Results on uniqueness of entire functions related to difference polynomial *†‡

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Abstract

In this paper, we deal with and improve one of the uniqueness results on two difference products of entire functions sharing one value by considering that the functions share the value zero, counting multiplicities. The research findings also include some IM-analogues of the theorems that we obtain, i.e., the nonzero value is allowed to be shared ignoring multiplicities. Meanwhile, we investigate the situation where the difference products share a nonzero polynomial instead, by confining its degree and generalize the previous concerning results. Moreover, we show by illustrating examples and a number of remarks that our results are best possible in certain senses.

1. Introduction

In this paper, by meromorphic functions we will always mean meromorphic functions in the complex plane. We adopt the standard notations of the Nevanlinna theory of meromorphic functions as explained in [6, 10, 19]. It will be convenient to let E denote any set of positive real numbers of finite linear measure, not necessarily the same at each occurrence. For a nonconstant meromorphic function h, we denote by $T(r,h)$ the Nevanlinna characteristic of h and by $S(r,h)$ any quantity satisfying $S(r,h)=o\{T(r,h)\}$, as $r\to\infty$ and $r\notin E$.

Let f and g be two nonconstant meromorphic functions, and let a be a value in the extended plane. We say that f and g share the value a CM, provided that f and g have the same a-points with the same multiplicities. We say that f and g share the value a IM, provided that f and g have the same a-points ignoring multiplicities (see [19]). We say that a is a small function of f, if a is a meromorphic function satisfying $T(r,a)=S(r,f)$ (see [25]). We say f and g sharing a function h CM(IM) if $f-h$ and $g-h$ share 0 CM(IM). Throughout this paper, we denote by $\rho(f)$ and $\rho_2(f)$ the order and the hyper-order of f respectively (see [6, 10, 19]).

Many research works on meromorphic functions whose differential polynomials share value or fixed points have been done (see [2, 13, 14, 18]). Recently the difference variant of the Nevanlinna theory has been established, see, e.g.[1, 2] and, in particular, in [3],

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by Halburd-Korhonen and by Chiang-Feng, independently. Using these theories, some mathematicians from Finland and China began to consider the uniqueness questions of entire functions sharing values with their shifts, and have done many fine works (see [7, 8, 17]). We recall the following results.

Theorem A ([14, Theorem 1.2]). Let f and g be transcendental entire functions of finite order, and c be *a nonzero complex constant, and let* $n \geq 6$. If $f^n f(z+\eta)$ and $g^n g(z+\eta)$ share 1 CM, then $fg = t_1$ or $f = t_2 g$ for some constants t_1 and t_2 that satisfies $t_1^{n+1} = t_2^{n+1} = 1$.

Let $P(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_0$ be a nonzero polynomial, where $a_n (\neq 0), a_{n-1}, \cdots, a_0$ are complex constants. Next we denote Γ_1, Γ_2 by $\Gamma_1 = m_1 + m_2$, $\Gamma_2 = m_1 + 2m_2$ respectively, where m_1 is the number of the simple zeros of $P(z)$ and m_2 is the number of the multiple zeros of $P(z)$. Throughout this paper we denote $d = GCD(\lambda_0, \lambda_1, \cdots, \lambda_n)$, where $\lambda_i = n+1$ if $a_i = 0$, $\lambda_i = i+1$ if $a_i \neq 0$.

Theorem B ([16, Theorem 2]). Let f and g be transcendental entire functions of finite order, c be a nonzero complex constant, and let $n > 2\Gamma_2 + 1$ be an integer. If $P(f(z))f(z+\eta)$ and $P(g(z))g(z+\eta)$ share 1 CM , then one of the following cases holds:

(i)$f = tg$, $t^d = 1$,

(ii)$R(f,g) \equiv 0$, where $R(w_1, w_2) = P(w_1)w_1(z+\eta) - P(w_2)w_2(z+\eta)$,

(iii)$f = e^{\alpha}$, $g = e^{\beta}$, where α and β are two polynomials and $\alpha + \beta = b$, b is a constant, $a_n^2 e^{(n+1)b} = 1$.

We recall the following example:

Example 1([16]). Let $P(z) = (z-1)^6(z+1)^6 z^{11}$, $f(z) = \sin z$, $g(z) = \cos z$ and $\eta = 2\pi$. It immediately yields that $n > 2\Gamma_2 + 1$ and $P(f(z))f(z+\eta) = P(g(z))g(z+\eta)$, and so $P(f(z))f(z+\eta)$ and $P(g(z))g(z+\eta)$ share 1 CM.

Clearly, we have $f \not\equiv tg$ for a constant t satisfying $t^m = 1$, where $m \in Z^+$. But f and g satisfy the algebraic equation $R(f,g) = 0$, where $R(w_1, w_2) = P(w_1(z))w_1(z+\eta) - P(w_2(z))w_2(z+\eta)$.

Moreover, we can see that f and g do not share 0 CM. Regarding this, one may ask, what can be said about the relationship between f and g, if f and g share 0 CM in Theorem B? In this direction, we will prove the following result:

Theorem 1. Let f, g be transcendental entire functions of finite orders such that f and g share 0 CM, let η be a nonzero complex number, and let $n > 2\Gamma_2 + 1$ be an integer. If $P(f(z))f(z+\eta)$ and $P(g(z))g(z+\eta)$ share 1 CM, then one of the following cases holds:

(i)$f = tg$, $t^d = 1$.

(ii)$f = e^{\alpha}$, $g = ce^{-\alpha}$, where α is a nonconstant polynomial, c is a constant satisfying $a_n^2 c^{n+1} = 1$.

Remark 1. From Example 1 we can see that the assumption that "f and g share 0 CM" in Theorem 1 is necessary.

The following result is an IM-analogue of Theorem 1 related to difference polynomials:

Theorem 2. Let f, g be two transcendental entire functions of finite orders such that f and g share 0 CM, η be a nonzero complex constant, and let $n > 3\Gamma_1 + 2\Gamma_2 + 4$ be an integer. If $P(f(z))f(z+\eta)$ and $P(g(z))g(z+\eta)$ share 1 IM , then one of the two cases holds:

(i)$f = tg$, $t^d = 1$,

(ii)$f = e^{\alpha}$, $g = ce^{-\alpha}$, α is a polynomial and c is a constant satisfying $a_n^2 c^{n+1} = 1$.

Remark 2. Suppose that the polynomial $P(z)$ in Theorem 2 is a nonzero monomial, say, $P(z) = z^n$, where $n > 11$. Then $m_1 = 0$, $m_2 = 1$, and so Γ_1 and Γ_2 in Theorem 2 satisfy $\Gamma_1 = 1$ and $\Gamma_2 = 2$ respectively. From this we can get Corollary 1 without the assumption that f and g share 0 CM:

Corollary 1. Let f, g be transcendental entire functions of finite order, and c be a non-zero complex constant and let $n > 11$ be an integer. If $f^n f(z+c)$ and $g^n g(z+c)$ share 1 IM, then $fg = t_1$ or $f = t_2 g$ for some constants t_1 and t_2 satisfying $t_1^{n+1} = t_2^{n+1} = 1$.

Next we let $P_0 \not\equiv 0$ be a nonzero polynomial, and set

$$F(z) = \frac{f(z)^n f(z+\eta)}{P_0(z)},\ G(z) = \frac{g(z)^n g(z+c)}{P_0(z)} \tag{1.1}$$

Theorem C ([12, Theorem 1.1]). Let f, g be transcendental entire functions of finite order, η be a nonzero complex number, n be an integer and $2\deg P_0 < n+1$. Suppose that $f^n(z)f(z+\eta)$ and $g^n(z)g(z+\eta)$ share P_0 CM. Then

(I) If $n \geq 4$ and that F is a Möbius transformation of G, then one of the following two cases holds: (i) $f = tg$, where t is a constant satisfying $t^{n+1} = 1$, (ii) $f = e^Q$ and $g = te^{-Q}$, where P_0 reduces to a nonzero constant c, say, and t is a constant such that $t^{n+1} = c^2$, Q is a nonconstant polynomial.

(II) If $n \geq 6$, then one of the above cases I(i) and I(ii) hold.

Theorem D ([12, Theorem 1.2]). Let f, g be transcendental entire functions of finite order, α be a meromorphic function such that $\rho(\alpha) < \rho(f)$ and $\alpha \not\equiv 0, \infty$. Suppose that η is a nonzero complex number, n and m are two positive integers, where $n \geq m+6$. If $f^n(z)(f^m(z)+1)f(z+\eta)$ and $g^n(z)(g^m(z)+1)g(z+\eta)$ share $\alpha(z)$ CM, then $f = tg$, where t is a constant satisfying $t^m = 1$.

The following results are IM-analogues of Theorem C and D related to difference polynomials:

Theorem 3. Let f, g be transcendental entire functions of finite order, let η be a nonzero complex constant and let n be an integer such that $\deg P_0 < n+1$. Suppose that $f^n(z)f(z+\eta)$ and $g^n(z)g(z+\eta)$ share P_0 IM. If $n \geq 4$ and that F is a Möbius transformation of G, or if $n > 11$, then one of the following cases holds:

(i) $f = tg$, $t^{n+1} = 1$,t is a constant,

(ii) $f = e^{\alpha}$ and $g = te^{-\alpha}$, where P_0 reduces to a nonzero constant c, say, and t is a constant such that $t^{n+1} = c^2$, α is a nonconstant polynomial.

Proceeding as in the proof of Theorem 1.2[12], we can get the following result by Lemma 9 in Section 2:

Theorem 4. Let f, g be transcendental entire functions of finite order, η be a nonzero complex constant, α be a meromorphic function such that $\rho(\alpha) < \rho(f)$ and $\alpha \not\equiv 0, \infty$, and let n and m be positive integers such that $n > 5m + 11$. If $f^n(z)(f^m(z) + 1)f(z + \eta)$ and $g^n(z)(g^m(z) + 1)g(z + \eta)$ share α IM, then $f = tg$, where t is a constant satisfying $t^m = 1$.

Next we let P_0 be a nonzero polynomial, and let $P(z) = a_n z^n + a_{n-1}z^{n-1} + \cdots + a_0$ be a nonzero polynomial, where $a_n(\neq 0), a_{n-1}, \cdots, a_0$ are complex constants. Set

$$F(z) = \frac{P(f)f(z+\eta)}{P_0(z)}, \quad G(z) = \frac{P(g)g(z+\eta)}{P_0(z)}, \ , \tag{1.2}$$

We continue to our study in this paper by establishing uniqueness theorems related to entire functions whose difference polynomials share a nonzero polynomial P_0 and obtain the results as follows.

Theorem 5. Let f, g be transcendental entire functions of finite orders such that f and g share 0 CM, η be a nonzero complex constant, and let n be an integer such that $\deg P_0 < n + 1$. Suppose that $P(f(z))f(z + \eta)$ and $P(g(z))g(z + \eta)$ share P_0 CM. If $n > 2\Gamma_1 + 1$ and that F is a Möbius transformation of G, or if $n > 2\Gamma_2 + 1$, then one of the following cases holds:

(i)$f = tg$, $t^d = 1$,

(ii)$f = e^{\alpha}$ and $g = te^{-\alpha}$, where P_0 reduces to a nonzero constant c, say, and t is a constant such that $t^{n+1} = c^2$, α is a nonconstant polynomial.

Remark 3. Suppose that the polynomial $P(z)$ in Theorem 5 is a nonzero monomial, say, $P(z) = z^n$, where $n > 5$. Then $m_1 = 0$, $m_2 = 1$, and so Γ_1 and Γ_2 in Theorem 5 satisfy $2\Gamma_1 + 1 = 3$ and $2\Gamma_2 + 1 = 5$ respectively. From this we can get Theorem C without the assumption that f and g share 0 CM.

Theorem 6. Let f, g be transcendental entire functions of finite orders such that f and g share 0 CM, η be a nonzero complex constant, and let n be an integer such that $\deg P_0 < n + 1$. Suppose that $P(f(z))f(z + \eta)$ and $P(g(z))g(z + \eta)$ share P_0 IM. If $n > 2\Gamma_1 + 1$ and that F is a Möbius transformation of G, or if $n > 3\Gamma_1 + 2\Gamma_2 + 4$, then one of the following cases holds:

(i)$f = tg$, $t^d = 1$,

(ii)$f = e^{\alpha}$ and $g = te^{-\alpha}$, where P_0 reduces to a nonzero constant c, say, and t is a constant such that $t^{n+1} = c^2$, α is a nonconstant polynomial.

Remark 4. Suppose that the polynomial $P(z)$ in Theorem 6 is a nonzero monomial, say, $P(z) = z^n$, where $n > 11$. Then $m_1 = 0$ and $m_2 = 1$, and so Γ_1 and Γ_2 in Theorem 5 satisfy $2\Gamma_1 + 1 = 3$ and $3\Gamma_1 + 2\Gamma_2 + 4 = 11$ respectively. From this we can get Theorem 3 without the assumption that f and g share 0 CM.

Theorem E ([16, Theorem 1]). Let f be a transcendental entire function of finite order and c be a fixed nonzero complex constant, let $P(z) = a_n z^n + a_{n-1}z^{n-1} + \cdots + a_0$ be a nonzero polynomial, where $a_n(\neq 0), a_{n-1}, \cdots, a_0$ are complex constants, and m is the number of the distinct zeros of $P(z)$. Then for $n > m$, $P(f)f(z+c) = a(z)$ has infinitely many solutions, where $a(z)$ is a small function of f.

Corresponding to the above result, we investigate the uniqueness of difference polynomials of entire functions, and obtain the next result.

Theorem 7. Let f, g be transcendental entire functions of finite non-integer orders such that f and g share 0 CM, η be a nonzero complex constant and let n be an integer. Suppose that $P(f(z))f(z+\eta)$ and $P(g(z))g(z+\eta)$ share $a(z)$ IM ,where $a(z)$ is a small function of f and g. If $n > 3\Gamma_1 + 2\Gamma_2 + 4$, then $f = tg$, $t^d = 1$.

Remark 5. The following example shows that Theorem 7 may fail to occur for entire functions of finite integer order.

Example 2. Let $P(z) = z^{10}$, $f(z) = (7z-3)^2(z+1)e^{(z-1)^3}$, $g(z) = (7z-3)^2(z+1)e^{-(z-1)^3}$, $a(z) = (7z-3)^{20}(z+1)^{10}(7z+4)^2(z+2)$ and $\eta = 1$. Clearly we see that f and g are of integer order, $n > 3\Gamma_1 + 2\Gamma_2 + 4$, and that $P(f)(z)f(z+\eta)$ and $P(g)g(z+\eta)$ share $a(z)$ IM. However, we get $f \not\equiv tg$ for a constant m such that $t^m = 1$, where $m \in Z^+$.

2. Some Lemmas

In this section, we state some lemmas which are important to prove the main results.

Lemma 1 ([5, Theorem 5.1]). Let f be a nonconstant meromorphic function and $\eta \in \mathcal{C}$. If f is of finite order, then

$$m\left(r, \frac{f(z+\eta)}{f(z)}\right) = O\left(\frac{T(r, f(z)) \log r}{r}\right)$$

for all r outside of a set E satisfying

$$\limsup_{r \longrightarrow \infty} \frac{\int_{E\cap[1,r)} dt/t}{\log r} = 0,$$

i.e., outside of a set E of zero logarithmic density. If $\rho_2(f) = \rho_2 < 1$ and $\varepsilon > 0$, then

$$m\left(r, \frac{f(z+\eta)}{f(z)}\right) = o\left(\frac{T(r, f(z))}{r^{1-\rho_2-\varepsilon}}\right),$$

for all r outside of a set of finite logarithmic measure, where and in what follows, ε is an arbitrary positive number.

Lemma 2 ([12, Lemma 2.3]). Let $f(z)$ be a nonconstant meromorphic function of order $\rho(f) < \infty$, let η be a nonzero complex number, and let $P(f)$ be defined as in (2.1). Suppose that $F(z) = P(f(z))f(z+\eta)$. Then

$$m(r, F(z)) = (n+1)m(r, f(z)) + o\left(\frac{T(r, f(z))}{r^{1-\varepsilon}}\right) + O(1),$$

for all r outside of a set of finite logarithmic measure.

Lemma 3 ([5, Lemma 8.3]). Let $T : [0, +\infty) \longrightarrow [0, +\infty)$ be a non-decreasing continuous function and let $s \in \mathcal{R}^+$. If the hyper-order of T is strictly less than one, i.e.,

$$\limsup_{r\to\infty} \frac{\log\log T(r)}{\log r} = \zeta < 1,$$

and $\delta \in (0, 1 - \zeta)$, then

$$T(r+s) = T(r) + o\left(\frac{T(r)}{r^\delta}\right),$$

where r runs to infinity outside of a set of finite logarithmic measure.

Lemma 4. Let f and g be two transcendental entire functions of finite orders, η be a nonzero complex constant, $a(z)$ be a small function of f and g, $P(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_0$ be a nonzero polynomial, where $a_0, a_1, \cdots, a_n (\neq 0)$ are complex constants; let $n > \Gamma_1$ be an integer. If $P(f)f(z+\eta)$ and $P(g)g(z+\eta)$ share $a(z)$ IM, then $\rho(f) = \rho(g)$.

Proof. Set

$$F(z) = \frac{P(f)f(z+\eta)}{a(z)}, \quad G(z) = \frac{P(g)g(z+\eta)}{a(z)}, \tag{2.1}$$

then from Lemma 2 and $f(z)$ is entire we get

$$T(r, F(z)) = (n+1)T(r, f(z)) + o\left(\frac{T(r, f(z))}{r^{1-\varepsilon}}\right) + O(\log r) \tag{2.2}$$

and

$$T(r, G(z)) = (n+1)T(r, g(z)) + o\left(\frac{T(r, g(z))}{r^{1-\varepsilon}}\right) + O(\log r), \tag{2.3}$$

as $r \longrightarrow \infty$ and $r \notin E$, where and in what follows, $E \subset [0, +\infty)$ is some subset with its logarithmic measure $\log mesE < \infty$. Since f, g are of finite orders, it follows from (2.2), (2.3) and the standard reasoning of removing exceptional set (see[13, Lemma 1.1.2]) that the same is true for F and G as well. By $\rho(f) < \infty$ we have $\rho_2(f) = 0$. Therefore, by a simple geometric observation and Lemma 2 we have

$$\begin{aligned}
\overline{N}(r, \frac{1}{f(z+\eta)}) &\leq \overline{N}(r + |\eta|, \frac{1}{f(z)}) \\
&\leq T(r + |\eta|, f(z)) + O(1) \\
&= T(r, f(z)) + o(\frac{T(r, f(z))}{r^\delta}),
\end{aligned}$$

as $r \longrightarrow \infty$ and $r \notin E$, where and in what follows, $\delta \in (0, 1)$ is a positive integer. This together with Lemma 3, the assumptions of Lemma 4 and the second fundamental theorem gives

$$\begin{aligned}
T(r, F(z)) &\leq \overline{N}(r, F(z)) + \overline{N}(r, \frac{1}{F(z)}) + \overline{N}(r, \frac{1}{F(z)-1}) + O(\log r) \\
&\leq \overline{N}(r, \frac{1}{P(f)}) + \overline{N}(r, \frac{1}{f(z+\eta)}) + \overline{N}(r, \frac{1}{G(z)-1}) \quad + O(\log r) \\
&\leq (\Gamma_1 + 1)T(r, f(z)) + T(r, G(z)) + o(\frac{T(r, f(z))}{r^\delta}) + O(\log r),
\end{aligned}$$

as $r \longrightarrow \infty$ and $r \notin E$. This together with (2.2) and (2.3) gives

$$\begin{aligned}(n+1)T(r,f(z)) &\leq (\Gamma_1+1)T(r,f(z))+(n+1)T(r,g(z))+o(\frac{T(r,f(z))}{r^{\delta}})\\ &\quad +o(\frac{T(r,g(z))}{r^{1-\varepsilon}})+O(\log r),\end{aligned}$$

i.e.,

$$\begin{aligned}(n-\Gamma_1)T(r,f(z)) &\leq (n+1)T(r,g(z))+o(\frac{T(r,f(z))}{r^{\delta}})+o(\frac{T(r,g(z))}{r^{1-\varepsilon}})\\ &\quad +O(\log r),\end{aligned}$$

as $r \longrightarrow \infty$ and $r \notin E$. From the above inequality, the condition $n > \Gamma_1$ and the standard reasoning of removing exceptional set we get

$$\rho(f) \leq \rho(g). \tag{2.4}$$

Similarly

$$\rho(g) \leq \rho(f). \tag{2.5}$$

From (2.4) and (2.5) we get $\rho(f) = \rho(g)$, this proves Lemma 4.

Lemma 5 ([11, Lemma 3]). Suppose that h is a nonconstant meromorphic function satisfying

$$\overline{N}(r,h)+\overline{N}(r,\frac{1}{h}) = S(r,h).$$

Let $f = a_0h^p + a_1h^{p-1} + \cdots + a_p$, and $g = b_0h^q + b_1h^{q-1} + \cdots + b_q$ be polynomials in h with coefficients $a_0, a_1, \cdots, a_p$, $b_0, b1, \cdots, b_q$ being small functions of h and $a_0b_0a_p \not\equiv 0$. If $q \leq p$, then $m(r, g/f) = S(r,h)$.

Lemma 6 ([9, Lemma 2.2]). Let $\varphi(r)$ be a nondecreasing, continuous function on R^+. Suppose that

$$0 < \rho < \limsup_{r\longrightarrow\infty} \frac{\log \varphi(r)}{\log r},$$

and set

$$I = \{t : t \in R^+, \quad \varphi(r) \geq r^{\rho}\}.$$

Then we have

$$\overline{\log dens I} = \limsup_{r\longrightarrow\infty} \frac{\int_{I\cap[1,r]} \frac{dr}{r}}{\log r} > 0.$$

Lemma 7 ([15, Lemma 7.1]). Let F and G be two nonconstant meromorphic functions such that G is a Möbius transformation of F. Suppose that there exists a subset $I \subset R+$ with its linear measure $Mes I = +\infty$ such that

$$\overline{N}(r,\frac{1}{F})+\overline{N}(r,F)+\overline{N}(r,\frac{1}{G})+\overline{N}(r,G) < (\lambda+o(1))T(r,f),$$

as $r \in I$ and $r \to \infty$, where $\lambda < 1$. If there exists a point $z_0 \in C$ such that $F(z_0) = G(z_0) = 1$, then $F = G$ or $FG = 1$.

Let h be a nonconstant meromorphic function. We denote by $\overline{N}(r,h)$ the counting function of simple poles of h, and by $\overline{N}_{(2}(r,h)$ the counting function of poles of h with multiplicities ≥ 2, each point in these counting functions is counted only once. Set

$$N_2(r,h) = \overline{N}(r,h) + \overline{N}_{(2}(r,h).$$

Lemma 8 ([19, Theorem 1.48 and 7.10]). Let F and G be two nonconstant meromorphic functions such that F, G share 1, ∞ CM. Suppose that there exists a subset $I \subset \mathrm{R}^+$ with its linear measure $mesI = +\infty$ such that

$$N_2\left(r, \frac{1}{F}\right) + N_2\left(r, \frac{1}{G}\right) + 2\overline{N}(r,F) < \lambda T(r) + S(r),$$

as $r \in I$ and $r \longrightarrow \infty$, where $\lambda < 1$, $T(r) = \max\{T(r,F), T(r,G)\}$ and $S(r) = o\{T(r)\}$, as $r \in I$ and $r \longrightarrow \infty$. Then $F = G$ or $FG = 1$.

Lemma 9 ([20, proof of Theorem 1]). Let f, g be nonconstant entire functions. If f and g share 1 IM, then one of the following cases holds:

(i)$T(r,f) \leq N_2(r,\frac{1}{f}) + N_2(r,\frac{1}{g}) + 2\overline{N}(r,\frac{1}{f}) + \overline{N}(r,\frac{1}{g}) + S(r,f) + S(r,g)$
the same inequality holding for $T(r,g)$;

(ii)$f = g$;

(iii)$fg = 1$.

Let F and G be two nonconstant meromorphic functions, let $a \in \mathcal{C} \cup \{\infty\}$, and let $\overline{N}_E(r,a)$ "count" those points in $\overline{N}(r, 1/(F-a))$, where a is taken by F and G with the same multiplicity, and each point is counted only once, and $\overline{N}_0(r,a)$ when ignoring multiplicities. $\overline{N}(r, 1/(F-\infty))$ means $\overline{N}(r,F)$. We say that F and G share the value a CM*, if

$$\overline{N}\left(r, \frac{1}{F-a}\right) - \overline{N}_E(r,a) = S(r,F), \quad \overline{N}\left(r, \frac{1}{G-a}\right) - \overline{N}_E(r,a) = S(r,G),$$

and F and G share the value a IM*, if $\overline{N}_E(r,a)$ is replaced by $\overline{N}_0(r,a)$.

Proceeding as in the proof of Theorem 1[20], Theorems 1.48 and 7.10[19], the above two lemmas still hold if CM(IM) is replaced by CM*(IM*).

3. Proof of Theorems

Proof of Theorem 2. Let $F = P(f)f(z+\eta)$, $G = P(g)g(z+\eta)$, then F and G share 1 IM. Applying Lemma 9 to F and G, we consider the following three cases:

Case 1. If

$$T(r,F) \leq N_2(r,\frac{1}{F}) + N_2(r,\frac{1}{G}) + 2\overline{N}(r,\frac{1}{F}) + \overline{N}(r,\frac{1}{G}) + O(\log r)$$

by Lemma 4 and Lemma 9, we have

$$\begin{aligned} T(r,F) &\leq N_2(r,\frac{1}{F}) + N_2(r,\frac{1}{G}) + 2\overline{N}(r,\frac{1}{F}) + \overline{N}(r,\frac{1}{G}) + O(\log r) \\ &= N_2(r,\frac{1}{P(f)f(z+\eta)}) + N_2(r,\frac{1}{P(g)g(z+\eta)}) + 2\overline{N}(r,\frac{1}{P(f)f(z+\eta)}) \end{aligned}$$

$$
\begin{aligned}
& +\overline{N}(r, \frac{1}{P(g)g(z+\eta)}) + O(\log r) \\
\leq\ & N_2(r, \frac{1}{P(f)}) + N_2(r, \frac{1}{f(z+\eta)}) + N_2(r, \frac{1}{P(g)}) + N_2(r, \frac{1}{g(z+\eta)}) \\
& +2\overline{N}(r, \frac{1}{P(f)}) + 2\overline{N}(r, \frac{1}{f(z+\eta)}) + \overline{N}(r, \frac{1}{P(g)}) + \overline{N}(r, \frac{1}{g(z+\eta)}) + O(\log r) \\
\leq\ & (2\Gamma_1 + \Gamma_2)T(r, f) + (\Gamma_1 + \Gamma_2)T(r, g) + 3T(r, \frac{1}{f(z+\eta)}) + 2N(r, \frac{1}{g(z+\eta)}) \\
& +O(\log r) \\
\leq\ & (2\Gamma_1 + \Gamma_2 + 3)T(r, f) + (\Gamma_1 + \Gamma_2 + 2)T(r, g) + o\left(\frac{T(r, f(z))}{r^\delta}\right) + O(\log r)
\end{aligned}
$$

From f is entire and Lemma 4, we deduce

$$(n+1)T(r, f) \leq (2\Gamma_1+\Gamma_2+3)T(r, f)+(\Gamma_1+\Gamma_2+2)T(r, g)+o\left(\frac{T(r, f(z))}{r^\delta}\right)+O(\log r). \quad (3.1)$$

Similarly, we obtain

$$(n+1)T(r, g) \leq (2\Gamma_1+\Gamma_2+3)T(r, g)+(\Gamma_1+\Gamma_2+2)T(r, f)+o\left(\frac{T(r, f(z))}{r^\delta}\right)+O(\log r). \quad (3.2)$$

Combining (3.1) and (3.2), we have

$$(n+1)[T(r, f) + T(r, g)] \leq (3\Gamma_1 + 2\Gamma_2 + 5)[T(r, f) + T(r, g)] + o\left(\frac{T(r, f(z))}{r^\delta}\right) + O(\log r).$$

which contradicts with $n > 3\Gamma_1 + 2\Gamma_2 + 4$.

Case 2. If $F = G$, that is

$$P(f(z))f(z+\eta) \equiv P(g(z))g(z+\eta). \quad (3.3)$$

Set $h = f/g$, then substituting $f = gh$ into (3.3), we deduce that

$$h(z+\eta)[a_n g^n(z)h^n(z) + a_{n-1}g^{n-1}(z)h^{n-1}(z) + \cdots + a_0] \equiv a_n g^n(z) + \cdots + a_0$$

where $a_n(\neq 0), a_{n-1}, \cdots, a_0$ are complex constants.

From above, we get

$$a_n g^n(z)[h(z+\eta)h^n(z) - 1] + a_{n-1}g^{n-1}(z)[h(z+\eta)h^{n-1}(z) - 1] + \cdots + a_0[h(z+\eta) - 1] \equiv 0, \quad (3.4)$$

We discuss the following subcases.

Case 2.1. Suppose that h is a constant. We claim $h^d = 1$, where d is defined as in Theorem B. Thus, $f \equiv tg$ for a constant t such that $t^d = 1$.

In fact, if a_n is the only nonzero coefficient, since g is transcendental entire function, we have $h^{n+1} = 1$.

If a_n is not the only nonzero coefficient, suppose that $h(z+\eta)h^n = h^{n+1} \neq 1$. By (3.4), we deduce $T(r, g) = S(r, g)$, which is a contradiction. Hence, $h^{n+1} = 1$. According to the similar discussion, we obtain that $h^{k+1} = 1$ when $a_k \neq 0$ for some $k = 0, \cdots, n$. Therefore, we get $f = tg$ for a constant t such that $t^d = 1$, where $d = GCD(\lambda_0, \lambda_1, \cdots, \lambda_n)$.

Case 2.2. Suppose that h is not a constant. We claim

$$h(z+\eta)h^n \equiv 1. \tag{3.4}$$

In fact, if a_n is the only nonzero coefficient, i.e., $P(f) = a_n f^n$. Therefore from $g \not\equiv 0$ we have $h(z+\eta)h^n \equiv 1$.

If a_n is not the only nonzero coefficient, suppose that $h(z+\eta)h^n \not\equiv 1$, by (3.4) we have

$$a_{n-1}g^{n-1}\frac{h(z+\eta)h^{n-1}-1}{h(z+\eta)h^n-1} + \cdots + a_0\frac{h(z+\eta)-1}{h(z+\eta)h^n-1} = -a_n g^n, \tag{3.5}$$

Let

$$H_i = \frac{h(z+\eta)h^{n-i}-1}{h(z+\eta)h^n-1},\ i = 1, 2, \cdots, n$$

Then we have

$$H_1 = \frac{h(z+\eta)h^{n-1}-1}{h(z+\eta)h^n-1} = \frac{\frac{h(z+\eta)}{h(z)}h^n - 1}{\frac{h(z+\eta)}{h(z)}h^{n+1} - 1}$$

Since f and g are entire functions and share 0 CM, from Lemma 5 we have $m(r, H_1) = S(r, h)$. Similarly, we have $m(r, H_i) = S(r, h)$, $i = 1, 2, \cdots, n$.

Thus by (3.5) and g is entire we deduce

$$T(r, g) = m(r, g) = S(r, g)$$

which is a contradiction. Therefore we have (3.4).

From the assumption that f and g are entire functions and share 0 CM, we can write $h = e^{\omega(z)}$, where $\omega(z)$ is a polynomial. Thus by (3.4), we have

$$e^{\omega(z+\eta)+n\omega(z)} \equiv 1.$$

Differentiating this yields

$$n\omega'(z) + \omega'(z+\eta) \equiv 0.$$

Since $\omega'(z)$ is a polynomial, we suppose $deg(\omega'(z)) = m$, and $z_1, \cdots, z_m$ are the zeros of $\omega'(z)$. Thus, $z_1+\eta, \cdots, z_m+\eta$ are also zeros of $\omega'(z)$. Therefore, $\omega' \equiv 0$, $\omega \equiv b$, where b is a constant. Immediately we have h is a constant, which contradicts with our assumption.

Case 3. If $FG \equiv 1$, that is

$$P(f)f(z+\eta)P(g)g(z+\eta) \equiv 1. \tag{3.6}$$

From the assumption that f and g are two nonconstant entire functions, we deduce by (3.6) that $P(f) \neq 0$, $P(g) \neq 0$.

By the second main theorem, we claim that $P(f) = a_n(f-a)^n$, $P(g) = a_n(g-a)^n$, where a is a complex constant. Otherwise, the Picard's exceptional values are at least three, which is a contradiction.

Hence, from the assumption that f and g be transcendental entire functions of finite order, we obtain that $f(z) = e^{\alpha(z)} + a$, $g(z) = e^{\beta(z)} + a$, where $\alpha(z)$ and $\beta(z)$ are two nonconstant polynomials.

By (3.6), we also get $f(z+\eta) \neq 0$, $g(z+\eta) \neq 0$. So $a = 0$, i.e.,$f(z) = e^{\alpha(z)}$, $g(z) = e^{\beta(z)}$, $P(z) = a_n z^n$, and $a_n^2 e^{n(\alpha(z)+\beta(z))+\alpha(z+\eta)+\beta(z+\eta)} \equiv 1$. Then we must have $\alpha + \beta \equiv c$, where c is a constant.

From this we can easily obtain that $f(z) = e^{\alpha(z)}$, $g(z) = ce^{-\alpha(z)}$, where $\alpha(z)$ is a polynomial, c is a constant satisfying $a_n^2 c^{n+1} = 1$.

This completes the proof of Theorem 2.

Proof of Theorem 5. First of all, we set (1.2). Then we have (2.2) and (2.3) From Lemma 4 and the assumptions of Theorem 5 we deduce $\rho(f) = \rho(g) = \rho(F) = \rho(G)$, and so F, G are of finite orders. We discuss the following two cases.

Case 1. Suppose that F is a Möbius transformation of G. Then it follows from (1.2) and the standard Valiron-Mokhon'ko lemma that

$$T(r, P(f)f(z+\eta)) = T(r, P(g)g(z+\eta)) + O(\log r).$$

Then from (2.2), (2.3) and the condition that f, g are transcendental entire functions we deduce

$$\frac{T(r,f)}{T(r,g)} \to 1, \quad \frac{T(r,F)}{T(r,f)} \to n+1, \quad r \to \infty, r \notin E.$$

By $\rho(f) < \infty$ we have $\rho_2(f) = 0$. This together with a Lemma 3 and a simple geometric observation gives

$$T(r, f(z+\eta)) \leq T(r+|\eta|, f(z)) = T(r, f(z)) + o\left(\frac{T(r,f(z))}{r^\delta}\right),$$

as $r \longrightarrow \infty$ and $r \notin E$. Therefore we have

$$\begin{aligned} \overline{N}(r, F(z)) + \overline{N}\left(r, \frac{1}{F(z)}\right) &\leq N_2\left(r, \frac{1}{P(f)}\right) + \overline{N}\left(r, \frac{1}{f(z+\eta)}\right) + O(\log r) \\ &\leq \Gamma_1 T(r, f(z)) + T(r, f(z+\eta)) + O(\log r) \\ &\leq (\Gamma_1 + 1)T(r, f(z)) + o\left(\frac{T(r,f(z))}{r^\delta}\right) + O(\log r), \end{aligned}$$

as $r \longrightarrow \infty$ and $r \notin E$. That is,

$$\overline{N}(r, F(z)) + \overline{N}\left(r, \frac{1}{F(z)}\right) \leq (\Gamma_1 + 1)T(r, f(z)) + o\left(\frac{T(r,f(z))}{r^\delta}\right) + O(\log r).$$

Similarly,

$$\overline{N}(r, G(z)) + \overline{N}\left(r, \frac{1}{G(z)}\right) \leq (\Gamma_1 + 1)T(r, g(z)) + o\left(\frac{T(r,g(z))}{r^\delta}\right) + O(\log r),$$

as $r \longrightarrow \infty$ and $r \notin E$. Then we have

$$\overline{N}\left(r, \frac{1}{F}\right) + \overline{N}(r, F) + \overline{N}\left(r, \frac{1}{G}\right) + \overline{N}(r, G) \leq \frac{2(\Gamma_1 + 1)}{n+1} T(r, F)(1 + o(1)), \tag{3.7}$$

as $r \longrightarrow \infty, r \notin E$. From (1.2), Lemma 3 and the second main theorem we get

$$\begin{aligned} T(r,F(z)) &\leq \overline{N}(r,F(z)) + \overline{N}\left(r,\frac{1}{F(z)}\right) + \overline{N}\left(r,\frac{1}{F(z)-1}\right) + O(\log r) \\ &\leq \overline{N}\left(r,\frac{1}{P(f)}\right) + \overline{N}\left(r,\frac{1}{f(z+\eta)}\right) + \overline{N}\left(r,\frac{1}{F(z)-1}\right) \\ &\quad + O(\log r) \\ &\leq \Gamma_1 T(r,f(z)) + T(r,f(z+\eta)) + \overline{N}\left(r,\frac{1}{F(z)-1}\right) + O(\log r) \\ &\leq \Gamma_1 T(r,f(z)) + T(r+|\eta|,f(z)) + \overline{N}\left(r,\frac{1}{F(z)-1}\right) + O(\log r) \\ &\leq \Gamma_1 T(r,f(z)) + \overline{N}\left(r,\frac{1}{F(z)-1}\right) + o\left(\frac{T(r,f(z))}{r^{\delta}}\right) \\ &\quad + O(\log r), \end{aligned}$$

as $r \longrightarrow \infty$ and $r \notin E$. This together with (2.2) gives

$$(n+1-\Gamma_1)T(r,f) \leq \overline{N}\left(r,\frac{1}{F-1}\right) + o\{T(r,f)\},$$

as $r \longrightarrow \infty, r \notin E$. From the inequality and the fact that F, G share 1 CM* we know that there exists a point $z_0 \in \mathcal{C}$ such that $F(z_0) = G(z_0) = 1$. Hence from (3.7), Lemma 7 and the condition $n > 2\Gamma_1 + 1$ we get $FG = 1$ or $F = G$. We discuss the following two subcases:

Case 1.1. Suppose that $F = G$. Then it follows from (1.2) that

$$P(f)f(z+\eta) = P(g)g(z+\eta).$$

With it and the similar arguments of Case 2 in the proof of Theorem 2 we can get our conclusion (1) with $n > 2\Gamma_1 + 1$.

Case 1.2. Suppose that $FG = 1$. By substituting (1.2) into $FG = 1$ we get

$$P(f(z))f(z+\eta)P(g(z))g(z+\eta) = P_0^2(z). \tag{3.8}$$

Then from f, g are transcendental entire functions, one may immediately see, $P(f)$, $P(g)$, f and g have at most finitely many zeros. Suppose that $P(u)$ has two zeros, say u_1, u_2, $u_1 \neq u_2$ then $P(f) = a_n(f-u_1)^{n_1}(f-u_2)^{n_2}$, where n_1, n_2, are positive integers and $n_1 + n_2 = n$. Therefore $f - u_1$, $f - u_2$ has at most finitely many zeros. Applying the second main theorem we immediately get a contradiction.

Next we consider $P(u)$ has only one zero. Then we may write $P(f) = a_n(f-a)^n$, where a is a complex constant. Here, from the assumption that f and g be transcendental entire functions of finite order, we obtain that

$$f(z) = \mu(z)e^{\alpha(z)} + a, \quad g(z) = \nu(z)e^{\beta(z)} + a, \tag{3.9}$$

and

$$f(z+\eta) = \mu_\eta(z)e^{\alpha_\eta(z)}, \quad g(z+\eta) = \nu_\eta(z)e^{\beta_\eta(z)}, \tag{3.10}$$

where where μ, ν, μ_η, and ν_η are nonzero polynomials, α and β are nonconstant polynomials. From the left of (3.9) and (3.10), we have $f(z+\eta) = \mu(z+\eta)e^{\alpha(z+\eta)} + a$ and

$$\mu_\eta(z)e^{\alpha_\eta(z)} = \mu(z+\eta)e^{\alpha(z+\eta)} + a$$

Considering that $\mu_\eta(z)$ and $\mu(z+\eta)$ are both polynomials, we must have $a = 0$ to avoid a contradiction comparing the quantity of the zeros on both sides.

Noticing that f and g share 0 CM, we get $\mu = \nu$. Therefore, we have $f = \mu e^\alpha$, $g = \mu e^\beta$. Substituting to (3.8), we have

$$a_n^2\mu^{2n}\mu^2(z+\eta)e^{n(\alpha+\beta)+\alpha(z+\eta)+\beta(z+\eta)} \equiv P_0^2.$$

Clearly, we must have $n(\alpha+\beta)+\alpha(z+\eta)+\beta(z+\eta) \equiv b$ and $\alpha+\beta \equiv d$,where b, d are nonzero constants. Therefore we get

$$a_n^2 b\mu^{2n}\mu^2(z+\eta) \equiv P_0^2. \tag{3.11}$$

If μ is not a constant, the degree of the left side of (3.11) is not less than $2(n+1)$. But the condition $\deg P_0 < n+1$ implies that the degree of the right side of (3.11) is less than $2(n+1)$, a contradiction. Hence μ and P_0 reduce to nonzero constants, say t_0 and c. Set $t = t_0 d$ then the assertion (2) now follows from (3.11).

Case 2. Suppose that $n > 2\Gamma_2 + 1$. From (1.2), (2.2), Lemma 2, Lemma 3 and the assumptions of Theorem 5 we get

$$\begin{aligned}
2\overline{N}(r,F(z)) + N_2\left(r,\frac{1}{F(z)}\right) &\leq N_2\left(r,\frac{1}{P(f)}\right) + N\left(r,\frac{1}{f(z+\eta)}\right) + O(\log r)\\
&\leq \Gamma_2 T(r,f(z)) + T(r,f(z+\eta)) + O(\log r)\\
&\leq \Gamma_2 T(r,f(z)) + T(r+|\eta|,f(z)) + O(\log r)\\
&\leq (\Gamma_2+1)T(r,f(z)) + o\left(\frac{T(r,f(z))}{r^\delta}\right) + O(\log r)\\
&= \frac{\Gamma_2+1}{n+1}T(r,F(z)) + o\left(\frac{T(r,f(z))}{r^{1-\varepsilon}}\right) + o\left(\frac{T(r,f(z))}{r^\delta}\right)\\
&\quad +O(\log r)\\
&\leq \frac{\Gamma_2+1}{n+1}T(r,F(z))(1+o(1))
\end{aligned}$$

i.e.,

$$2\overline{N}(r,F(z)) + N_2\left(r,\frac{1}{F(z)}\right) \leq \frac{\Gamma_2+1}{n+1}T(r,F(z))(1+o(1)). \tag{3.12}$$

In the same way,

$$2\overline{N}(r,G(z)) + N_2\left(r,\frac{1}{G(z)}\right) \leq \frac{\Gamma_2+1}{n+1}T(r,G(z))(1+o(1)), \tag{3.13}$$

as $r \longrightarrow \infty$ and $r \notin E$. From (3.12) and (3.13) we have

$$N_2\left(r,\frac{1}{F}\right) + N_2\left(r,\frac{1}{G}\right) + 2\overline{N}(r,F) \leq \frac{2(\Gamma_2+1)}{n+1}T(r) + o\{T(r)\}, \tag{3.14}$$

as $r \longrightarrow \infty$ and $r \notin E$, where $T(r) = \max\{T(r,F), T(r,G)\}$. From (3.14), Lemma 8 and the condition $n > 2\Gamma_2 + 1$ we have $FG = 1$ or $F = G$. Next in the same manner as in subcases 1) and 2) we get the conclusion. This completes the proof of Theorem 5.

Proof of Theorem 7. First of all, we set (2.1). Then F and G share 1 IM except the zeros and poles of $a(z)$. From (2.1) we have (2.2) and (2.3). From Lemma 4 and the assumptions of Theorem 7 we deduce $\rho(f) = \rho(g) = \rho(F) = \rho(G)$, and so F, G are of finite orders. Applying Lemma 9 to F and G, we consider the following three cases:

Case 1. If

$$T(r,F) \leq N_2(r, \frac{1}{F}) + N_2(r, \frac{1}{G}) + 2\overline{N}(r, \frac{1}{F}) + \overline{N}(r, \frac{1}{G}) + O(\log r)$$

by Lemma 4 and Lemma 9, we can get a contradiction with $n > 3\Gamma_1 + 2\Gamma_2 + 4$ as it go through in proof of Theorem 2.

Case 2. Suppose that $F = G$. Then it follows from (2.1) that

$$P(f)f(z+\eta) = P(g)g(z+\eta).$$

With it and the similar arguments of Case 2 in the proof of Theorem 2 we can get our conclusion with $n > 3\Gamma_1 + 2\Gamma_2 + 4$, i.e., we get $f = tg$ for a constant t such that $t^d = 1$, where $d = GCD(\lambda_0, \lambda_1, \cdots, \lambda_n)$.

Case 3. Suppose that $FG = 1$. By substituting (2.1) into $FG = 1$ we get

$$P(f)f(z+\eta)P(g)g(z+\eta) = a^2(z). \tag{3.15}$$

Since f and g are entire then we have $a(z)$ is entire. From (3.15) and f, g are transcendental entire functions of finite order, one may immediately see,

$$N\left(r, \frac{1}{P(f)}\right) = O(\log r), \quad N\left(r, \frac{1}{P(g)}\right) = O(\log r).$$

Suppose that $P(u)$ has two zeros, say u_1, u_2, $u_1 \neq u_2$ then $P(f) = a_n(f-u_1)^{n_1}(f-u_2)^{n_2}$, where n_1, n_2, are positive integers and $n_1 + n_2 = n$. Therefore

$$N\left(r, \frac{1}{f-u_1}\right) = O(\log r), \quad \left(r, \frac{1}{f-u_2}\right) = O(\log r).$$

Applying the second main theorem we immediately get a contradiction.

Next we consider $P(u)$ has only one zero. Then we may write $P(f) = a_n(f-a)^n$, where a is a complex constant. Therefore, from the assumption that f and g are transcendental entire functions of finite order and that f and g share 0 CM, we can obtain

$$f(z) = P_1(z)e^{\alpha(z)} + a, \quad g(z) = P_1(z)e^{\beta(z)} + a, \tag{3.16}$$

where α and β are nonconstant polynomials. Therefore, from (3.16) we have $\rho(f) = \deg(\alpha)$, which contradicts the assumption that $\rho(f)$ is not a positive integer.

Thus we complete the proof of Theorem 7.

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