal right ideal of R to R is a left multiplication. They studied some properties of these rings and their applications. In [18], Sanh et al. transferred this notion to modules. A right R-module N is called *M-principally injective* (briefly, M-p-injective) if every homomorphism from an M-cyclic submodule of M to N can be extended to one from M to N. A right R-module M is called *quasi-principally injective* (briefly, quasi p-injective) if M is M-principally injective. Quasi-p-injective modules were defined first by Wisbauer in [24] under the terminology of *semi-injective* modules, but there are no details. Following [13], a module M is called *principally quasi-injective* if every homomorphism from a cyclic submodule of M to M can be extended to an endomorphism of M. Since an M-cyclic submodule of M needs not to be cyclic, the notion of quasi-p-injective modules is different from that was defined in [13].

As a generalization of injective modules, the class of pseudo injective modules have been studied by Singh and Jain in 1967 [11], Teply (1975)[22], Jain and Singh (1975)[11], Wakamatsu (1979)[23]. Recently, Hai Quang Dinh ([6]) introduced the notion of pseudo M-injective modules (the original terminology is M-pseudo-injective). A right R-module N is called *pseudo M-injective* if for every submodule A of M, any monomorphism $\alpha : A \to N$ can be extended to a homomorphism $M \to N$. A right R-module N is called *pseudo-injective* if N is pseudo-N-injective. In 2009, Sanh et al., introduced the notion of pseudo-M-p-injective modules and studied the endomorphism rings of quasi-pseudo-p-injective modules. A right R-module N is called *pseudo-M-p-injective* if every monomorphism from an M-cyclic submodule of M to N can be extended to a homomorphism from M to N, or equivalently, for any homomorphism $\alpha \in \mathrm{End}(M)$, every monomorphism from $\alpha(M)$ to N can be extended to a homomorphism from M to N (see [16]). A module M is called *quasi-pseudo-p-injective* if M is pseudo-M-p-injective. A ring R is called right *pseudo P-injective* if R_R is quasi-pseudo-p-injective. Following [8], a right R-module M is said to be *generalized principally injective* (briefy gp-injective), if for any $0 \in x \in R$, there exists an $n \in \mathbb{N}$ such that $x^n \neq 0$ and any R-homomorphism from x^nR into M can be extended to one from R_R to M. A ring R is called right GP-injective if R_R is GP-injective. The concept of

GP-injective modules was introduced in [12] to study the class of von Neumann regular rings, V-rings, self-injective rings and their generalizations. In [2], Chen et al. studied some properties of GP-injective rings. In particular, they gave some characterizations of GP-injective ring with special chain conditions. In 2009, Sanh et al. introduced the notion of pseudo-M-gp-injective modules. A right R-module N is called for *pseudo-M-gp-injective* if for each homomorphism $0 \neq \alpha \in \mathrm{End}(M)$, there exists $n \in \mathbb{N}$ such that $\alpha^n \neq 0$ and every monomorphism from $\alpha^n(M)$ to N can be extended to a homomorphism from M to N ([17]). A module M is called *quasi-pseudo-gp-injective* if M is pseudo-M-gp-injective. A ring R is called right pseudo GP-injective if R_R is quasi-pseudo-gp-injective. In this paper, we continue studying more properties of pseudo-p-injective modules, pseudo-gp-injective modules and the endomorphism rings of pseudo-p-injective modules.

2 On pseudo-M-gp-injective

Firstly, we give a new characterization of pseudo-M-gp-injective modules.

Theorem 2.1 *Let M, N be right R-modules. Then following conditions are equivalent:*

(1) N is pseudo-M-gp-injective.

(2) For each $0 \neq s \in \mathrm{End}(M)$, there exists $n \in \mathbb{N}$ such that $s^n \neq 0$ and

$$\{f \in \mathrm{Hom}(M, N) |\ \mathrm{Ker} f = \mathrm{Ker} s^n\} \subseteq \mathrm{Hom}(M, N)s^n.$$

(3) For each $0 \neq s \in \mathrm{End}(M)$, there exists $n \in \mathbb{N}$ such that $s^n \neq 0$ and

$$\{f \in \mathrm{Hom}(M, N) |\ \mathrm{Ker} f = \mathrm{Ker} s^n\} = \{f \in \mathrm{Hom}(M, N) |\ \mathrm{Ker} f \cap \mathrm{Im} s^n = 0\}s^n.$$

Proof. (1) $\Rightarrow$ (2). Suppose that $0 \neq s \in \mathrm{End}(M)$. Since N is pseudo-M-gp-injective, there exists $n \in \mathbb{N}$ such that $s^n \neq 0$ and every monomorphism from $s^n(M)$ to N can be extended to a homomorphism from M to N. Let $f \in \mathrm{Hom}(M, N)$ such that $\mathrm{Ker} f = \mathrm{Ker} s^n$. We consider homomorphism

$$\varphi : s^n(M) \to N \ \text{ via } \ \varphi(s^n(m)) = f(m),\ \forall m \in M.$$

It is easy to see that φ is a monomorphism. By our assumption, there exists a homomorphism $h : M \to N$ such that $h\iota = \varphi$, where ι is the inclusion map from $s^n(M) \to M$, which implies that $f = hs^n \in \mathrm{Hom}(M, N)s^n$.

(2) $\Rightarrow$ (3). It is clear that

$$\{f \in \mathrm{Hom}(M, N) |\ \mathrm{Ker} f \cap \mathrm{Im} s^n = 0\}s^n \subseteq \{f \in \mathrm{Hom}(M, N) |\ \mathrm{Ker} f = \mathrm{Ker} s^n\}.$$

Let $g \in \text{Hom}(M, N)$ such that $\text{Ker}g = \text{Ker}s^n$. Then by (2), there exists a homomorphism $h : M \to N$ such that $g = hs^n$. It follows that $\text{Ker}h \cap \text{Im}s^n = 0$. Hence, $g \in \{f \in \text{Hom}(M, N)|\ \text{Ker}f \cap \text{Im}s = 0\}s^n$.

(3) $\Rightarrow$ (1). For each $0 \neq s \in \text{End}(M)$, by (3), there exists $n \in \mathbb{N}$ such that $s^n \neq 0$ and

$$\{f \in \text{Hom}(M, N)|\ \text{Ker}f = \text{Ker}s^n\} = \{f \in \text{Hom}(M, N)|\ \text{Ker}f \cap \text{Im}s^n = 0\}s^n.$$

Assume that $\phi : s^n(M) \to N$ is a monomorphism. Then $\text{Ker}(\phi s^n) = \text{Ker}s^n$. Hence there is $h \in \text{Hom}(M, N)$ such that $\phi s^n = hs^n$. It gives $h\iota = \phi$, where ι is the inclusion map, proving that N is pseudo-M-gp-injective. □

From the above theorem, we get some characterizations of quasi-pseudo-gp-injective modules.

Corollary 2.2 *Let M be right R-module and $S = \text{End}(M)$. The following conditions are equivalent:*

(1) M is quasi-pseudo-gp-injective;

(2) For each $0 \neq s \in S$, there exists $n \in \mathbb{N}$ such that $s^n \neq 0$ and

$$\{f \in S|\ \text{Ker}f = \text{Ker}s^n\} \subseteq Ss^n;$$

(3) For each $0 \neq s \in S$, there exists $n \in \mathbb{N}$ such that $s^n \neq 0$ and

$$\{f \in S|\ \text{Ker}f = \text{Ker}s^n\} = \{f \in S|\ \text{Ker}f \cap \text{Im}s^n = 0\}s^n.$$

Corollary 2.3 *Let M, N be right R-modules. The following conditions are equivalent:*

(1) N is pseudo-M-p-injective;

(2) For each $s \in \text{End}(M)$,

$$\{f \in \text{Hom}(M, N)|\ \text{Ker}f = \text{Ker}s\} \subseteq \text{Hom}(M, N)s;$$

(3) For each $s \in \text{End}(M)$,

$$\{f \in \text{Hom}(M, N)|\ \text{Ker}f = \text{Ker}s\} = \{f \in \text{Hom}(M, N)|\ \text{Ker}f \cap \text{Im}s = 0\}s.$$

Proposition 2.4 *Let N be pseudo-M-p-injective. Then for any elements $s, \alpha \in \text{End}(M)$, we have:*

$$\{\beta \in \text{Hom}(M, N)|\text{Ker}\beta \cap \text{Im}s = \text{Ker}\alpha \cap \text{Im}s\} =$$

$$\{\gamma \in \text{Hom}(M, N)|\text{Ker}\gamma \cap \text{Im}(\alpha s) = 0\}\alpha + \{\delta \in \text{Hom}(M, N)|\delta s = 0\}.$$

Proof. Let

$$\mathcal{A} = \{\beta \in \mathrm{Hom}(M, N) | \mathrm{Ker}\beta \cap \mathrm{Im}s = \mathrm{Ker}\alpha \cap \mathrm{Im}s\}$$
$$\mathcal{B} = \{\gamma \in \mathrm{Hom}(M, N) | \mathrm{Ker}\gamma \cap \mathrm{Im}(\alpha s) = 0\}$$
$$\mathcal{C} = \{\delta \in \mathrm{Hom}(M, N) | \delta s = 0\}$$

It is easy to see that $\mathcal{B}\alpha + \mathcal{C} \subseteq \mathcal{A}$. Conversely, let $\beta \in \mathrm{Hom}(M, N)$ such that $\mathrm{Ker}\beta \cap \mathrm{Im}s = \mathrm{Ker}\alpha \cap \mathrm{Im}s$ ($\beta \in \mathcal{A}$). It follows that $\mathrm{Ker}(\alpha s) = \mathrm{Ker}(\beta s)$. By Corollary 2.3, there exists $\gamma \in \mathcal{B}$ such that $\beta s = \gamma\alpha s$ or $(\beta - \gamma\alpha)s = 0$. It means $\beta - \gamma\alpha \in \mathcal{C}$, which implies that $\beta \in \mathcal{B}\alpha + \mathcal{C}$. $\square$

Proposition 2.5 *If $M = M_1 \oplus M_2$ is quasi-pseudo-p-injective, then M_1 is M_2-p-injective.*

Proof. Let $M = M_1 \oplus M_2$ be quasi-pseudo-p-injective and $s \in \mathrm{End}(M_2)$. Let $f : s(M_2) \to M_1$ be a homomorphism. Consider homomorphism $g : s(M_2) \to M$ defined by $g(a) = f(a) + a$ for all $a \in s(M_2)$. Then g is a monomorphism. By [16, Proposition 1.3], M is pseudo-M_2-p-injective, whence g extends to a homomorphism $\bar{g} : M_2 \to M$. Let $\pi : M \to M_1$ be the canonical projection. Then $\pi\bar{g} : M_2 \to M$ extends f. Thus M_1 is M_2-p-injective, as required. $\square$

Corollary 2.6 *For any integer $n \geq 2$, if M^n is quasi-pseudo-p-injective, then M is quasi-p-injective.*

Proposition 2.7 *Let M and N be modules and $X = M \oplus N$. The following conditions are equivalent:*

(1) N is pseudo-M-p-injective.

(2) For each M-cyclic submodule K of X with $K \cap M = K \cap N = 0$, there exists $C \leq X$ such that $K \leq C$ and $N \oplus C = X$.

Proof.(1) $\Rightarrow$ (2). Let K be a submodule of X which is M-cyclic with $K \cap M = K \cap N = 0$, and $\pi_M : M \oplus N \to M$ and $\pi_N : M \oplus N \to N$ be the canonical projections. We can check that $N \oplus K = N \oplus \pi_M(K)$ and hence $\pi_M(K) \simeq K$, proving that $\pi_M(K)$ is a M-cyclic submodule of M. Let $\varphi : \pi_M(K) \to \pi_N(K)$ be a homomorphism defined as follows: for $k = m + n \in K$ (with $m \in M, n \in N$), $\varphi(m) = n$. It is easy to see that φ is a monomorphism. Since N is pseudo M-p-injective, there is a homomorphism $\bar{\varphi} : M \to N$ extending φ. Let $C = \{m + \bar{\varphi}(m) |\ m \in M\}$. Then $X = N \oplus C$ and $K \leq C$.

(2) $\Rightarrow$ (1). Let $s \in End(M)$ and $\varphi : s(M) \to N$ be a monomorphism. Put $K = \{s(m) - \varphi(s(m)) |\ m \in M\}$. Then $K \cap M = 0$ and $N \oplus K = N \oplus \pi_M(K) = N \oplus s(M)$. It is easy to see that K is M-cyclic. By assumption, there exists a submodule C of X containing K with $N \oplus C = X$. Let $\pi : N \oplus C \to N$ be the natural projection. Then the restriction $\pi|_M$ extends φ, proving (1). $\square$

3 On quasi-pseudo-gp-injective rings and modules

From Corollary 2.3, we have some characterizations of quasi-pseudo-p-injective modules.

Theorem 3.1 *The following conditions are equivalent for module M with $S = \mathrm{End}(M)$:*

(1) M is quasi-pseudo-p-injective;

(2) If $\mathrm{Ker}f = \mathrm{Ker}g$ with $f, g \in S = \mathrm{End}(M)$, then $Sf = Sg$;

(3) If $f \in S = \mathrm{End}(M)$ and $\alpha, \beta : f(M) \to M$ is monomorphisms, then $\alpha = s\beta$ for some $s \in S$.

Proof. (1) $\Rightarrow$ (2). By Corollary 2.3.

(2) $\Rightarrow$ (3). Assume that $0 \neq f \in S$ satisfies (2). Let $\alpha, \beta : f(M) \to M$ be monomorphisms. Then $\mathrm{Ker}(\alpha f) = \mathrm{Ker}(\beta f)$. By our assumption, there exists $s \in S$ such that $\alpha f = s\beta f$, which implies that $\alpha = s\beta$.

(3) $\Rightarrow$ (1). Let $s \in S$ and $\varphi : s(M) \to M$ be a monomorphism. Let $\iota : s(M) \to M$ be the inclusion. By (3), there exists $\bar{\varphi} \in S$ such that $\varphi = \bar{\varphi}\iota$ showing that $\bar{\varphi}$ extends φ. Thus M is quasi-pseudo p-injective. $\square$

Corollary 3.2 *The following conditions are equivalent for ring R:*

(1) R is right pseudo P-injective;

(2) If $r(x) = r(y)$ with $x, y \in R$, then $Rx = Ry$.

We have the following relations:
quasi-p-injective $\Rightarrow$ quasi-pseudo-p-injective $\Rightarrow$ quasi-pseudo-gp-injective.

Example 3.3 *i)* Let F be an algebraically closed field and x, y be indeterminates. Let $R = F(y)[x]$ such that $xf - fx = df/dy$, $f \in F(y)$ (see [20, Example]). Then the R-module $M = R/(x(x + y)(x + y - 1/y))R$ is quasi-pseudo-p-injective but not quasi-p-injective by [20, Example].

ii) Let $K = F(y_1, y_2, ...)$ and $L = F(y_2, y_3, ...)$ with F a field, and $\rho : K \to L$ be an isomorphism via $\rho(y_i) = y_{i+1}$ and $\rho(c) = c$ for all $c \in F$ (see [4, Exmaple 1]. Let $K[x_1, x_2; \rho]$ be the ring of twisted left polynomials over K where $x_i k = \rho(k)x_i$ for all $k \in K$ and for $i = 1, 2$. Set $R = K[x_1, x_2; \rho]/(x_1^2, x_2^2)$. Then R_R is quasi-pseudo-gp-injective which is not quasi-pseudo-p-injective.

Next we study some properties of quasi-pseudo-gp-injective, self-generator modules and their endomorphism rings.

Theorem 3.4 *Let M be a right R-module with $S = \text{End}(M)$. Then*

1. *(1) If S is a right pseudo GP-injective ring, then M is quasi-pseudo-gp-injective.*
2. *(2) If M is quasi-pseudo-gp-injective and self-generator, then S is a right pseudo GP-injective ring.*

Proof. (1). Let $f \in S$. Since S is right pseudo GP-injective, there exists $n \in \mathbb{N}$ such that $f^n \neq 0$ and if $r_S(f^n) = r_S(g)$ for some $g \in S$, then $g \in Sf^n$ by Corollary 2.2. Assume that $Kerf^n = Kerg$ with $g \in S$. Then $r_S(f^n) = r_S(g)$ and hence $g \in Sf^n$. Thus M is quasi-pseudo-gp-injective by Corollary 2.2.

(2). Let $0 \neq f \in S$. Since M is quasi-pseudo-gp-injective, there exists $n \in \mathbb{N}$ such that $f^n \neq 0$ and if $\text{Ker}(f^n) = \text{Ker}(g)$ with $g \in S$, then $g \in Sf^n$. Let $g \in S$ with $r_S(f^n) = r_S(g)$. Since M is a self-generator, we get $\text{Ker}f^n = \text{Ker}g$. By our assumption, $g \in Sf^n$ and so S is right pseudo GP-injective. □

Corollary 3.5 *Let M be a right R-module with $S = \text{End}(M)$. Then*

1. *(1) If S is a right pseudo P-injective ring, then M is quasi-pseudo-p-injective.*
2. *(2) If M is a quasi-pseudo-p-injective module which is a self-generator, then S is a right pseudo P-injective ring.*

For a right R-module M, $S = \text{End}(M)$ we denote:

$$W(S) = \{s \in S |\ \text{Ker}(s) \text{ is essential in } M\}.$$

Lemma 3.6 *Let M_R be a quasi-pseudo-gp-injective module which is a self-generator, $S = \text{End}(M)$. If $a \notin W(S)$, then* $\text{Ker}(a) < \text{Ker}(a - ata)$ *for some $t \in S$.*

Proof. If $a \notin W(S)$, then $\text{Ker}(a)$ is not an essential submodule of M. Hence there exists $0 \neq m \in M$ such that $mR \cap \text{Ker}(a) = 0$. Since M is a self-generator, there exists $\lambda \in S$ such that $0 \neq \lambda(M) \leq mR$. Hence $\text{Ker}(a) \cap \lambda(M) = 0$. It follows that $a\lambda \neq 0$. Since M is quasi-pseudo-gp-injective, there exists $n \in \mathbb{N}$ such that $(a\lambda)^n \neq 0$ and if $\text{Ker}(a\lambda)^n = \text{Ker}g$ with $g \in S = \text{End}(M)$, then $g \in S(a\lambda)^n$. From $\text{Ker}(a) \cap \lambda(M) = 0$ we also have $\text{Ker}((a\lambda)^n) = \text{Ker}(\lambda(a\lambda)^{n-1})$. Hence $\lambda(a\lambda)^{n-1} \in S(a\lambda)^n$. Therefore $\lambda(a\lambda)^{n-1} = s(a\lambda)^n$ for some $s \in S$, which implies that $Im(\lambda(a\lambda)^{n-1}) \leq Ker(a - asa)$. It follows that $\text{Ker}(a) < \text{Ker}(a - asa)$, since $\text{Im}(\lambda(a\lambda)^{n-1}) \nleq \text{Ker}(a)$ and $(a\lambda)^n \neq 0$. □

Lemma 3.7 *Assume that M is quasi-pseudo-gp-injective module which is a self-generator. Then $J(S) = W(S)$.*

Proof. Let $a \in J(S)$. If $a \notin W(S)$, then by the proof of Lemma 3.6, there exist a positive integer n and $\lambda,\ t \in S$ such that $(a\lambda)^n \neq 0$ and $(1-at)(a\lambda)^n = 0$. Note that $1-at$ is left invertible, so $(a\lambda)^n = 0$, a contradiction. Conversely, let $a \in W(S)$. Then, for each $t \in S$, $ta \in W(S)$ and hence $1 - ta \neq 0$. Since M is a quasi-pseudo-p-injective module, there exists $n \in \mathbb{N}$ such that $(1-ta)^n \neq 0$ and if $\mathrm{Ker}(1-ta)^n = \mathrm{Ker}g$ for some $g \in S = \mathrm{End}(M)$, then $g \in S(1-ta)^n$. Put $u = (1-ta)^n$, $1-u = v$ for some $v \in W(S)$. Since $\mathrm{Ker}(v) \cap \mathrm{Ker}(1-v) = 0$, we have $\mathrm{Ker}(1-v) = 0$. Then $\mathrm{Ker}(u) = \mathrm{Ker}(1_S)$. It follows that $Su = S$ and hence $(1-ta)^n$ is left invertible, proving our lemma. □

Corollary 3.8 *If R is right pseudo GP-injective, then $J(R) = Z(R_R)$.*

Recall that a module M is said to satisfy the *generalized C2-condition (or GC2)* (see [25]) if for any $N \simeq M$ with $N \leq M$, N is a direct summand of M.

Theorem 3.9 *If M is quasi-pseudo-gp-injective, then M satisfies GC2.*

Proof. Let $S = \mathrm{End}(M)$. Assume that $\mathrm{Ker}s = 0$ with $s \in S$. We need to prove that $S = Ss$. Since M is quasi-pseudo-gp-injective, there exists $n \in \mathbb{N}$ such that $s^n \neq 0$ and $\mathrm{Ker}s^n = \mathrm{Ker}g$ with $g \in S$, which would imply that $g \in Ss^n$. Note that $\mathrm{Ker}s = 0 = \mathrm{Ker}1_S$. It follows that $1_S \in Ss^n \leq Ss$, whence $S = Ss$. Thus M is GC2 by [25, Theorem 3]. □

Corollary 3.10 *If R is right pseudo GP-injective, then R is right GC2.*

Proposition 3.11 *Let M be a quasi-pseudo-p-injective module which is a self-generator and $S = \mathrm{End}(M)$. If every complement submodule of M is M-cycilc, then $S/J(S)$ is von Neumann regular.*

Proof. We have $J(S) = W(S)$ by Lemma 3.7. For all $\lambda \in S$, let L be a complement of $\mathrm{Ker}\lambda$. We consider the map $\phi : \lambda(L) \to M$ defined by $\phi(\lambda(x)) = x$ for all $x \in L$. Then ϕ is a monomorphism and $\lambda(L) \simeq L$ which implies $\lambda(L)$ is a M-cyclic submodule of M. Since M is quasi-pseudo-p-injective, there exists $\theta \in S$, which is an extension of ϕ. Then $\mathrm{Ker}\lambda + L \leq \mathrm{Ker}(\lambda\theta\lambda - \lambda)$, and we see that $\mathrm{Ker}\lambda \oplus L \leq^e M$. Consequently $\lambda\theta\lambda - \lambda \in W(S) = J(S)$. □

Theorem 3.12 *Let M be a quasi-pseudo-gp-injective module which is a self-generator and $S = \mathrm{End}(M)$. Then the following conditions are equivalent:*

(1) S is right perfect;

(2) For any infinite sequence $s_1, s_2, \cdots \in S$, the chain

$$\mathrm{Ker}(s_1) \leq \mathrm{Ker}(s_2 s_1) \leq \cdots$$

is stationary.

Proof.(1) $\Rightarrow$ (2). Let $s_i \in S$, $i = 1, 2....$ Since S is right perfect, S satisfies DCC on finitely generated left ideals. So the chain $Ss_1 \geq Ss_2s_1 \geq ...$ terminates. Thus there exists $n > 0$ such that $Ss_ns_{n-1}...s_1 = Ss_ks_{k-1}...s_1$ for all $k > n$. It follows that $\mathrm{Ker}(s_ns_{n-1}...s_1) = \mathrm{Ker}(s_ks_{k-1}...s_1)$ for all $k > n$.

(2) $\Rightarrow$ (1). We first prove that $S/W(S)$ is a von Neumann regular ring. Let $a_1 \notin W(S)$. Then by Lemma 3.6, there is $c_1 \in S$ such that $\mathrm{Ker}(a_1) < \mathrm{Ker}(a_1 - a_1c_1a_1)$. Put $a_2 = a_1 - a_1c_1a_1$. If $a_2 \in W(S)$, then we have $\bar{a}_1 = \bar{a}_1\bar{c}_1\bar{a}_1$, i.e., $\bar{a}_1$ is a regular element of $S/W(S)$. If $a_2 \notin W(S)$, there exists $a_3 \in S$ such that $\mathrm{Ker}(a_2) < \mathrm{Ker}(a_3)$ with $a_3 = a_2 - a_2c_2a_2$ for some $c_2 \in S$ by the preceding proof. Repeating the above-mentioned process, we get a strictly ascending chain

$$\mathrm{Ker}(a_1) < \mathrm{Ker}(a_2) < ...,$$

where $a_{i+1} = a_i - a_ic_ia_i$ for some $c_i \in S$, $i = 1, 2....$ Let

$$b_1 = a_1, b_2 = 1 - a_1c_1, ..., b_{i+1} = 1 - a_ic_i, ...,$$

then

$$a_1 = b_1, a_2 = b_2b_1, ..., a_{i+1} = b_{i+1}b_i...b_2b_1, ...$$

and we have the following strictly ascending chain

$$\mathrm{Ker}(b_1) < \mathrm{Ker}(b_2b_1) < ...,$$

which contradicts the hypothesis. Hence there exists a positive integer m such that $a_{m+1} \in W(S)$, i.e., $a_m - a_mc_ma_m \in W(S)$. This shows that $\bar{a}_m$ is a regular element of $S/W(S)$, and hence $\bar{a}_{m-1}, \bar{a}_{m-2}, ..., \bar{a}_1$ are regular elements of $S/W(S)$, i.e., $S/W(S)$ is von Neumann regular. We have $J(S) = W(S)$ by Lemma 3.7, proving that $S/J(S)$ is von Neumann regular. Thus S is right perfect by [5, Lemma 1.9]. □

Lemma 3.13 *Let M be a right R-module and $S = End(M)$. Then*

(1) $l_S(A(M)) = l_S(A)$ for all $A \subseteq S$ with $A(M) = \sum_{s \in A} s(M)$.

(2) $l_S(r_M(l_S(A))) = l_S(A)$ for all $A \subseteq S$.

Proof.(1). Let $a \in l_S(A)$, $a \cdot A = 0$. Therefore $a \cdot s = 0$ or $a(s(M)) = 0$ for all $s \in A$. This implies that $a \in l_S(A(M))$. Hence $l_S(A) \leq l_S(A(M))$. Conversely, for every $a \in l_S(A(M))$, we have $a.s(M) = 0$ for all $s \in A$. This implies that $a \in l_S(A)$.

(2). It is clear that $l_S(r_M(l_S(A))) \geq l_S(A)$. Conversely, for all $s \in l_S(A)$, $s.A(M) = 0$. This implies that $A(M) \leq r_M(l_S(A))$. Thus

$$l_S(A(M)) \geq l_S(r_M(l_S(A))).$$

By (1) we get the result. □

Let $\emptyset \neq A \subset S = \text{End}(M)$. Put

$$\text{Ker}A = \bigcap_{f \in A} \text{Ker}f = \{m \in M | f(m) = 0 \ \forall f \in A\}.$$

If $X \leq M$ and $X = \text{Ker}A$ for some $\emptyset \neq A \subset S$, X is called an M-annihilator.

Proposition 3.14 *Let M_R be a quasi-pseudo-gp-injective, self-generator module and $S = End(M_R)$. If M_R satisfies ACC on M-annihilators, then S is semiprimary.*

Proof. Now we will claim that S satisfies ACC on right annihilators or DCC on left annihilators. Indeed, we consider the descending chain

$$l_S(A_1) \geq l_S(A_2) \geq ... \text{ where } A_i \subseteq S,$$

then

$$r_M(l_S(A_1)) \leq r_M(l_S(A_2)) \leq ....$$

By our assumption, there exists $n \in \mathbb{N}$ such that $r_M(l_S(A_n)) = r_M(l_S(A_k))$ for all $k > n$, and so $l_S r_M(l_S(A_n)) = l_S r_M(l_S(A_k))$. By Lemma 3.13, $l_S(A_n) = l_S(A_k)$ for all $k > n$. This shows that S satisfies DCC on left annihilators or ACC on right annihilators. Therefore $J(S)$ is nilpotent by [14, Lemma 3.29] and Lemma 3.7. It follows that S is semiprimary by Theorem 3.12. □

Corollary 3.15 *If R is right pseudo GP-injective and satisfies ACC on right annihilators, then R is semiprimary.*

For quasi-pseudo-p-injective modules, we have

Theorem 3.16 *Let M_R be a quasi-pseudo-p-injective module and $S = End(M_R)$. If M satisfies ACC on M-annihilators, then S is semiprimary.*

Proof. Consider the chain $Sf_1 \geq Sf_2 \geq \cdots$ of cyclic left ideals of S. Then we have $\text{Ker}f_1 \leq \text{Ker}f_2 \leq \cdots$. By hypothesis, there exists $n \in \mathbb{N}$ such that $\text{Ker}f_n = \text{Ker}f_{n+k}$, $\forall k \in \mathbb{N}$. It follows that $Sf_n = Sf_{n+k}$ $\forall k \in \mathbb{N}$. Thus R is right perfect.

Consider the ascending chain $r_M(J(S)) \leq r_M(J(S)^2) \leq \cdots$. By assumption, there is $n \in \mathbb{N}$ such that $r_M(J(S)^n) = r_M(J(S)^{n+k})$ for all $k \in \mathbb{N}$. Let $B = J(S)^n$.

Then we get $r_M(B) = r_M(B^2)$. Assume $J(S)$ is not nilpotent. Then $B^2 \neq 0$ and the non-empty set

$$\{\mathrm{Ker} g |\ g \in B \text{ and } Bg \neq 0\}$$

has a maximal element $\mathrm{Ker} g_0,\ g_0 \in B$. The relation $BBg_0 = 0$ would imply that $\mathrm{Im} g_0 \leq r_M(B^2) = r_M(B)$ and hence $Bg_0 = 0$, contradicting to the choice of g_0. Therefore we can find an $h \in B$ with $Bhg_0 \neq 0$. However, since $\mathrm{Ker} g_0 \leq \mathrm{Ker}(hg_0)$, the maximality of $\mathrm{Ker} g_0$ implies that $\mathrm{Ker} g_0 = \mathrm{Ker} hg_0$. Since M is quasi-pseudo-p-injective, this implies that $Sg_0 = Shg_0$, i.e. $g_0 = shg_0$ for some $s \in S$ or $g_0(1 - sh) = 0$. Since $sh \in B \leq J(S)$, this gives $g_0 = 0$, a contradiction. Thus $J(S)$ must be nilpotent. □

Following [14], a ring R is called *directly finite* if $ab = 1$ in R implies that $ba = 1$.

Proposition 3.17 *A right pseudo P-injective ring R is directly finite if and only if all monomorphisms $R_R \to R_R$ are isomorphisms.*

Proof. Assume that $\varphi : R_R \to R_R$ is a monomorphism. Let $a = \varphi(1)$. Then $r(a) = 0 = r(1)$ and so $Ra = R$ by Corollary 2.2. Hence $ba = 1$ for some $b \in R$, so $ab = 1$ by hypothesis, and so φ is onto. Conversely, let $ab = 1$ in R. Therefore the homomorphism $\alpha : R \to R,\ \alpha(r) = br,\ \forall r \in R$ is monomorphism. By hypothesis α is an epimorphism. There exists $c \in R$ such that $1 = \alpha(c) = bc$. It follows that $a = c$ and $ba = 1$. □

The series of higher left socles $\{S^l_\alpha\}$ of the ring R are defined inductively as follows: $S^l_1 = Soc({}_RR)$, and $S^l_{\alpha+1}/S^l_\alpha = Soc(R/S^l_\alpha)$ for each ordinal $\alpha \geq 1$.

Motivated by [3, Lemma 9 (ii)], we have the following proposition.

Proposition 3.18 *If R is a right pseudo GP-injective ring and satisfies ACC on essential left ideals, then*

(1) $r(J) \leq^e R_R$,

(2) J *is nilpotent,*

(3) $J = lr(J)$.

Proof. (1) Since R has ACC on essential left ideals, R/S_l is a left Noetherian ring. Then, there exists $k > 0$ such that $S^l_k = S^l_{k+1} = \cdots$ and R/S^l_k is a right Noetherian ring. Now we will claime that $S^l_k \leq^e R_R$. In fact, assume that $xR \cap S^l_k = 0$ for some $0 \neq x \in R$. Let $\bar{R} = R/S^l_k$ and $l_{\bar{R}}(\bar{a})$ be maximal in the set $\{l_{\bar{R}}(\bar{y}) |\ 0 \neq y \in xR\}$. Since $S^l_k = S^l_{k+1}$, we get $Soc({}_{\bar{R}}\bar{R}) = 0$, and so $\bar{R}\bar{a}$ is not simple as left $\bar{R}$-module. Thus there exists $t \in R$ such that $0 \neq \bar{R}\bar{t}\bar{a} < \bar{R}\bar{a}$.

If $\bar{a}\bar{t}\bar{a} = \bar{0}$, then $ata \in aR \cap S_k^l = 0$, and so $ata = 0$. From this fact and pseudo GP-injectivity of R, we see that if $r(ta) = r(b)$, $b \in R$ then $Rta = Rb$ by Corollary 2.2. If $r(a) = r(ta)$, then $Ra = Rta$, a contradiction. Thus $r(a) < r(ta)$. Then there exixts $b \in R$ such that $ab \neq 0$ and $tab = 0$. That means $0 \neq ab \in xR$ and $l_{\bar{R}}(\bar{a}) < l_{\bar{R}}(\overline{ab})$. This contradicts to the maximality of $l_{\bar{R}}(\bar{a}_0)$.

If $\bar{a}\bar{t}\bar{a} \neq \bar{0}$, then $0 \neq \bar{R}\bar{a}\bar{t}\bar{a} < \bar{R}\bar{a}$. Since R is right pseudo GP-injective, there exists $m \in \mathbb{N}$ such that $(ata)^m \neq 0$ and if $r((ata)^m) = r(b)$, $b \in R$ then $b \in R(ata)^m$. It follows that $r(a) < r((ata)^m)$. Let $c \in r((ata)^m) \setminus r(a)$. Then $0 \neq ac \in xR$, $(\bar{a}\bar{t}\bar{a})^{m-1}\bar{a}\bar{t} \in l_{\bar{R}}(\overline{ac}) \setminus l_{\bar{R}}(\bar{a})$, a contradiction.

Thus $S_k^l \leq^e R_R$ and hence $r(J) \leq^e R_R$ (since $S_k^l \leq r(J)$).

(2). By [3, Lemma 9 (ii)].

(3). Since $r(J) \leq^e R_R$, $lr(J) \leq Z_r = J$. □

A module M_R is called *extending (or CS)* if every submodule of M is essential in a direct summand of M. A ring R is called right CS if R_R is CS (see [7]). Following [10], a module M is called *NCS* if there are no nonzero complement submodules which is small in M. A ring R is *right NCS* if R_R is NCS. Clearly every CS module is NCS, but the converse is not true, as we can see that the $\mathbb{Z}$-module $\mathbb{Z}_2 \oplus \mathbb{Z}_8$ is NCS but not CS. On the other hand, let K be a division ring and V be a left K-vector space of infinite dimension. Let $S = End_K(V)$. Take $R = \begin{pmatrix} S & S \\ S & S \end{pmatrix}$, then R is right NCS but not right CS.

Proposition 3.19 *If R is a left Noetherian, right pseudo P-injective and right NCS ring, then R is left Artinian.*

Proof. First, we prove that $\bar{R} = R/J$ is a regular ring. Assume that $a \notin J$. Since $J = lr(J) = Z_r$, there exists a nonzero complement right ideal I of R such that $r(a) \cap I = 0$ by Lemma 3.18. We claim that there exists $b \in I$ such that $ab \notin J$. Suppose on the contrarily that $aI \leq J$. Then $aIr(J) = 0$. Since $r(a) \cap I = 0, Ir(J) \leq I \cap r(a) = 0$. Thus $I \leq lr(J) = J$. It follows that I is small in R_R, a contradiction. Hence we have $b \in I$ such that $r(a) \cap bR = 0$ and $ab \notin J$. It follows that $r(b) = r(ab)$. Hence $Rb = Rab$ and so $b = cab$ for some $c \in R$. This implies that $\bar{b} \in r_{\bar{R}}(\bar{a} - \bar{a}\bar{c}\bar{a})$, where $\bar{r} = r + J \in R/J$ for any $r \in R$. Since $\overline{ab} \neq \bar{0}$, we see that $r_{\bar{R}}(\bar{a}) < r_{\bar{R}}(\bar{a} - \bar{a}\bar{c}\bar{a})$. If $a - aca \in J$, then a is a regular element of R. If $a - aca \notin J$, let $a_1 = a - aca$. Then $r(a_1)$ is not essential in R_R. By the same way, we get $a_2 = a_1 - a_1c_1a_1$ for some $c_1 \in R$ and $r_{\bar{R}}(\bar{a}_1) < r_{\bar{R}}(\bar{a}_2)$. If $a_2 \in J$, then a_1 is a regular element of R. It follows that a is a regular element of R. If $a_2 \notin J$, we have $a_3 = a_2 - a_2c_2a_2$ for some $c_2 \in R$ and $r_{\bar{R}}(\bar{a}_2) < r_{\bar{R}}(\bar{a}_3)$. Continuing this process, we get $a_k \in R$, $k = 1, 2, \ldots$. Since R is left noetherian and $Jac(\bar{R}) = 0$, $\bar{R}$ is a semiprime and left Goldie ring. By [9, Lemma 5.8], $\bar{R}$ satisfies ACC on right annihilators. Hence there exists some positive integer m such that $a_m \in J$, and thus a is also a regular element of R. Since $\bar{a}$ is an arbitrary nonzero element of $\bar{R}$, we see that $\bar{R}$ is a regular ring. Then $\bar{R}$ is semisimple because R is left noetherian. Moreover, by Lemma 3.18, J is nilpotent and so R is semiprimary. Thus R is left artinian. □

4 On maximal ideals

In this section, we study the endomorphism ring of quasi-pseudo-gp-injective modules.

Let $S = \mathrm{End}_R(M)$ be the endomorphism ring of a right R-module M. Following [19], an element $u \in S$ is called a *right uniform element* of S if $u \neq 0$ and $u(M)$ is a uniform submodule of M. An element $u \in R$ is called right uniform if uR is a uniform right ideal (see [14]). In this section, we generalize some results of Sanh and Shum for quasi-p-injective modules; Nicholson and Yousif for p-injective rings to quasi-pseudo-gp-injective modules.

First, we need the following lemma:

Lemma 4.1 *Let M be a quasi-pseudo-gp-injective module and $S = End(M)$. Then for any right uniform element u of S, the set*

$$A_u = \{s \in S | \mathrm{Ker} s \cap \mathrm{Im} u \neq 0\}$$

is the unique maximal left ideal of S containing $l_S(\mathrm{Imu})$.

Proof. Clearly, A_u is a left ideal of S. It is easy to see that $l_S(\mathrm{Im}u) \leq A_u$ and $A_u \neq S$ (because $1 \notin A_u$). We now claim that A_u is maximal. In fact, for any $s \in S \setminus A_u$, we have $\mathrm{Im}u \cap \mathrm{Ker}s = 0$, whence $su \neq 0$. There exists $m \in \mathbb{N}$ such that $(su)^m \neq 0$ and if $\mathrm{Ker}(su)^m = \mathrm{Ker}(g)$, $g \in S$ then $g \in S(su)^m$. Since $\mathrm{Ker}((su)^m) = \mathrm{Ker}u$, we get $S(su)^m = Su$. Then there exists $t \in S$ such that $(1 - t(su)^{m-1}s)u = 0$. It follows from $S = l_S(u) + Ss$, that A_u is maximal in S. It remains to show that A_u is unique. In fact, assume that there is another maximal left ideal L of S containing $l_S(\mathrm{Im}u)$ and $L \neq A_u$. Repeating above process we also have $S = L$, a contradiction. □

Corollary 4.2 ([19, Lemma 1]) *Let M be a quasi-p-injective module and $S = End(M)$. Then for any right uniform element u of S, the set*

$$A_u = \{s \in S | \mathrm{Ker} s \cap \mathrm{Im} u \neq 0\}$$

is the unique maximal left ideal of S containing $l_S(\mathrm{Imu})$.

Corollary 4.3 *Let R be right pseudo GP-injective. If $u \in R$ is a right uniform element, define*

$$M_u = \{x \in R | r(x) \cap uR \neq 0\}.$$

Then M_u is the unique maximal left ideal which contains $l(u)$.

The following lemma is a generalization of Lemma 3 in [19] .

Lemma 4.4 *Let M be a quasi-pseudo-p-injective module, $S = \mathrm{End}(M_R)$ and $W = \oplus_{i=1}^{n} u_i(M)$ a direct sum of uniform submodule $u_i(M)$ of M. If $A \leq S$ is a maximal left ideal which is not of the form A_u for some right uniform element u of S, then there is $\psi \in A$ such that $\mathrm{Ker}(1-\psi) \cap W$ is essential in W.*

Proof. Since $A \neq A_{u_1}$, we can take $k \in A \setminus A_{u_1}$. Then $\mathrm{Im} u_1 \cap \mathrm{Ker} k = 0$, whence $ku_1 \neq 0$. There exists $m \in \mathbb{N}$ such that $(ku_1)^m \neq 0$ and if $\mathrm{Ker}(ku_1)^m = \mathrm{Ker}(g)$, $g \in S$ then $g \in S(ku_1)^m$. It is easy to see that $\mathrm{Ker}(ku_1)^m = \mathrm{Ker}(u_1)$ and hence $S(ku_1)^m = Su_1$. Consequently we have $u_1 = \alpha_1(ku_1)^m$ for some $\alpha_1 \in S$. Let $\varphi_1 = \alpha_1(ku_1)^{m-1}k \in SA \subset A$. Then $(1-\varphi_1)u_1 = 0$. This shows that $\mathrm{Ker}(1-\varphi_1) \cap u_1(M) = u_1(M) \neq 0$. If $\mathrm{Ker}(1-\varphi_1) \cap u_i(M) \neq 0$ for all $i \geq 2$, then we are done and in this case $\oplus_{i=1}^{n}(\mathrm{Ker}(1-\varphi_1) \cap u_i(M)) \leq^e W$. Without loss of generality, we now assume that $\mathrm{Ker}(1-\varphi_1) \cap u_2(M) = 0$. It follows that $(1-\varphi_1)(u_2(M)) \simeq u_2(M)$ is uniform. Since $A \neq A_{(1-\varphi_1)u_2}$, we can take any $h \in A \setminus A_{(1-\varphi_1)u_2}$. By using the above argument, there exists $\alpha_2 \in S$ such that $(1-\varphi_1)u_2 = \alpha_2 h(1-\varphi_1)u_2$. It follows that

$$(1-(\alpha_2 h + \varphi_1 - \alpha_2 h\varphi_1))u_2 = 0.$$

Let $\varphi_2 = \alpha_2 h + \varphi_1 - \alpha_2 h\varphi_1$. Then $(1-\varphi_2)u_i = 0$ for $i = 1, 2$. Continuing this way, we eventually obtain a $\psi \in A$ such that $Ker(1-\psi) \cap u_i(M) \neq 0$ for all $i = 1, \ldots, n$. In other words, we have shown that $\mathrm{Ker}(1-\psi) \cap W$ is essential in W as required. □

The following theorem describes the properties of the endomorphism ring $S = \mathrm{End}(M_R)$ of a quasi pseudo p-injective module M_R.

Theorem 4.5 *Let M be a quasi-pseudo-gp-injective, self-generator module with finite Goldie dimension and $S = \mathrm{End}(M_R)$.*

(1) If $I \subset S$ is a maximal left ideal, then $I = A_u$ for some right uniform element $u \in S$.

(2) S is semilocal.

Proof. Since M is a self-generator which has finite Goldie dimension, there exist elements $u_1, u_2, ..., u_n$ of S such that $W = u_1(M) \oplus u_2(M) \oplus \cdots \oplus u_n(M)$ is essential in M, where each $u_i(M)$ is uniform. Moreover, M is a quasi-p-injective module, we have $J(S) = W(S) = \{s \in S | \text{ Ker}(s) \text{ is essential in } M\}$ by Lemma 3.7.

(1). Suppose on the contrary that I is not of the form A_u for some right uniform element of $u \in S$. Then by Lemma 4.4, there exists a $\varphi \in I$ such that $\text{Ker}(1-\varphi) \cap W$ is essential in W. It follows that $1-\varphi \in J(S) \subset I$, a contradiction. Hence $I = A_u$ for some right uniform element $u \in S$.

(2). If $\varphi \in A_{u_1} \cap A_{u_2} \cap \cdots \cap A_{u_n}$, then $\text{Ker}(\varphi) \cap u_i(M) \neq 0$ for each i. Hence $\text{Ker}(\varphi)$ is essential in M. Therefore $\varphi \in J(S)$, i.e., $A_{u_1} \cap \cdots \cap A_{u_n} = J(S)$. This shows that $S/J(S)$ is semisimple. □

As a consequence, we immediately get the following result for the right pseudo GP-injective rings.

Corollary 4.6 *Let R be a right pseudo GP-injective ring which has right finite Goldie dimension. Then*

(1) If $I \subset R$ is a maximal left ideal, then $I = A_u$ for some right uniform element $u \in R$.

(2) R is semilocal.

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