Asymptotic properties of multivariate order statistics with random index

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Abstract

The class of limit distribution functions (df's) of multivariate order statistics from independent and identical random vectors with random sample size is fully characterized. Two cases in this study are considered, the first case is when the random sample size is assumed to be independent of all the basic random vectors, and the second case is when the interrelation of the random size and the basic random vectors is not restricted.

Keywords: Weak convergence; random sample size; multivariate order statistics.

1 Introduction

The pioneer papers Finkelstein (1953), Tiago de Oliveira (1958), Gumbel (1960) and Galambos (1975) gave the foundations for the multivariate approach to extreme value distributions. Following these works several bivariate and trivariate extreme value models began to appear in the literature. In fact, many works, e.g., Mikhailov (1974), Tiago de Oliveira (1975), Marshall and Olkin (1983), Galambos (1987), Takahashi (1994), Barakat (1990, 1997, 2001) and Barakat, et al. (2004, 2012) have been devoted to study the asymptotic behaviour, as well as the conditions of the convergence of the bivariate and trivariate extremes. In the last two decades much attention has been paid to multivariate order statistics, especially the models for multivariate extremes based on extreme value theory. These models have attracted a great deal of attention particularly in the area of environmental extremes. For example, in the analysis of environmental extreme value data there is often need to study joint inter-site extreme behaviour: examples are joint flooding at various see ports, or at various rain gauges. If measurements of m characteristics are taken on the same members of the population, then the observed random quantities follow some type of multivariate distribution. Let this distribution of a random vector $\underline{X} = (X_1, X_2, ..., X_m)$ be $F(\underline{x}) = F(x_1, x_2, ..., x_m)$. Consider a sequence of n independent $m-$dimensional random vectors $\underline{X}_j = (X_{1,j}, X_{2,j}, ..., X_{m,j}), j = 1, 2, ..., n$, with common df $F(\underline{x}) = P(X_{1,j} < x_1, X_{2,j} < x_2, ..., X_{m,j} < x_m)$ and the survival function $G(\underline{x}) = P(\underline{X} \geq \underline{x}) = P(X_{1,j} \geq x_1, X_{2,j} \geq x_2, ..., X_{m,j} \geq x_m)$. Let $F_{\underline{t}}(\underline{x}_{\underline{t}})$ and $G_{\underline{t}}(\underline{x}_{\underline{t}})$ be the possible marginals of $F(\underline{x})$ and $G(\underline{x})$, respectively, where $\underline{t} = (t_1, t_2, ..., t_k), k = 1, 2, ..., m-1, 1 \leq t_k \leq m$, and $\underline{x}_{\underline{t}} = (x_{t_1}, x_{t_2}, ..., x_{t_k})$. The order statistics of the kth marginal random sample $X_{k,1}, X_{k,2}, ..., X_{k,n}$, $k = 1, 2, ..., m$, are $X_{k,1:n} \leq X_{k,2:n} \leq ... \leq X_{k,n:n}$. Write $\underline{Z}_{\underline{k}:n}$ and $\underline{W}_{\underline{k}:n}$ to denote the random vectors $(X_{1,n-k_1+1:n}, ..., X_{m,n-k_m+1:n})$ and $(X_{1,k_1:n}, ..., X_{m,k_m:n})$, respectively, where $\underline{k} = (k_1, ..., k_m)$ are any positive integers (independent of n). Clearly, any investigation of $\underline{Z}_{\underline{k}:n}$ can be easily carried over $\underline{W}_{\underline{k}:n}$ by turning to $-\underline{X}_j$. In many biological, agricultural and military activities problem it is almost impossible to have a fixed sample size, because some observations are always lost for various reasons. Therefore, the sample size n itself is considered frequently to be a random variable (rv) ν_n. In this case the random vector $\underline{Z}_{\underline{k}:\nu_n}$ is called the random extreme vector. In this paper, the asymptotic behavior of the vector $\underline{Z}_{\underline{k}:\nu_n}$ is investigated, assuming that the random sample size itself is a positive integer-valued rv ν_n, which weakly converges ($\xrightarrow[n]{w}$) to a nondegenerate limit, as $n \to \infty$. Subsequently, operations and relations for vectors are understood componentwise. Given $\underline{a}$, $\underline{b}$, $\underline{x}$, $\underline{y} \in \mathbb{R}^m$, let $\underline{a}\underline{x} + \underline{b} = (a_1x_1 + b_1, ..., a_mx_m + b_m)$ and $\underline{x} \leq \underline{y}$ means $x_j \leq y_j, j = 1, ..., m$. Moreover, for any suitable normalizing constants $\underline{a}_n = (a_{1,n}, ..., a_{m,n}) > (0, 0, ..., 0) = \underline{0}$ and $\underline{b}_n = (b_{1,n}, ..., b_{m,n})$, let $F^{\langle n \rangle}(\underline{x}) = F(\underline{a}_n\underline{x} + \underline{b}_n)$, $G^{\langle n \rangle}(\underline{x}) = G(\underline{a}_n\underline{x} + \underline{b}_n)$, $F_{\underline{t}}^{\langle n \rangle}(\underline{x}_{\underline{t}}) = F_{\underline{t}}(\underline{a}_n\underline{x}_{\underline{t}} + \underline{b}_n)$, $G_{\underline{t}}^{\langle n \rangle}(\underline{x}_{\underline{t}}) = G_{\underline{t}}(\underline{a}_n\underline{x}_{\underline{t}} + \underline{b}_n)$ and $H_{\underline{k}:n}^{\langle n \rangle}(\underline{x}) = H_{\underline{k}:n}(\underline{a}_n\underline{x} + \underline{b}_n) = P(\underline{Z}_{\underline{k}:n} < \underline{a}_n\underline{x} + \underline{b}_n)$. Finally, we adopt the abbreviations $\max(a_1, ..., a_n) = \bigvee_{i=1}^n a_i$ and $\min(a_1, ..., a_n) = \bigwedge_{i=1}^n a_i$.

The key ingredient in getting a suitable exact expression of $H_{\underline{k}:n}$ is the realization of the event $E = \{X_{j,n-k_j+1:n} \leq x_j, j = 1, ..., m\}$ under certain collection of conditions $\mathcal{C}_n$ as follows: Let $\underline{x} = (x_1, ..., x_m)$ be a fixed point in $\mathbb{R}^m$ and define for $i_k \in \{0, 1\}, k = 1, 2, ..., m$, the random events $E_{\underline{i}}(\underline{x}) = \{\underline{X} \in \mathbb{R}^m : X_k \geq x_k, \text{if } i_k = 0 \text{ and } X_k < x_k, \text{ if } i_k = 1, k = 1, 2, ..., m\}$. The derivation of the df of $H_{\underline{k}:n}$ for all rank vector $\underline{k}$ is mainly based on the finding the probabilities of the events $E_{\underline{i}}(\underline{x})$. Barakat and Nigm (2012) calculated these probabilities by considering the set of indices $\underline{i}$ and denote $I_{\underline{i}}^0$ and $I_{\underline{i}}^1$ the sets of ranks associated with the null and units subindex values, respectively (e.g., if $\underline{i} = (0, 0, 1, 1)$, we get $I_{\underline{i}}^0 = I_{(0,0,1,1)}^0 = (1, 2)$ and $I_{\underline{i}}^1 = I_{(0,0,1,1)}^1 = (3, 4)$). In this case the probabilities of these events can be easily evaluated for $\underline{i} \neq (1,1,...,1)$ in terms of the marginal survival function of $\underline{X}$ as $P(E_{\underline{i}}(\underline{x})) = P_{\underline{i}}(\underline{x}) = G_{I^0_{\underline{i}}}(\underline{x}_{I^0_{\underline{i}}}) + \sum_{I \subseteq I^1_{\underline{i}}} (-1)^{card(I)} G_{I^0_{\underline{i}} \cup I}(\underline{x}_{I^0_{\underline{i}} \cup I})$, where $card(I)$ refers to the cardinality (number of elements) of the set I. Barakat and Nigm (2012) have used the above idea to introduce the following two theorems.

Theorem 1.1. *For any extreme rank vector* $\underline{k} = (k_1, ..., k_m)$ *in* $\mathbb{R}^m$, *we get*

$$H_{\underline{k}:n}(\underline{x}) = n! \sum_{i_{1_1}=0}^{k_1-1} ... \sum_{i_{1_m}=0}^{k_m-1} \sum_{r_{\underline{i}} \in \mathcal{C}_n} \prod_{\underline{i} \in \{0,1\}^m} \frac{(G_{I^0_{\underline{i}}}(\underline{x}_{I^0_{\underline{i}}}) + \sum_{I \subseteq I^1_{\underline{i}}} (-1)^{card(I)} G_{I^0_{\underline{i}} \cup I}(\underline{x}_{I^0_{\underline{i}} \cup I}))^{r_{\underline{i}}}}{r_{\underline{i}}!}, \tag{1.1}$$

where $\mathcal{C}_n = \{r_{\underline{i}} \in \mathbb{N}^+ : \sum_{\underline{i} \in \{0,1\}^m} r_{\underline{i}} = n, \ \sum_{\underline{i} \in \Omega_j} r_{\underline{i}} = i_{1_j}, \ j = 1, 2, ..., m\}$ *and* $\Omega_j = \{\underline{i} = (i_1, i_2, ..., i_m) \in \{0,1\}^m : i_j = 0\}$.

Theorem 1.2. *For any suitable normalizing vectors of constants* $\underline{a}_n = (a_{1,n},...,a_{m,n}) > \underline{0}$ *and* $\underline{b}_n \in \mathbb{R}^m$ *and an extreme rank vector* $\underline{k} = (k_1, ..., k_m)$, *we have*

$$H^{\langle n \rangle}_{\underline{k}:n}(\underline{x}) \xrightarrow[n]{w} H_{\underline{k}}(\underline{x}), \tag{1.2}$$

where $H_{\underline{k}}(\underline{x})$ *is a nondegenerate df if and only if, for all* $\underline{x}$ *for which the univariate marginals of* $H_{\underline{k}}(\underline{x})$ $(H_{k_1}(x_1) = H_{\underline{k}}(x_1, \underline{\infty}), H_{k_2}(x_2) = H_{\underline{k}}(\infty, x_2, \underline{\infty}), ..., H_{k_m}(x_m) = H_{\underline{k}}(\underline{\infty}, x_m))$ *are positive, the limits*

$$n\, G^{\langle n \rangle}_{I^0_{\underline{i}}}(\underline{x}_{I^0_{\underline{i}}}) \to h_{I^0_{\underline{i}}}(\underline{x}_{I^0_{\underline{i}}}), \ \forall\ \underline{i} \in \{0,1\}^m, \ \text{as } n \to \infty, \tag{1.3}$$

$$n\, G^{\langle n \rangle}_{I^0_{\underline{i}} \cup I}(\underline{x}_{I^0_{\underline{i}} \cup I}) \to h_{I^0_{\underline{i}} \cup I}(\underline{x}_{I^0_{\underline{i}} \cup I}), \ \forall\ I \subseteq I^1_{\underline{i}}, \ \underline{i} \in \{0,1\}^m / \underline{1}, \ \text{as } n \to \infty, \tag{1.4}$$

are finite, and the function

$$H_{\underline{k}}(\underline{x}) = H_{\underline{1}}(\underline{x}) \sum_{i_{1_1}=0}^{k_1-1} \sum_{i_{1_2}=0}^{k_2-1} ... \sum_{i_{1_m}=0}^{k_m-1} \sum_{r_{\underline{i}} \in \mathcal{C}}$$

$$\cdot \prod_{\underline{i} \in \{0,1\}^m / \underline{1}} \frac{(h_{I^0_{\underline{i}}}(\underline{x}_{I^0_{\underline{i}}}) + \sum_{I \subseteq I^1_{\underline{i}}} (-1)^{card(I)} h_{I^0_{\underline{i}} \cup I}(\underline{x}_{I^0_{\underline{i}} \cup I}))^{r_{\underline{i}}}}{r_{\underline{i}}!} \tag{1.5}$$

is a nondegenerate df, where $H_{\underline{1}}(\underline{x}) = \exp(\sum_{j=1}^{m} (-1)^j \sum_{1 \le i_1 < i_2 < ... < i_j \le m} h_{i_1,...,i_j}(x_{i_1}, ..., x_{i_j}))$ *(note that* $h_{i_1,...,i_m}(x_{i_1}, ..., x_{i_m}) = h(\underline{x}) = h_{I^0_{\underline{1}}}(\underline{x})$*) and* $\mathcal{C} = \{r_{\underline{i}} \in \mathbb{N}^+ : \sum_{\underline{i} \in \Omega_j} r_{\underline{i}} = i_{1_j}, \ j = 1, 2, ..., m\}$. *The actual limit df of* $\underline{Z}_{\underline{k}:n}$ *is the one given by (1.5). Moreover, the components of* $\underline{Z}_{\underline{k}:n}$ *are asymptotically independent if and only if* $h_{i_1,i_2}(x_{i_1}, x_{i_2}) \equiv 0$, *for all* $1 \le i_1 < i_2 \le m$.

Remark 1.1. If, we let each component x_j of $\underline{x}$ in $H_{\underline{k}}(\underline{x})$, except $x_t, 1 \leq t, j \leq m, t \neq j$, tend to infinity, we obtain the tth univariate marginal of $H_{\underline{k}}(\underline{x})$. Namely, $H_{k_t}(x_t) = 1 - \Gamma_{k_t}(h_t(x_t)) = \sum_{i_t=0}^{k_t-1} \frac{h_t^{i_t}(x_t)}{i_t!} e^{-h_t(x_t)}, t = 1, 2, ..., m$. In view of the Extremal Types Theorem (see, Galambos, 1987) the function $h_t(x_t)$ can take one and only one of the three types $h_t^{(1)}(x_t) = x_t^{-\alpha}, x_t > 0$; $h_t^{(2)}(x_t) = (-x_t)^{\alpha}, x_t \leq 0$; and $h_t^{(3)}(x_t) = e^{-x_t}$, where $\alpha > 0$.

Remark 1.2. For all integers $k_1, k_2, ..., k_m$, the limit $H_{\underline{k}}(\underline{x})$ is continuous in $\underline{x}$ (since, the univariate marginals $H_{k_t}(x_t),\ t = 1, 2, ..., m$, are all differentiable). Hence, the convergence in (1.2) is uniform with respect to $\underline{x}$. We conclude this section by introducing a lemma, which is due to Helly and it will be used in the next section. Moreover, it is independent of the number of dimensions (see Feller, 1979).

Lemma 1.1. *Every sequence of df's $\{F_n\}$ possesses a subsequence $\{F_{n_k}\}$, that converges to an extended df F (by the extended df we mean that $F(\infty) - F(-\infty) \leq 1$). Moreover, a necessary and sufficient condition for such a limit to be a proper one is that $\{F_n\}$ is stochastically bounded (for the definition see Feller, 1979, p. 247). Finally, $F_n \xrightarrow[n]{w} F$ if and only if the limit of every convergent subsequence equals F.*

2 Asymptotic Properties of the Random Extreme Vector, when the Random Sample Size and the Basic rv's are Independent

Throughout this section we deal with the weak convergence of different multivariate extreme order statistics, when the sample size itself is a rv ν_n, which is assumed to be independent of the basic random vector. Consider now the following assumptions:

$$H_{\underline{k}:n}^{\langle n \rangle}(\underline{x}) \xrightarrow[n]{w} H_{\underline{k}}(\underline{x}), \qquad [\mathcal{A}]$$

$$A_n(nx) = P(\frac{\nu_n}{n} < x) \xrightarrow[n]{w} A(x), \qquad [\mathcal{B}]$$

where $H_{\underline{k}}(\underline{x})$ is a nondegenerate df and $A(x)$ is a df with $A(+0) = 0$,

$$H_{\underline{k}:\nu_n}^{\langle n \rangle}(\underline{x}) = P(\underline{Z}_{\underline{k}:\nu_n}^{\langle n \rangle} < \underline{x}) \xrightarrow[n]{w} \Psi_{\underline{k}}(\underline{x}) = \int_0^{\infty} \widetilde{H}_{\underline{k}}(zh_{I_{\underline{i}}^0}(\underline{x}_{I_{\underline{i}}^0}), zh_{I_{\underline{i}}^0 \cup I}(\underline{x}_{I_{\underline{i}}^0 \cup I}))dA(z), \qquad [\mathcal{C}]$$

where

$$\widetilde{H}_{\underline{k}}(zh_{I_{\underline{i}}^0}(\underline{x}_{I_{\underline{i}}^0}), zh_{I_{\underline{i}}^0 \cup I}(\underline{x}_{I_{\underline{i}}^0 \cup I})) = H_{\underline{1}}(z, \underline{x}) \sum_{i_{1_1}=0}^{k_1-1} \cdots \sum_{i_{1_m}=0}^{k_m-1} \sum_{r_{\underline{i}} \in C}$$

$$\prod_{\underline{i} \in \{0,1\}^m / \underline{1}} \frac{(zh_{I_{\underline{i}}^0}(\underline{x}_{I_{\underline{i}}^0}) + \sum_{I \subseteq I_{\underline{i}}^1} (-1)^{card(I)} zh_{I_{\underline{i}}^0 \cup I}(\underline{x}_{I_{\underline{i}}^0 \cup I}))^{r_{\underline{i}}}}{r_{\underline{i}}!},$$

$$H_{\underline{1}}(z,\underline{x})=\exp\left(\sum_{j=1}^{m}(-1)^j\sum_{1\le i_1<i_2<...<i_j\le m} zh_{i_1,...,i_j}(x_{i_1},...,x_{i_j})\right).$$

Theorem 2.1. *The following implications hold:*

$$(1)\quad [\mathcal{A}]+[\mathcal{B}]\Rightarrow[\mathcal{C}],\qquad (2)\quad [\mathcal{A}]+[\mathcal{C}]\Rightarrow[\mathcal{B}],\qquad (3)\quad [\mathcal{B}]+[\mathcal{C}]\Rightarrow[\mathcal{A}],$$

where in the third implication, $A(x)$ *is assumed to be a nondegenerate df.*

Remark 2.1. The continuity of the limit df $H_{\underline{k}}(\underline{x})$ in $[\mathcal{A}]$ implies the continuity of the limit $\Psi_{\underline{k}}(\underline{x})$. Hence the convergence in $[\mathcal{C}]$ is uniform with respect to $\underline{x}$.

Remark 2.2. It is natural to look for the limitations on ν_n, under which we get the relation $H_{\underline{k}}(\underline{x})=\Psi_{\underline{k}}(\underline{x})$, $\forall\underline{x}$. In view of Theorem 2.1, the last equation is satisfied if and only if the df $A(z)$ is degenerate at one, which means the asymptotically almost randomlessness of ν_n. In practice, this limitation is satisfied, when the rv ν_n has a Poisson distribution with mean n, or ν_n has a binomial distribution$(p=1-\frac{1}{n},n)$. Moreover, in view of Theorems 1.2 and 2.1, we deduce that the components of the vector $\underline{Z}^{(n)}_{\underline{k}:\nu_n}=(\underline{Z}_{\underline{k}:\nu_n}-\underline{b}_n)/\underline{a}_n$ are asymptotically independent if and only if $h_{i_1,i_2}(x_{i_1},x_{i_2})\equiv 0$, for all $1\le i_1<i_2\le m$, and $A(x)$ is degenerate df at one. Throughout the proof, due to Remark 2.2, we assume that $A(x)$ is a nondegenerate df.

Proof of the implication $[\mathcal{A}]+[\mathcal{B}]\Rightarrow[\mathcal{C}]$: First, we note that $H^{\langle n\rangle}_{\underline{k}:n}(\underline{x})$ can be written in the form (see Barakat and Nigm, 2009)

$$H^{\langle n\rangle}_{\underline{k}:n}(\underline{x})=\sum_{i_{1_1}=0}^{k_1-1}\sum_{i_{1_2}=0}^{k_2-1}...\sum_{i_{1_m}=0}^{k_m-1}\sum_{r_{\underline{i}}\in C_n^-} F^{\langle n\rangle^{n-\sum_{\underline{i}\in\{0,1\}^m/\underline{1}} r_{\underline{i}}}}(\underline{x})(1+o(1))$$

$$\prod_{\underline{i}\in\{0,1\}^m/\underline{1}}\frac{(nG^{\langle n\rangle}_{I^0_{\underline{i}}}(\underline{x}_{I^0_{\underline{i}}})+\sum_{I\subseteq I^1_i}(-1)^{card(I)}nG^{\langle n\rangle}_{I^0_{\underline{i}}\cup I}(\underline{x}_{I^0_{\underline{i}}\cup I}))^{r_{\underline{i}}}}{r_{\underline{i}}!},\tag{2.1}$$

where $\mathcal{C}_n^-=\{r_{\underline{i}}\in\mathbb{N}^+:\sum_{\underline{i}\in\{0,1\}^m/\underline{1}} r_{\underline{i}}\le n,\ \sum_{\underline{i}\in\Omega_j} r_{\underline{i}}=i_{1_j}, j=1,2,...,m\}\to\mathcal{C}=\{r_{\underline{i}}\in\mathbb{N}^+:\sum_{\underline{i}\in\Omega_j} r_{\underline{i}}=i_{1_j},\ j=1,2,...,m\}$, as $n\to\infty$,

$$F^{\langle n\rangle^{n-\sum_{\underline{i}\in\{0,1\}^m/\underline{1}} r_{\underline{i}}}}(\underline{x})=(1+\sum_{j=1}^{m}(-1)^j\sum_{1\le i_1<i_2<...<i_j\le m}\frac{nG^{\langle n\rangle}_{i_1,...,i_j}(x_{i_1},...,x_{i_j})}{n})^{n-\sum_{\underline{i}\in\{0,1\}^m/\underline{1}} r_{\underline{i}}}.$$

Now, by using the total probability theorem and by the independence of ν_n and $\underline{X}_1,\underline{X}_2,...,\underline{X}_n$, we get

$$H^{\langle n\rangle}_{\underline{k}:\nu_n}(\underline{x})=\sum_{S=\vee_{i=1}^m k_i}^{\infty} H^{\langle n\rangle}_{\underline{k}:s}(\underline{x})P(\nu_n=s).\tag{2.2}$$

Assume that $z = [\frac{S}{n}]$, where $[\theta]$ denotes the greatest integer part of θ. Thus, the relations (1.1) and (2.1) show that the sum in (2.2) is a Riemann sum of the integral

$$H^{\langle n\rangle}_{\underline{k}:\nu_n}(\underline{x}) = \int_0^\infty M^{\langle n\rangle}_{\underline{k}:n}(\underline{x}, z)dA_n(nz), \tag{2.3}$$

where

$$M^{\langle n\rangle}_{\underline{k}:n}(\underline{x}, z) = \sum_{i_{1_1}=0}^{k_1-1}\sum_{i_{1_2}=0}^{k_2-1}\cdots\sum_{i_{1_m}=0}^{k_m-1}\sum_{r_{\underline{i}}\in C^-_{zn}} F^{\langle n\rangle\, zn-\sum_{\underline{i}\in\{0,1\}^m/\underline{1}} r_{\underline{i}}}(\underline{x})(1+o(1))$$

$$\prod_{\underline{i}\in\{0,1\}^m/\underline{1}} \frac{(znG^{\langle n\rangle}_{I^0_{\underline{i}}}(\underline{x}_{I^0_{\underline{i}}}) + \sum_{I\subseteq I^1_{\underline{i}}}(-1)^{card(I)}znG^{\langle n\rangle}_{I^0_{\underline{i}}\cup I}(\underline{x}_{I^0_{\underline{i}}\cup I}))^{r_{\underline{i}}}}{r_{\underline{i}}!}.$$

Appealing to the condition $[\mathcal{A}]$, Theorem 1.2 and Remark 1.2, we get

$$M^{\langle n\rangle}_{\underline{k}:n}(\underline{x}, z) \xrightarrow[n]{w} \widetilde{H}_{\underline{k}}(zh_{I^0_{\underline{i}}}(\underline{x}_{I^0_{\underline{i}}}), zh_{I^0_{\underline{i}}\cup I}(\underline{x}_{I^0_{\underline{i}}\cup I})), \tag{2.4}$$

where the convergence is uniform with respect to $\underline{x}$ over any finite interval of z. Now, let ζ be a continuity point of $A(x)$ such that $1 - A(\zeta) < \epsilon$. Then

$$\int_\zeta^\infty \widetilde{H}_{\underline{k}}(zh_{I^0_{\underline{i}}}(\underline{x}_{I^0_{\underline{i}}}), zh_{I^0_{\underline{i}}\cup I}(\underline{x}_{I^0_{\underline{i}}\cup I}))\, dA(z) \le 1 - A(\zeta) < \epsilon. \tag{2.5}$$

Moreover, due to (2.5) and the condition $\mathcal{B}$), we get for sufficiently large n

$$\int_\zeta^\infty M^{\langle n\rangle}_{\underline{k}:n}(\underline{x}, z)\, dA_n(nz) \le 1 - A_n(n\zeta) \le (1 - A(\zeta)) + (A(\zeta) - A_n(n\zeta)) < 2\varepsilon. \tag{2.6}$$

In order to estimate the difference $H^{\langle n\rangle}_{\underline{k}:\nu_n}(\underline{x}) - \Psi(\underline{x})$, we first estimate

$$\int_0^\zeta M^{\langle n\rangle}_{\underline{k}:n}(\underline{x}, z)\, dA_n(nz) - \int_0^\zeta \widetilde{H}_{\underline{k}}(zh_{I^0_{\underline{i}}}(\underline{x}_{I^0_{\underline{i}}}), zh_{I^0_{\underline{i}}\cup I}(\underline{x}_{I^0_{\underline{i}}\cup I}))dA(z).$$

By the triangle inequality

$$\left|\int_0^\zeta M^{\langle n\rangle}_{\underline{k}:n}(\underline{x}, z)\, dA_n(nz) - \int_0^\zeta \widetilde{H}_{\underline{k}}(zh_{I^0_{\underline{i}}}(\underline{x}_{I^0_{\underline{i}}}), zh_{I^0_{\underline{i}}\cup I}(\underline{x}_{I^0_{\underline{i}}\cup I}))dA(z)\right| \tag{2.7}$$

$$\le \left|\int_0^\zeta M^{\langle n\rangle}_{\underline{k}:n}(\underline{x}, z)\, dA_n(nz) - \int_0^\zeta \widetilde{H}_{\underline{k}}(zh_{I^0_{\underline{i}}}(\underline{x}_{I^0_{\underline{i}}}), zh_{I^0_{\underline{i}}\cup I}(\underline{x}_{I^0_{\underline{i}}\cup I}))dA_n(nz)\right|$$

$$+\left|\int_0^\zeta \widetilde{H}_{\underline{k}}(zh_{I^0_{\underline{i}}}(\underline{x}_{I^0_{\underline{i}}}), zh_{I^0_{\underline{i}}\cup I}(\underline{x}_{I^0_{\underline{i}}\cup I}))dA_n(nz) - \int_0^\zeta \widetilde{H}_{\underline{k}}(zh_{I^0_{\underline{i}}}(\underline{x}_{I^0_{\underline{i}}}), zh_{I^0_{\underline{i}}\cup I}(\underline{x}_{I^0_{\underline{i}}\cup I}))dA(z)\right|,$$

where the convergence in (2.4) is uniform over the finite interval $[0, \zeta]$. Therefore, for arbitrary $\epsilon > 0$ and for sufficiently large n,

$$\left|\int_0^\zeta [M^{\langle n\rangle}_{\underline{k}:n}(\underline{x}, z) - \widetilde{H}_{\underline{k}}(zh_{I^0_{\underline{i}}}(\underline{x}_{I^0_{\underline{i}}}), zh_{I^0_{\underline{i}}\cup I}(\underline{x}_{I^0_{\underline{i}}\cup I}))]dA(z)\right| \le \varepsilon(A_n(n\zeta) - A_n(0)) \le \epsilon. \tag{2.8}$$

In order to estimate the third term in (2.7), we construct Riemann sums which are close to the integral there. Let N be a fixed number and $0 = \zeta_0 < \zeta_1 < ... < \zeta_N = \zeta$ be continuity points of $A(x)$. Furthermore, let N and ζ_i be such that

$$\Big|\int_0^\zeta \widetilde{H}_{\underline{k}}(zh_{I^0_{\underline{i}}}(\underline{x}_{I^0_{\underline{i}}}), zh_{I^0_{\underline{i}}\cup I}(\underline{x}_{I^0_{\underline{i}}\cup I}))dA_n(nz)$$

$$- \sum_{i=0}^{N} \widetilde{H}_{\underline{k}}(\zeta_i h_{I^0_{\underline{i}}}(\underline{x}_{I^0_{\underline{i}}}), \zeta_i h_{I^0_{\underline{i}}\cup I}(\underline{x}_{I^0_{\underline{i}}\cup I}))(A_n(n\zeta_i) - A_n(n\zeta_{i-1}))\Big| < \epsilon$$

and

$$\Big|\int_0^\zeta \widetilde{H}_{\underline{k}}(zh_{I^0_{\underline{i}}}(\underline{x}_{I^0_{\underline{i}}}), zh_{I^0_{\underline{i}}\cup I}(\underline{x}_{I^0_{\underline{i}}\cup I}))dA_n(nz)$$

$$- \sum_{i=0}^{N} \widetilde{H}_{\underline{k}}(\zeta_i h_{I^0_{\underline{i}}}(\underline{x}_{I^0_{\underline{i}}}), \zeta_i h_{I^0_{\underline{i}}\cup I}(\underline{x}_{I^0_{\underline{i}}\cup I}))(A(\zeta_i) - A(\zeta_{i-1}))\Big| < \epsilon.$$

Since, by the assumption $A_n(n\zeta_i) \xrightarrow[n]{w} A(\zeta_i),\ 0 \leq i \leq N$, the two Riemann sums are closer to each other than ϵ for all n sufficiently large. Thus, once again by the triangle inequality, the absolute value of the difference of the integrals is smaller than 3ϵ. Combining this fact with (2.8), the left hand side of (2.7) becomes smaller than 4ϵ for all large n. Therefore, in view of (2.5), (2.6) and (2.4),

$$\Big|H^{\langle n\rangle}_{\underline{k}:\nu_n}(\underline{x}) - \Psi(\underline{x})\Big| < \Big|\int_0^\zeta M^{\langle n\rangle}_{\underline{k}:n}(\underline{x}, z)\, dA_n(nz) - \int_0^\zeta \widetilde{H}_{\underline{k}}(zh_{I^0_{\underline{i}}}(\underline{x}_{I^0_{\underline{i}}}), zh_{I^0_{\underline{i}}\cup I}(\underline{x}_{I^0_{\underline{i}}\cup I}))\, dA(z)\Big|$$

$$+ \int_\zeta^\infty M^{\langle n\rangle}_{\underline{k}:n}(\underline{x}, z)\, dA_n(nz) + \int_\zeta^\infty \widetilde{H}_{\underline{k}}(zh_{I^0_{\underline{i}}}(\underline{x}_{I^0_{\underline{i}}}), zh_{I^0_{\underline{i}}\cup I}(\underline{x}_{I^0_{\underline{i}}\cup I}))\, dA(z) < 7\epsilon.$$

This completes the proof of the first part of the theorem. □

Proof of the implication $[\mathcal{A}] + [\mathcal{C}] \Rightarrow [\mathcal{B}]$**:** Starting with (2.4), we select a subsequence $\{n'\}$ of $\{n\}$ for which $A_{n'}(n'z)$ converges weakly to an extended df $A'(z)$ (i.e., $A'(\infty) - A'(0) \leq 1$ and such a subsequence exists by the compactness of df's). Then, by repeating the first part of the theorem for the subsequence $\{n'\}$, with the exception that we choose ζ so that $A'(\infty) - A'(\zeta) < \epsilon$, we get $\Psi(\underline{x}) = \int_0^\infty e^{-u(z,\underline{x})}\, dA'(z)$. Since the two limits $\Psi_{\underline{k}}(\underline{x})$ and $\widetilde{H}_{\underline{k}}(\underline{x})$ are df's, we get $\Psi_{\underline{k}}(\underline{\infty}) = 1 = \int_0^\infty dA'(z) = A'(\infty) - A'(0)$, which implies that $A'(z)$ is a df. Now, if $A_n(nz)$ did not converge weakly, then we can select two subsequences $\{n'\}$ and $\{n''\}$ such that $A_{n'}(n'z) \xrightarrow[n']{w} A'(z)$ and $A_{n''}(n''z) \xrightarrow[n'']{w} A''(z)$, where $A'(z)$ and $A''(z)$ are df's. In this case, we get

$$\Psi_{\underline{k}}(\underline{x})=\int_0^\infty \widetilde{H}_{\underline{k}}(zh_{I^0_{\underline{i}}}(\underline{x}_{I^0_{\underline{i}}}), zh_{I^0_{\underline{i}}\cup I}(\underline{x}_{I^0_{\underline{i}}\cup I}))dA'(z)=\int_0^\infty \widetilde{H}_{\underline{k}}(zh_{I^0_{\underline{i}}}(\underline{x}_{I^0_{\underline{i}}}), zh_{I^0_{\underline{i}}\cup I}(\underline{x}_{I^0_{\underline{i}}\cup I}))dA''(z).$$

Thus, let $(x_2 \to \infty, ..., x_m \to \infty)$ (say), we get

$$\int_0^\infty \Gamma_{k_1}(zh_1(x_1))dA'(z) = \int_0^\infty \Gamma_{k_1}(zh_1(x_1))dA''(z). \tag{2.9}$$

Let $L'(t)=\int_0^\infty \Gamma_{k_1}(tz)\,dA'(z)$ and $L''(t)=\int_0^\infty \Gamma_{k_1}(tz)\,dA''(z)$, where $\Gamma_k(tz)=\sum\limits_{r=k}^{\infty}\frac{(tz)^r}{r!}e^{-tz}$. Evidently $L'(t)$ and $L''(t)$ are analytic functions on the region $\mathcal{R}=\{t:0<|t|<\infty\}\cap\{t:$ $Real(t)>0\}$. In view of (2.9) we deduce that

$$L'(h_1(x_1))=L''(h_1(x_1)),\ \forall \text{ real values of } x_1. \tag{2.10}$$

Since the function $h_1(x_1)$ is continuous and $h_1(-\infty)=+\infty$, $h_1(+\infty)=0$, the equation (2.10) shows that the two analytic functions L' and L'' coincide on some interval contained in $\mathcal{R}$. Thus by the uniqueness theory of analytic functions, we deduce that L' and L'' are coincide on the region $\mathcal{R}$, which implies $A'(z)=A''(z)$. This completes the proof of this part. □

Proof of the implication $[\mathcal{B}]+[\mathcal{C}]+\Rightarrow[\mathcal{A}]$**:** We can assume, in view of Remark 2.1, without any loss of generality, that the df $\Psi_{\underline{k}}(\underline{x})$ is continuous. Therefore, in view of Lemma 5.2.1 in Galambos (1987), the condition $[\mathcal{C}]$ will be satisfied for all univariate marginals of $\Psi_{\underline{k}}(\underline{x})$, i.e. we have

$$H^{\langle n\rangle}_{k_i:\nu_n}(x_i)\ \xrightarrow[n]{w}\ \Psi_{k_i}(x_i),\ i=1,2,.....,m, \tag{2.11}$$

where $\Psi_{k_i}(x_i)$ is the ith univariate marginal df of $\Psi_{\underline{k}}(\underline{x})$. We shall now prove

$$H^{\langle n\rangle}_{k_i,n}(x_i)\ \xrightarrow[n]{w}\ H_{k_i}(x_i),\ i=1,2,..,m. \tag{2.12}$$

In view of Lemma 1.1, we first show that the sequences $\{Z^{\langle n\rangle}_{k_i:n}\}$, $i=1,2,...,m$, are stochastically bounded . If we assume the contrary, we would find $\varepsilon_{i,1},\varepsilon_{i,2}>0$ such that at least one of the relations

(a) $\quad\overline{\lim_{n\to\infty}}\ P(Z^{\langle n\rangle}_{k_i:n}\geq x_i)\geq\varepsilon_{i,1}>0,\ \ \forall\ \ x_i>0,\ i=1,...,m,$

(b) $\quad\overline{\lim_{n\to\infty}}\ P(Z^{\langle n\rangle}_{k_i:n}<x_i)\geq\varepsilon_{i,2}>0,\ \ \forall\ \ x_i<0,\ i=1,...,m,$

is satisfied. The assertions (a) and (b) mean that the sequence $\{Z^{\langle n\rangle}_{k_i:n}\}_n$, $i=1,...,m$, is not stochastically bounded at the left $(-\infty)$ and at the right $(+\infty)$, respectively. Let the assumption (a) be true. Since $A(x)$ is nondegenerate df, we find $\varepsilon_0>0$ and $\beta>0$ such that

$$P\left(\frac{\nu_n}{n}\geq\beta\right)\geq\varepsilon_0,\ \text{for sufficiently large } n. \tag{2.13}$$

Using the well known inequality, for $i=1,2,...,m$,

$$P\left(Z^{\langle n\rangle}_{k_i,\ell}\geq x_i\right)\geq P\left(Z^{\langle n\rangle}_{k_i,j}\geq x_i\right),\quad\forall\ell\geq j. \tag{2.14}$$

We thus get the following inequalities, for sufficiently large n,

$$P\left(Z^{\langle n\rangle}_{k_i:\nu_n})\geq x_i\right)\geq\sum_{s\geq[n\beta]}P\left(Z^{\langle n\rangle}_{k_i:s}\geq x_i\right)P(\nu_n=s)$$

$$\geq P\left(Z^{\langle n\rangle}_{k_i:[n\beta]} \geq x_i\right) P\left(\nu_n \geq [n\beta]\right) \geq \varepsilon_0 P\left(Z^{\langle n\rangle}_{k_i:[n\beta]} \geq x_i\right),\ i = 1, 2, ...m$$

(note that $P\left(\nu_n \geq [n\beta]\right) \geq P\left(\nu_n \geq n\beta\right)$). Therefore,

$$\overline{\lim_{n\to\infty}}\ P(Z^{\langle n\rangle}_{k_i:\nu_n} \geq x_i) \geq \varepsilon_0\ \overline{\lim_{n\to\infty}}\ P(Z^{(n)}_{k_i:[n\beta]} \geq x_i).$$

Now, if we find $\varepsilon'_t > 0$ such that $\overline{\lim_{n\to\infty}}\ P(Z^{\langle n\rangle}_{k_i:[n\beta]} \geq x_i) \geq \varepsilon'_i > 0$, we get $\overline{\lim_{n\to\infty}}\ P(Z^{\langle n\rangle}_{k_i:\nu_n} \geq x_i) \geq \epsilon_0 \varepsilon'_i > 0$, which contradicts the right stochastic boundedness of the sequence $\{Z^{\langle n\rangle}_{k_i:\nu_n}\}_n$ and consequently contradicts the relation (2.11). However, if such an $\varepsilon'_i > 0$ does not exist we have $\overline{\lim_{n\to\infty}}\ P(Z^{\langle n\rangle}_{k_i:[n\beta]} \geq x_i) = 0$, which in view of the first relation (3.10) of Lemma 3.1 in Barakat (1997) leads to the following chain of implications ($\forall\ x_i > 0$) $P(Z^{\langle n\rangle}_{k_i:[n\beta]} \geq x_i) \to 0 \Rightarrow \Gamma_{k_i}([n\beta]G_{i:n}(x_i)) \to 0 \Rightarrow [n\beta]G_{i:n}(x_i) \to 0 \Rightarrow nG_{i:n}(x_i) \to 0$ (since $nG_{i:n}(x_i) \to 0) \Rightarrow \Gamma_{k_i}(nG_{i:n}(x_i)) \to 0 \Rightarrow P(Z^{\langle n\rangle}_{k_i:n} \geq x_i) \to 0$, which contradicts the assumption (a). Consider the assumption (b). Since $A(x)$ is a df we can find a positive integer γ and real number $\alpha > 0$ such that

$$P(\nu_n \leq \gamma) \geq \alpha, \qquad \text{for sufficiently large n.} \tag{2.15}$$

Therefore, in view of (2.15) and the inequality (2.14), we have

$$P(Z^{\langle n\rangle}_{k_i:\nu_n} < x_i) \geq \sum_{s=\vee_{i=1}^m k_i}^{\gamma n} P(Z^{\langle n\rangle}_{k_i:s} < x_i) P(\nu_n = s) \geq P(Z^{\langle n\rangle}_{k_i:\gamma n} < x_i) P(\frac{\nu_n}{n} \leq \gamma)$$

$$\geq \alpha P(Z^{\langle n\rangle}_{k_i:\gamma n} < x_i),\ i = 1, 2, ..., m.$$

Hence, we get $\overline{\lim_{n\to\infty}}\ P(Z^{\langle n\rangle}_{k_i:\nu_n} < x_i) \geq \alpha\ \overline{\lim_{n\to\infty}}\ P(Z^{\langle n\rangle}_{k_i:\gamma n} < x_i)$. By using the second relation (3.11) of Lemma 3.1 in Barakat (1997) and applying the same argument as in the case (a), it is easy to show that the last inequality leads to a contradiction (the last inequality, in view of the assumption (b)), which yields that the sequences $\{Z^{\langle n\rangle}_{k_i:n}\}_n,\ i = 1, 2, ..., m$, are not stochastically bounded at the left. This completes the proof that the sequences $\{Z^{\langle n\rangle}_{k_i:n}\}_n,\ i = 1, 2, ..., m,$ are stochastically bounded. Now, if $H^{\langle n\rangle}_{k_i:n}(x_i)$ did not converge weakly, then we could select two subsequences $\{n'\}$ and $\{n''\}$ such that $H^{\langle n'\rangle}_{k_i:n'}(x_i)$ would converge weakly to $H'_{k_i}(x_i)$ and $H^{\langle n''\rangle}_{k_i:n''}(x_i)$ to another df $H''_{k_i}(x_i), i = 1, 2, ..., m$. In this case we get (by repeating the first part of Theorem 2.1 for the univariate case and for the two subsequences $\{n'\}, \{n''\}$)

$$\Psi_{k_i}(x_i) = \int_0^\infty (1 - \Gamma_{k_i}(zh'_i(x_i)))dA(z) = \int_0^\infty (1 - \Gamma_{k_i}(zh''_i(x_i)))dA(z).$$

However, Lemma 3.2 in Barakat (1997) shows that the last equalities, cannot hold unless $h'_i(x_i) \equiv h''_i(x_i)$. Hence the relation (2.12) is proved.

For accomplishing the proof of the last part of Theorem 2.1, we have to prove the relation (1.4). Since (2.12) implies (1.3), the elementary inequality $G_n(\underline{x}) \leq G_{i:n}(x_i),\ i = 1, 2, ..., m,$ yields that, the sequence $\{G_n(\underline{x})\}_n$ is bounded. Therefore, we can select a subsequence $\{n'\}$ of $\{n\}$, for which the relation (1.4) is satisfied. Let us repeat the first part of Theorem 2.1 for this subsequence. We get

$$\Psi_{\underline{k}}(\underline{x}) = \int_0^\infty \widetilde{H}_{\underline{k}}(zh_{I_{\underline{i}}^0}(\underline{x}_{I_{\underline{i}}^0}), zh_{I_{\underline{i}}^0\cup I}(\underline{x}_{I_{\underline{i}}^0\cup I}))dA(z), \qquad (2.16)$$

where the function $h_{I_{\underline{i}}^0}(\underline{x}_{I_{\underline{i}}^0})$ in (2.16) may depend on the actual subsequence $\{n'\}$. Observing, however, that

$$\frac{\partial}{\partial h_{I_{\underline{i}}^0}(\underline{x}_{I_{\underline{i}}^0})}\widetilde{H}_{\underline{k}}(h_{I_{\underline{i}}^0}(\underline{x}_{I_{\underline{i}}^0}), h_{I_{\underline{i}}^0\cup I}(\underline{x}_{I_{\underline{i}}^0\cup I})) > 0, \qquad \text{if } m \text{ is even},$$

$$\frac{\partial}{\partial h_{I_{\underline{i}}^0}(\underline{x}_{I_{\underline{i}}^0})}\widetilde{H}_{\underline{k}}(h_{I_{\underline{i}}^0}(\underline{x}_{I_{\underline{i}}^0}), h_{I_{\underline{i}}^0\cup I}(\underline{x}_{I_{\underline{i}}^0\cup I})) < 0, \qquad \text{if } m \text{ is odd},$$

for all values of $h_{I_{\underline{i}}^0}(\underline{x}_{I_{\underline{i}}^0})$, for which $0 < h_{I_{\underline{i}}^0}(\underline{x}_{I_{\underline{i}}^0}) < \bigwedge_{i=1}^m h_i(x_i) < \infty,\ i = 1, 2, ..., m.$ Hence, the function $\widetilde{H}_{\underline{k}}(h_{I_{\underline{i}}^0}(\underline{x}_{I_{\underline{i}}^0}), h_{I_{\underline{i}}^0\cup I}(\underline{x}_{I_{\underline{i}}^0\cup I}))$ is strictly monotone in $h_{I_{\underline{i}}^0}(\underline{x}_{I_{\underline{i}}^0})$, or in other words, the function $\widetilde{H}_{\underline{k}}(h_{I_{\underline{i}}^0}(\underline{x}_{I_{\underline{i}}^0}), h_{I_{\underline{i}}^0\cup I}(\underline{x}_{I_{\underline{i}}^0\cup I}))$ is uniquely determined by $h_{I_{\underline{i}}^0}(\underline{x}_{I_{\underline{i}}^0})$, where $h_i,\ i = 1, ..., m,$ are fixed and $0 < h_{I_{\underline{i}}^0}(\underline{x}_{I_{\underline{i}}^0}) < \bigwedge_{i=1}^m h_i < \infty$. This fact, with Lemma 3.3 in Barakat (1997), lead to a contradiction if we assume that the limit $h_{I_{\underline{i}}^0}(\underline{x}_{I_{\underline{i}}^0})$ depends on the subsequence $\{n'\}$ and at the same time consider the representation (2.16). Hence, the proof of Theorem 2.1 is completed. □

3 Asymptotic Properties of the Random Extreme Vector, when the Interrelation of the Random Size and the Basic rv's is not Restricted

When the interrelation between the random index and the basic variables is not restricted, parallel theorem of Theorem 2.1 may be proved by replacing the condition $[\mathcal{B}]$ by a stronger one. Namely, the weak convergence of the df $A_n(nx)$ must be replaced by the convergence in probability of the rv $\frac{\nu_n}{n}$ to a positive rv τ. However, the key ingredient of the proof of this parallel result is to prove the mixing property, due to Rényi (see, Barakat and Nigm, 1996) of the sequence of order statistics under consideration. In the sense of Rényi a sequence $\{X_n\}$ of rv's is called mixing if for any event $\mathcal{E}$ of positive probability, the conditional df of $\{X_n\}$, under the condition $\mathcal{E}$, converges weakly to a nondegenerate df, which does not depend on $\mathcal{E}$, as $n \to \infty$. The following lemma proves this property for the sequence $\{Z^{\langle n\rangle}_{\underline{k}:n}\}_n$.

Lemma 3.1. *Under the condition* $[\mathcal{A}]$, *the sequence* $\{Z^{\langle n\rangle}_{\underline{k}:n}\}_n$ *is mixing.*

Proof. The lemma will be proved if one shows the relation $P(Z^{\langle n\rangle}_{\underline{k}:n} < \underline{x} \mid Z^{\langle \ell\rangle}_{\underline{k}:\ell} < \underline{x}) \xrightarrow[n]{w} H_{\underline{k}}(\underline{x})$, for all integers $\ell = (\bigvee_{i=1}^m k_i), (\bigvee_{i=1}^m k_i) + 1, ...$ The sufficiency of the above relation can easily be proved as a direct multivariate extension of Lemma 6.2.1. of Galambos (1987). However, this relation is equivalent to

$$P(Z^{\langle n\rangle}_{\underline{k}:n} \geq \underline{x} \mid Z^{\langle \ell\rangle}_{\underline{k}:\ell} \geq \underline{x}) \xrightarrow[n]{w} T_{\underline{k}}(\underline{x}), \tag{3.1}$$

where, $T_{\underline{k}}(\underline{x})$ is the survival function of the limit df $H_{\underline{k}}(\underline{x})$, i.e.,

$$T_{\underline{k}}(\underline{x}) = 1 + \sum_{j=1}^{m-1}(-1)^j \sum_{1\leq \imath_1 \leq \imath_2 \leq ... \leq \imath_j \leq m} H_{\underline{k}}(\underline{c}_{\underline{\imath}}) + (-1)^m H_{\underline{k}}(\underline{x}),$$

where $\underline{\imath} = (\imath_1, ..., \imath_j), \underline{c}_{\underline{\imath}} = (c_1, ..., c_m), c_{\imath_j} = x_{\imath_j}$ and $c_s = \infty$, $\forall s \neq \imath_j, j = 1, 2, ..., m$. Therefore, our lemma will be established if one proves the relation (3.1). Now, we can write

$$\begin{aligned} P(Z^{\langle n\rangle}_{\underline{k}:n} \geq \underline{x} \mid Z^{\langle \ell\rangle}_{\underline{k}:\ell} \geq \underline{x}) &= P(Z^{\langle n\rangle}_{\underline{k}:n} \geq \underline{x}, Z^{\langle n\rangle}_{\underline{k}:\ell} < \underline{x} \mid Z^{\langle \ell\rangle}_{\underline{k}:\ell} \geq \underline{x}) \\ &+ P(Z^{\langle n\rangle}_{\underline{k}:n} \geq \underline{x}, Z^{\langle n\rangle}_{\underline{k}:\ell} \geq \underline{x} \mid Z^{\langle \ell\rangle}_{\underline{k}:\ell} \geq \underline{x}). \end{aligned} \tag{3.2}$$

Bearing in mind that all vectors are independent and identical, the first term in (3.2) can be written in the form

$$\begin{aligned} P(Z^{\langle n\rangle}_{\underline{k}:n} \geq \underline{x}, Z^{\langle n\rangle}_{\underline{k}:\ell} < \underline{x} \mid Z^{\langle \ell\rangle}_{\underline{k}:\ell} \geq \underline{x}) &= P(Z^{*\langle n\rangle}_{\underline{k}:(n-\ell)} \geq \underline{x}, Z^{\langle n\rangle}_{\underline{k}:\ell} < \underline{x} \mid Z^{\langle \ell\rangle}_{\underline{k}:\ell} \geq \underline{x}) \\ &= P(Z^{*\langle n\rangle}_{\underline{k}:(n-\ell)} \geq \underline{x}) - P(Z^{*\langle n\rangle}_{\underline{k}:(n-\ell)} \geq \underline{x}, Z^{\langle n\rangle}_{\underline{k}:\ell} \geq \underline{x} \mid Z^{\langle \ell\rangle}_{\underline{k}:\ell} \geq \underline{x}), \end{aligned}$$

where

$$Z^{*\langle n\rangle}_{\underline{k}:(n-\ell)} = (Z^{*\langle n\rangle}_{k_1:(n-\ell)}, Z^{*\langle n\rangle}_{k_2:(n-\ell)}, ..., Z^{*\langle n\rangle}_{k_m:(n-\ell)}),$$

$$Z^{*\langle n\rangle}_{k_1:(n-\ell)} = ((k_1\text{th largest of } X_{1,(\ell+1)}, X_{1,(\ell+2)}, ..., X_{1,n}) - b_{1,n})/a_{1,n},$$

$$Z^{*\langle n\rangle}_{k_2:(n-\ell)} = ((k_2\text{th largest of } X_{2,(\ell+1)}, X_{2,(\ell+2)}, ..., X_{2,n}) - b_{2,n})/a_{2,n},$$

$$\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots,$$

$$Z^{*\langle n\rangle}_{k_m:(n-\ell)} = ((k_m\text{th largest of } X_{m,(\ell+1)}, X_{m,(\ell+2)}, ..., X_{m,n}) - b_{m,n})/a_{m,n}.$$

Therefore, in view of (3.2), we have

$$P(Z^{\langle n\rangle}_{\underline{k}:n} \geq \underline{x} \mid Z^{\langle \ell\rangle}_{\underline{k}:\ell} \geq \underline{x}) = P(Z^{*\langle n\rangle}_{\underline{k}:(n-\ell)} \geq \underline{x}) - \Delta_n(\underline{x}), \tag{3.3}$$

where $\Delta_n(\underline{x}) = P(Z^{\langle n\rangle}_{\underline{k}:n} \geq \underline{x}, Z^{\langle \ell\rangle}_{\underline{k}:\ell} \geq \underline{x} \mid Z^{\langle \ell\rangle}_{\underline{k}:\ell} \geq \underline{x}) - P(Z^{*\langle n\rangle}_{\underline{k}:(n-\ell)} \geq \underline{x}, Z^{\langle n\rangle}_{\underline{k}:\ell} \geq \underline{x} \mid Z^{\langle \ell\rangle}_{\underline{k}:\ell} \geq \underline{x})$. By using the well-known inequalities $Z^{*\langle n\rangle}_{\underline{k}:(n-\ell)} \leq Z^{\langle n\rangle}_{\underline{k}:n}$ and $P(E_2 \bigcap E_3) - P(E_1 \bigcap E_3) \leq P(E_2) - P(E_1)$, for any three events E_1, E_2 and E_3, for which $E_1 \subseteq E_2$, we get

$$0 \leq \Delta_n(\underline{x}) P(Z^{\langle \ell\rangle}_{\underline{k}:\ell} \geq \underline{x}) \leq P(Z^{\langle n\rangle}_{\underline{k}:n} \geq \underline{x}) - P(Z^{*\langle n\rangle}_{\underline{k}:(n-\ell)} \geq \underline{x}). \tag{3.4}$$

On the other hand, by virtue of the condition $[\mathcal{A}]$, it is easy to prove that

$$\lim_{n\to\infty} P(Z^{*\langle n\rangle}_{\underline{k}:(n-l)} \geq \underline{x}) = \lim_{n\to\infty} P(Z^{\langle n\rangle}_{\underline{k}:(n-l)} \geq \underline{x}) = T_{\underline{k}}(\underline{x}) \tag{3.5}$$

(not that $nG^{\langle n\rangle}_i(x_i) \to h_i(x_i) \Rightarrow (n-1)G^{\langle n\rangle}_i(x_i) \to h_i(x_i)$, $\forall\ X_i's$ for which $h_i(x_i) < \infty$, $i = 1, 2, ..., m$ and $nG^{\langle n\rangle}_{\underline{t}}(\underline{x}_{\underline{t}}) \to h_{\underline{t}}(\underline{x}_{\underline{t}}) \Rightarrow (n-1)G^{\langle n\rangle}_{\underline{t}}(\underline{x}_{\underline{t}}) \to h_{\underline{t}}(\underline{x}_{\underline{t}}), \forall \underline{x}_{\underline{t}}$'s for which $h_{\underline{t}}(\underline{x}_{\underline{t}}) < \infty$). By combining the relations (3.3)-(3.5), the proof of the relation (3.1) follows immediately. Hence the required result. □

Considering the facts that the normalizing constants, which may be used in the multivariate extreme case are the same as those for the univariate case, and the limit df $H_{\underline{k}}(\underline{x})$ is continuous, we can easily by using Lemma 3.1, show that the proof of the following theorem follows without any essential modifications as a direct multivariate extension of the proof of Theorem 2.1 in Barakat and El Shindidy (1990).

Theorem 3.1. *Consider the condition*

$$\nu_n/n \to \tau, \text{ in probability, as } n \to \infty, \tag{$[\mathcal{B}']$}$$

where τ *is a positive rv. Under the conditions of Theorem 2.1, we have the implication* $[\mathcal{A}] + [\mathcal{B}'] \Rightarrow [\mathcal{C}]$.

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References

[1] Barakat, H. M. (1990). Limit theorems for lower-upper extreme values from two-dimensional distribution function. J. Stat. Plann. Inference, 24, 69-79.

[2] Barakat, H. M. (1997). Asymptotic properties of bivariate random extremes. J. Stat. Plann. Inference, 61, 203-217.

[3] Barakat, H. M. (2001). Asymptotic distribution theory of bivariate order statistics. Ann. Inst. Stat. Math. Vol., 53, No., 3, 487-497.

[4] Barakat, H. M. and M. A. El Shandidy (1990). On the limit distribution of the extremes of a random number of independent random variables. J. Statist Plann. Inference, 26, 353-361.

[5] Barakat. H. M. and Nigm, E. M. (1996). The mixing property of order ststistics with some applications. Bull. Malaysian Math. Soc. (Second Series) 19, 39-52.

[6] Barakat, H. M. and Nigm, E. M. (2012). On multivariate order statistics. Asymptotic theory and computing moments. Kuwait J. Sci. Eng. 39 (IA), 113-127.

[7] Barakat, H. M., Nigm, E. M. and Askar, M. M. (2004). On the trivariate extreme value distributions. J. Stat. Plann. Inference, 118, No. 1-2, 19-35.

[8] Barakat, H. M., Nigm, E. M. and Elsawah, A. M. (2012). Asymptotic distributions of the generalized and the dual generalized extremal quotient. Bull. Malays. Math. Sci. Soc. (2), accepted.

[9] Feller, W. (1979). An introduction to probability theory and its applications. Vol. 2, John Wiley & Sons. Inc. (Wiley Eastern University edition).

[10] Finkelstein, B. V. (1953). On the limiting distributions of the extreme terms of a variational series of a two-dimensional random quantity. Dokl. Akad. SSSR (N.S.) 91, 209-211 (in Russian).

[11] Galambos, J. (1975). Order statistics of samples from multivariate distributions. J. Amer. Stat. Assoc. 70, 674-680.

[12] Galambos, J. (1987). The asymptotic theory of extreme order statistics, Wiley, New York (1st ed.). Krieger, FL (2nd ed.).

[13] Gumbel, E.J. (1960). Distributions del valeurs extremes en plusiers dimensions, Publications de L'Institute de Statistique. Vol. 9, 171-173.

[14] Marshall, A. W. and Olkin I. (1983). Domain of attraction of multivariate extreme value distributions. Ann. Prob., Vol. 11, No. 1, 168-177.

[15] Mikhailov, V. G. (1974). Asymptotic independence of vector components of multivariate extreme order statistics. Teor. Verojatnost. i primen., 19, 817-821.

[16] Takahashi, R. (1994). Domains of attraction of multivariate extreme value distributions. J. Res. Nat. Inst. Stand. Technol. Vol. 99, No. 4, 551-554.

[17] Tiago de Oliveira, J. (1958). Extremal distributions. Rev. Fac. Ciencias Lisbon, Ser 2, A, 7, 219-227.

[18] Tiago de Oliveira, J. (1975). Bivariate extremes: extensions 40th session, I.S.I., Warsaw, Vol. 46, Book 2, 241-251.