Singular orbits in cohomogeneity one pseudo Euclidean space $\mathbb{R}^4_\nu$

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Abstract. The purpose of this paper is to study cohomogeneity one pseudo Euclidean space $\mathbb{R}^4_\nu$, $0 \leqslant \nu \leqslant 4$, under a proper action of a connected Lie subgroup $G \subset Iso(\mathbb{R}^4_\nu)$. Among other results, when there is a singular orbit B, we determine the representation of the acting group G in $SO_\circ(\nu, 4-\nu) \ltimes R^4$ up to conjugacy, then the orbits up to isometry. In particular, we prove that B is an affine pseudo Euclidean subspace with $dimB \geqslant min\{\nu, 4-\nu\}$.

Keywords: *Cohomogeneity one , pseudo Euclidean space.*

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1 Introduction

Felix Klein is best known for his work in non-Euclidean geometry, for his work on the connections between geometry and group theory. According to his approach, a geometry is a G-space M, that is, a set M together with a group G of transformations of M. This approach provides a powerful link between geometry and algebra. If the group G acts transitively on M, that is, for any two points p and q in M there exists a transformation in G which maps p to q, then M is called a homogeneous G-space. So if the action is not transitive, we have nonhomogeneous geometries. One special case of nonhomogeneous geometries is when the action of the transformation group G has an orbit of codimension one in M, in which case the action is said to be of cohomogeneity one.

Cohomogeneity one Riemannian manifolds have been studied by many mathematicians, (see [2, 5, 6, 13, 15, 16, 19, 21, 23, 24, 25]) and currently it is still a very active subject. When the metric is indefinite there exist no much papers in the literature (see [3, 4]). In fact there are substantial differences between these two cases. A main difference is that in the Riemannian case, where G is closed in $Iso(M)$, the action is proper which is vital in the study of the subject, while in the indefinite case, this assumption in general does not imply that the action is proper, so the study becomes much more difficult. Also some of the results and techniques of the definite metric fails for the indefinite metric. In this paper, we study cohomogeneity one pseudo-Euclidean space $\mathbb{R}^4_\nu$, $0 \leqslant \nu \leqslant 4$, under the proper action of a closed and connected Lie subgroup G of $Iso(\mathbb{R}^4_\nu)$. We would like to determine the acting group G up to conjugacy and consequently the orbits up to isometry. Explicitly we prove that if there is a singular orbit B, then $dimB \geqslant min\{\nu, 4-\nu\}$ and B is an affine Riemannian, time-like or Lorentzian subspace of $\mathbb{R}^4_\nu$. In particular, there is neither degenerate nor light-like orbit.

2 Preliminaries

A smooth manifold M is called of cohomogeneity one under an action of a Lie group G if an orbit has codimension one. If M is a pseudo-Riemannian manifold and G is a closed Lie subgroup of $Iso(M)$ which acts isometrically and by cohomogeneity one on M, then M is called a cohomogeneity one pseudo-Riemannian manifold. For a general theory of cohomogeneity one pseudo-Riemannian manifolds we refer to [2, 3, 4, 8, 21, 23]. Here we remind some of the indispensable backgrounds.

Definition 2.1 ([*10*, p.53]). *An action of a Lie group G on a smooth manifold M is said to be proper if the mapping $\varphi : G \times M \to M \times M$, $(g, x) \mapsto (g.x, x)$ is proper.*

If there is a proper action of a Lie group G on a manifold M, then the orbit space M/G equipped with the quotient topology is Hausdorff, each orbit is a closed submanifold of M, and each isotropy (stabilizer) subgroup is a compact Lie subgroup of G (see [10, p.149]). The following theorem makes a link between proper G-manifolds and Riemannian G-manifolds

Theorem 2.2 ([22, p.77-78]) *There is a proper action of a Lie group G on a finite dimensional manifold M if and only if there is a Riemannian metric on M such that G is a closed subgroup of $Iso(M)$.*

An action of a Lie group G on a manifold M is called effective if $\bigcap_{x\in M} G_x = \{e\}$, where G_x denotes the isotropy subgroup at $x \in M$. Throughout the paper we assume that the action is effective and proper. A result by Mostert (see [18]), for the compact Lie groups, and Berard Bergery (see [5]), for the general case, says that the orbit space M/G is homeomorphic to one of the spaces

$$\mathbb{R} \quad , \quad S^1 \quad , \quad [0, +\infty) \quad , \quad [0, 1] \ .$$

Consider the canonical projection map $M \to M/G$ to the orbit space. Given a point $x \in M$, we say that the orbit $G(x)$ is *principal* (resp. *singular*) if the corresponding image in the orbit space M/G is an internal (resp. boundary) point. A point x whose orbit is principal (resp. singular) will be called *regular* (resp. *singular*). All principal orbits are diffeomorphic to each other, each singular orbit is of dimension less than or equal to $n-1$, where $n = dimM$, and the union of regular points is open and dense in M ([10, p.152]). A singular orbit of dimension $n-1$ is called an *exceptional* orbit. Note that no exceptional orbit is simply connected, and if M is simply connected no exceptional orbit may exist.

Remark 2.3 *Let G be a connected Lie group and M be a smooth manifold. Suppose that G acts on M properly and by cohomogeneity one . Then by Theorem 2.2, there exists a Riemannian metric g' on M such that G is a Lie subgroup of $Iso(M, g')$, and so the action is isometric with respect to the metric g'. If x and y are regular and singular points, respectively, then $G_x \subsetneqq G_y$ by Proposition 4.1 of [2].*

Throughout the following $\mathbb{R}^n_\nu$ denotes the n-dimensional real vector space $\mathbb{R}^n$ with a scalar product of signature $(\nu, n-\nu)$ given by

$$\langle x, y\rangle = -\sum_{i=1}^{\nu} x_i y_i + \sum_{i=\nu+1}^{n} x_i y_i \quad . \tag{1}$$

The set of all linear isometries $\mathbb{R}^n_\nu \to \mathbb{R}^n_\nu$ is the same as the set $O(\nu, n-\nu)$ of all matrices $g \in GL(n, \mathbb{R})$ that preserve the scalar product defined above. The identity component of $O(\nu, n-\nu)$ is denoted by $SO_\circ(\nu, n-\nu)$. Each maximal compact subgroup of $SO_\circ(\nu, n-\nu)$ is conjugate to $SO(\nu) \times SO(n-\nu)$ (see [14]).

Now we give a few facts of the theory of Lie groups which will be needed in the sequel.

Lemma 2.4 ([9, p.51]) *A simply connected solvable Lie group is diffeomorphic to $\mathbb{R}^n$, $n = dimG$.*

Lemma 2.5 ([9, p.52]) *Let G be a connected Lie group. Then the following conditions are equivalent:*
(i) The Lie group G is diffeomorphic to $\mathbb{R}^n$, $n = dimG$.
(ii) The maximal compact subgroup of G is trivial.

Lemma 2.6 ([17]) *Let G be a compact, or connected and semisimple, Lie group. Then any smooth representation of G by affine transformations of $\mathbb{R}^n$ admits a fixed point.*

Let $\mathfrak{g}$ and $\mathfrak{h}$ be Lie algebras. A derivation of $\mathfrak{g}$ is a linear map $D : \mathfrak{g} \to \mathfrak{g}$ satisfying

$$D[X, Y] = [DX, Y] + [X, DY] \quad , \quad \forall X, Y \in \mathfrak{g}$$

The vector space consists of derivations of $\mathfrak{g}$ with the bracket operation $[D_1, D_2] = D_1D_2 - D_2D_1$ is a Lie algebra which is denoted by $Der(\mathfrak{g})$. Let $\varphi : \mathfrak{g} \to Der(\mathfrak{h})$ be a Lie algebra homomorphism. The semi-direct sum $\mathfrak{g} \oplus_\varphi \mathfrak{h}$ is the direct sum of vector spaces $\mathfrak{g}$ and $\mathfrak{h}$ with the bracket operation

$$[(X,Y)\ ,\ (X',Y')] = ([X,X']_{\mathfrak{g}}\ ,\ [Y,Y']_{\mathfrak{h}} + \varphi(X)Y' - \varphi(X')Y)$$

for all $X, X' \in \mathfrak{g}$ and $Y, Y' \in \mathfrak{h}$.

Theorem 2.7 ([9, p.213]) *An arbitrary three dimensional connected real Lie group is isomorphic to one of the following pairwise nonisomorphic Lie groups:*

$$\mathbb{R}^3\ ,\ \mathbb{R}^2\times\mathbb{T}\ ,\ \mathbb{R}\times\mathbb{T}^2\ ,\ \mathbb{T}^3\ ,\ N_3(\mathbb{R})\ ,\ N_3^*\ ,\ R_2\times\mathbb{R}\ ,\ R_2\times\mathbb{T}\ ,\ R_3\ ,\ R_{3,\lambda}(\lambda\neq 0)$$
$$R'_{3,\lambda}(\lambda\neq 0)\ ,\ E^0(2)\ ,\ E_k\ (k\in\mathbb{N})\ ,\ SU(2)\ ,\ SO(3)\ ,\ \mathcal{A}\ ,\ A_1(m)\ (m\in\mathbb{N}).$$

Remark 2.8 Let's describe the stated Lie groups in Theorem 2.7 in detail.

The Lie group $N_3(\mathbb{R})$ is three dimensional Heisenberg group which is simply connected and nilpotent (see [9, p.54]) and $N_3^* = N_3(\mathbb{R})/Z(N_3(\mathbb{Z}))$ where $Z(N_3(\mathbb{Z}))$ is the group of integer points of the center $Z(N_3(\mathbb{R})) \cong \mathbb{R}$ of the group $N_3(\mathbb{R})$. The Lie group N_3^* is not linearizable (see example 7 of [9, p.23]). $R_2 = Aff_\circ(\mathbb{R})$ where $Aff_\circ(\mathbb{R})$ is the connected component of the group of affine transformations of the real line $\mathbb{R}^1$ (see example 3 of [9, p.49]) and the Lie groups $R_3\ ,\ R_{3,\lambda}\ ,\ R'_{3,\lambda}$ correspond to the Lie algebras $\mathfrak{r}_3(\mathbb{R})\ ,\ \mathfrak{r}_{3,\lambda}(\mathbb{R})\ ,\ \mathfrak{r}'_{3,\lambda}(\mathbb{R})$ which are of the form $\mathbb{R}\oplus_\varphi\mathbb{R}^2$, where the matrix $A = \varphi(1)$ is, respectively, $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$, $\begin{bmatrix} 1 & 0 \\ 0 & \lambda \end{bmatrix}$, $\begin{bmatrix} \lambda & -1 \\ 1 & \lambda \end{bmatrix}$. The Lie groups $R_3\ ,\ R_{3,\lambda}\ ,\ R'_{3,\lambda}$ are simply connected and solvable (see example 2 of [9, p.48]) so the maximal compact subgroup of them are trivial by Lemmas 2.4 and 2.5. The Lie group E_k is the k-fold covering of the Lie group $E_1 = E^0(2) = SO(2) \ltimes \mathbb{R}^2$. The Lie group $\mathcal{A}$ is the universal covering group of $SL(2,\mathbb{R})$ which is not linearizable (see example 6 of [9, p.23]) and $A_1(m)$ is the m-fold covering of the Lie group $A_1(1) = PSL(2,\mathbb{R})$ which is linearizable only for $m = 1, 2$ (see [9, p.152]).

3 Main results

Let $\mathbb{R}^4_\nu$, $0 \leqslant \nu \leqslant 4$, be of cohomogeneity one under the proper action of a connected Lie subgroup $G \subset Iso(\mathbb{R}^4_\nu)$. The following theorem has been proved in [4] (see Lemmas 3.2 to 3.8 of [4]). To facilitate the reader we prove it directly.

Theorem 3.1 *Let $\mathbb{R}^n_\nu$, $0 \leqslant \nu \leqslant n$, be of cohomogeneity one under the proper action of a connected Lie subgroup $G \subset Iso(\mathbb{R}^n_\nu)$.*

(a) *If $1 \leqslant \nu \leqslant n-1$, then G is not compact.*

(b) *The orbit space* $\mathbb{R}^n_\nu/G$ *is a one dimensional Hausdorff space homeomorphic to* $\mathbb{R}$ *or* $[0,+\infty)$. *In particular, there is at most one singular orbit.*

(c) *Suppose that* B *is a singular orbit and* $H = G_b$*, the isotropy subgroup at a point* $b \in B$. *Then* H *is a maximal compact subgroup of* G*, and* B *is diffeomorphic to* $\mathbb{R}^k$*, for some* $0 \leqslant k \leqslant n-2$.

Proof. (a) If G is compact then each (principal) orbit is compact, but there is no compact pseudo-Riemannian hypersurface in $\mathbb{R}^n_\nu$ for $1 \leqslant \nu \leqslant n-1$ (see [20, p.125]), so G is not compact.

(b) The orbit space $\mathbb{R}^n_\nu/G$ is homeomorphic to one of the spaces (see [5])

$$(i)\ \mathbb{R} \quad , \quad (ii)\ S^1 \quad , \quad (iii)\ [0,+\infty) \quad , \quad (iv)\ [0,1]\ .$$

By Proposition 3.3 of [23] there is at most one singular orbit, and so the case (iv) can not occur. We claim that the case (ii) is also not admitted. If $M/G \cong S^1$ then the canonical projection map $\pi : \mathbb{R}^n_\nu \to S^1$ is a fibration with fibre G/K, where K is the stabilizer of a regular point ([1]). By Theorem 4.41 of [12, p.379] there is a long exact sequence of homotopy groups

$$\to \pi_m(G/K, x_\circ) \to \pi_m(M, x_\circ) \to \pi_m(S^1, b_\circ) \to \pi_{m-1}(G/K, x_\circ) \to \cdots \to \pi_0(M, x_\circ) \to 0$$

where $x_\circ \in \pi^{-1}(b_\circ)$ and $b_\circ \in S^1$.
Hence $\pi_0(G/K, x_\circ) \cong \mathbb{Z}$ and this contradicts the connectedness of G/K.

(c) Since the action is proper, so H is compact. Suppose that H is not maximal compact in G. Hence $H \subsetneqq H'$, where H' is a compact Lie subgroup of G. There is a point $x_\circ \in \mathbb{R}^n_\nu$ which is fixed under the action of H', so under H, by Lemma 2.6. We note that $G(x_\circ)$ is necessarily a singular orbit by Remark 2.3, and $x_\circ$ does not belong to the orbit B, since otherwise H and H' would be conjugate, and hence equal. The unique geodesic γ through b and $x_\circ$ is left pointwise fixed under the action of H, since b and $x_\circ$ are left fixed. By the properness of the action $M_\circ$, the union of regular points, is open and dense in $\mathbb{R}^n_\nu$ (see ([10, p.152]). So $\gamma(t_0)$ is a regular point, for some $t_0 \in \mathbb{R}$, and is left fixed under the action of H. Hence $H \subset G_{\gamma(t_0)}$, which is a contradiction by Remark 2.3.

Now we show that B is diffeomorphic to $\mathbb{R}^k$, for some $0 \leqslant k \leqslant n-2$. Let $b \in B$ and $dim B = k$. As there is at most one singular orbit and the action is proper, by using Theorem 2.2 one gets that $\mathbb{R}^n_\nu$ is homeomorphic to $G \times_{G_b} V$, where V is an $(n-k)$-dimensional vector space (see [1]). Hence $\mathbb{R}^n_\nu$ is a fibre bundle with base G/G_b . Thus $\mathbb{R}^n_\nu$ and $G(b)$ are of the same homotopy type, therefore $G(b)$ is diffeomorphic to $\mathbb{R}^k$ by the following lemma. $\square$

Lemma 3.2 ([10, p.137]). *Let G be a connected Lie group and H a Lie subgroup of G. If $\pi_n(G/H) = 0$ for each $n \geqslant 0$, where π_n is the n-th homotopy group, then the manifold G/H is diffeomorphic to $\mathbb{R}^m$, where $m = dimG/H$.*

Two isometric actions on a pseudo-Riemannian manifold M are said to be orbit equivalent if there exists an isometry of M mapping the orbits of one of these actions onto the orbits of the other.

3.1 The case $M = \mathbb{R}^4$

If $M = \mathbb{R}^4$, i.e. $\nu = 0$, and G is a connected Lie subgroup of $Iso(\mathbb{R}^4)$ which acts by cohomogeneity one on $\mathbb{R}^4$, then by Theorem 3.1 of [19] either each principal orbit is isometric to $\mathbb{R}^3$ and there does not exist any singular orbit or each principal orbit is isometric to $S^k(r) \times \mathbb{R}^{3-k}$, $1 \leqslant k \leqslant 3$, k is fixed for all orbits, and the unique singular orbit is isometric to $\mathbb{R}^{3-k}$.

Proposition 3.3 *Let $\mathbb{R}^4$ be of cohomogeneity one under the isometric action of a connected, closed subgroup $G \subset Iso(\mathbb{R}^4)$. If there is a singular orbit B, then one of the following cases occurs:*

(i) If $dimB = 0$, then G is conjugate to one of the following Lie groups

$$SO(4)\ ,\ U(2)\ ,\ SU(2)\ ,\ Sp(1)\ ,\ Sp(1)Sp(1)\ ,\ Sp(1)U(1)\ .$$

(ii) If $1 \leqslant dimB \leqslant 2$, then G is conjugate to one of the the following Lie groups

$$\begin{aligned}
G_1 &= \left\{ \left(\begin{bmatrix} 1 & 0 \\ 0 & SO(3) \end{bmatrix}, \begin{bmatrix} x \\ 0 \end{bmatrix} \right) \mid x \in R \right\} \\
G_2 &= \left\{ \left(\begin{bmatrix} SO(2) & 0 \\ 0 & SO(2) \end{bmatrix}, \begin{bmatrix} 0 \\ X \end{bmatrix} \right) \mid X \in R^2 \right\} \\
G_3 &= \left\{ \left(\begin{bmatrix} R(\theta) & 0 \\ 0 & R(c\theta) \end{bmatrix}, \begin{bmatrix} 0 \\ X \end{bmatrix} \right) \mid X \in R^2\ ,\ \theta \in \mathbb{R} \right\}
\end{aligned}$$

where c is a fixed real number and $R(\theta) = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$.

Proof : (i) If $dimB = 0$, then without loss of generality we may assume that $B = \{0\}$, so $G_0 = G$ and so G is a compact subgroup of $SO(4)$, and by Theorem 3.1 of [19] G acts on S^3 transitively. But for an n-dimensional real vector space V, if $G \subseteq SO(n)$ is a compact connected Lie group which acts transitively on the unit sphere $S^{n-1} \subset V$, then G is any element of the following list which is called the Borel list (see [7]).

G	$SO(n)$	$U(n)$	$SU(n)$	$Sp(n)$	$Sp(n)Sp(1)$	$Sp(n)U(1)$	$Spin(9)$	$Spin(7)$	G_2
V	$\mathbb{R}^n$	$\mathbb{R}^{2n}$	$\mathbb{R}^{2n}$	$\mathbb{R}^{4n}$	$\mathbb{R}^{4n}$	$\mathbb{R}^{4n}$	$\mathbb{R}^{16}$	$\mathbb{R}^8$	$\mathbb{R}^7$

Hence for $V = \mathbb{R}^4$ one gets that G is one of the mentioned Lie groups in Proposition 3.3-(i).

(ii) If $dimB = 2$, then B is isometric to $\mathbb{R}^2$ and without loss of generality we may assume that $B = \{0\} \times \mathbb{R}^2 \subset \mathbb{R}^2 \oplus \mathbb{R}^2$. Let

$$B^{\perp} = \{x \in \mathbb{R}^4 \mid \langle x, y\rangle = 0 \ , \ \forall y \in B\}$$

where $\langle ., .\rangle$ denotes the usual dot product on $\mathbb{R}^4$. The relation $G(B) \subseteq B$ implies that $G(B^{\perp}) \subseteq B^{\perp}$, and so $p_1(G)(B^{\perp}) \subseteq B^{\perp}$, where $p_1 : SO(4) \ltimes R^4 \to SO(4)$ is the projection on the first factor. Hence each element of $p_1(G)$ is of the form

$$\begin{bmatrix} C & \\ & D \end{bmatrix} \in SO(4)$$

where C and D belong to $SO(2)$, and so $p_1(G) \subseteq diag(SO(2) \times SO(2))$. By the fact that $G(0) = B$, each element of G should be of the form

$$\left(\begin{bmatrix} C & \\ & D \end{bmatrix}, \begin{bmatrix} 0 \\ X \end{bmatrix} \right)$$

where $X \in R^2$. On the other hand, for each $x \in B^{\perp}$, $G(x) = S^1(r) \times \mathbb{R}^2$ implies that $p_1(G)$ acts on $S^1 \times \{0\}$ transitively, thus according to the dimension of $p_1(G)$, if $dimp_1(G) = 2$ then G is conjugate to G_2 and if $dimp_1(G) = 1$ then G is conjugate to G_3.

If $dimB = 1$, by a similar discussion one gets that G is conjugate to G_1. $\square$

3.2 The case $M = \mathbb{R}^4_1$

Let $M = \mathbb{R}^4_\nu$, $1 \leqslant \nu \leqslant 2$, and G be a connected Lie subgroup of $Iso(\mathbb{R}^4_\nu)$ which acts on $\mathbb{R}^4_\nu$ isometrically. Then there may be an orbit which is not closed in M and so the orbit space is not Hausdorff and hence the definition of principal and singular orbits (see preliminaries section) can not be used. Thus in the following subsections we assume that the action is proper. First we determine the acting Lie group G up to conjugacy and then we specify the orbits up to isometry and the orbit space up to homeomorphism.

Lemma 3.4 *The Lie algebra of any two dimensional non-abelian Lie subgroup of* $Iso_\circ(\mathbb{R}^2_1) = SO_\circ(1,1) \ltimes \mathbb{R}^2$ *is conjugate to*

$$\left\{ \left(\begin{bmatrix} 0 & t \\ t & 0 \end{bmatrix}, \begin{bmatrix} s \\ \beta s \end{bmatrix} \right) \mid s, t \in \mathbb{R} \right\}$$

where $\beta = 1$ *or* -1.

Proof : Consider the Lie algebra of the Lie group $Iso_\circ(\mathbb{R}^2_1)$ as follows

$$\mathfrak{I} = \mathfrak{so}(1,1) \oplus_\tau \mathbb{R}^2 = \left\{ \left(\begin{bmatrix} 0 & t \\ t & 0 \end{bmatrix}, \begin{bmatrix} s \\ u \end{bmatrix} \right) \mid s, t, u \in \mathbb{R} \right\}$$

where the representation τ of $\mathfrak{so}(1,1)$ on $\mathbb{R}^2$ is the standard representation and the bracket is defined as follows:

$$[(A,a),(B,b)] = (AB - BA, Ab - Ba)$$

Let $\mathfrak{g}$ be a non-abelian two dimensional Lie subalgebra of $\mathfrak{I}$. Since $\mathfrak{g}$ is not abelian so $p_1(\mathfrak{g}) \neq \{0\}$. By the fact that $dim\mathfrak{g} = 2$, one gets that u , s and t are not independent and hence one of them is a function of the others. Since $\mathfrak{g}$ is a Lie algebra, this function has to be linear. By choosing a new coordinate, one may assume that $u := u(s,t)$ is a linear function $u : \mathbb{R}^2 \longrightarrow \mathbb{R}$ and hence there exist fixed real numbers α , β such that $u(s,t) = \alpha t + \beta s$, i.e. $\mathfrak{g}$ as a vector space is

$$\mathfrak{g} = \left\{ \left(\begin{bmatrix} 0 & t \\ t & 0 \end{bmatrix}, \begin{bmatrix} s \\ \alpha t + \beta s \end{bmatrix} \right) \mid s, t \in \mathbb{R} \right\}$$

Take the following vectors

$$X = \left(\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 \\ \alpha \end{bmatrix} \right) \quad , \quad Y = \left(\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 \\ \beta \end{bmatrix} \right)$$

then the relation $[X,Y] \in \mathfrak{g}$ implies that $\beta = \pm 1$. Finally, if

$$a = \left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} \alpha \\ 0 \end{bmatrix} \right) \in Iso_\circ(\mathbb{R}^2_1)$$

then $a\mathfrak{g}a^{-1}$ is equal to

$$\begin{pmatrix} 1 & 0 & \alpha \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \left\{ \begin{pmatrix} 0 & t & s \\ t & 0 & \alpha t + \beta s \\ 0 & 0 & 0 \end{pmatrix} \mid s, t \in \mathbb{R} \right\} \begin{pmatrix} 1 & 0 & -\alpha \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \left\{ \begin{pmatrix} 0 & t & s \\ t & 0 & \beta s \\ 0 & 0 & 0 \end{pmatrix} \mid s, t \in \mathbb{R} \right\}$$

which is the desired Lie algebra. □

Corollary 3.5 *If G is a two dimensional non-abelian Lie subgroup of $Iso_\circ(\mathbb{R}^2_1)$ which acts on $\mathbb{R}^2_1$ isometrically. Then the action is neither proper nor transitive. There are three orbits, a light-like orbit of dimension one and two other orbits of dimension two. Each two dimensional orbit is not closed in $\mathbb{R}^2_1$, so the orbit space (that is a set with three points), with the quotient topology, is not Hausdorff.*

Lemma 3.6 *If G is a two dimensional abelian Lie subgroup of $Iso_\circ(\mathbb{R}^2_1)$, then it is the pure translation Lie subgroup.*

Proof : Let G be a two dimensional abelian Lie subgroup of $Iso_\circ(\mathbb{R}^2_1)$ and $\mathfrak{g}$ be its Lie algebra. Let $p_1 : \mathfrak{g} \longrightarrow \mathfrak{so}(1,1)$ be the projection map to the first factor. If $p_1(\mathfrak{g}) \neq \{0\}$ then by the proof of Lemma 3.4 $\mathfrak{g}$ should be conjugate to

$$\left\{ \left(\begin{bmatrix} 0 & t \\ t & 0 \end{bmatrix}, \begin{bmatrix} s \\ \beta s \end{bmatrix} \right) \mid s, t \in \mathbb{R} \right\}$$

which is non-abelian, a contradiction. Thus $p_1(\mathfrak{g}) = \{0\}$, i.e. G is a pure translation Lie group. □

In the proof of the following Theorem, whenever we use $SO(2)$ or $SO(3)$ as a Lie subgroup of $SO_\circ(1,3) \ltimes \mathbb{R}^4$ it is meant that $SO(2) = diag(I_{2\times 2}, SO(2)) \times \{0\}$, and $SO(3) = diag(1, SO(3)) \times \{0\}$.

Theorem 3.7 *Let $\mathbb{R}^4_1$ be of cohomogeneity one under the proper action of a connected and closed Lie subgroup $G \subset Iso(\mathbb{R}^4_1)$. If there is a singular orbit, then G is conjugate to one of the following Lie groups:*

$$\begin{aligned} H_1 &= \left\{ \left(\begin{bmatrix} I_{2\times 2} & \\ & SO(2) \end{bmatrix}, \begin{bmatrix} X \\ 0 \end{bmatrix} \right) \mid X \in R^2 \right\} \subset Iso(\mathbb{R}^4_1) \\ H_2 &= \left\{ \left(\begin{bmatrix} 1 & \\ & SO(3) \end{bmatrix}, \begin{bmatrix} x \\ 0 \end{bmatrix} \right) \mid x \in R \right\} \subset Iso(\mathbb{R}^4_1) \end{aligned}$$

Proof : Let $B = G(y)$ be the singular orbit, for some $y \in \mathbb{R}^4_1$. The Lie subgroup G_y is compact by the properness of the action and connected by Proposition 17 of [20, p.309]. Since each maximal compact Lie subgroup of $SO_\circ(1,3)$ is conjugate to $SO(3)$, so $G_y \subset SO(3)$ up to conjugacy and hence one of the following cases occurs

$$(i)\ G_y = \{I\} \qquad (ii)\ G_y = SO(2) \qquad (iii)\ G_y = SO(3)\ .$$

If $G_y = \{I\}$ then by Remark 2.3, up to conjugacy, $G_x \subsetneq G_y$ for each regular point $x \in \mathbb{R}^4_1$ which is not obviously possible. Hence $G_y \neq \{I\}$. We follow the proof by considering the following two cases.

Case 1 : $G_y = SO(2)$, then for each regular point $x \in \mathbb{R}^4_1$, G_x is conjugate to a proper Lie subgroup of G_y, so $dimG_x = 0$ hence $dimG = dimG/G_x = 3$ and $dimG(y) = 2$, thus $G(y)$ is diffeomorphic to $\mathbb{R}^2$ (see Lemma 3.4 of [4]). Let $G = S \ltimes L$ be a Levi decomposition of G. If S is not trivial, then the semisimple group S fixes some point $x_\circ \in \mathbb{R}^4_1$ by Lemma 2.6, so by the properness of the action S should be compact. On the other hand G_y is a maximal compact subgroup of G by Theorem 3.1, so L is conjugate to some subgroup of $SO(2)$ which is in contrast to the fact that S is semisimple, hence S is trivial which implies that G is solvable. Thus $G = G_y \ltimes F$ by Theorem 7.1 of [9, p.66], where F is a two dimensional simply connected normal Lie subgroup of G. We claim that G_y is a normal Lie subgroup of G. Assume that Π is the two dimensional subspace of $\mathbb{R}^4_1$ on which G_y acts by rotations, and so fixes $\Pi^\perp$ pintwise, where

$$\Pi^\perp = \{x \in \mathbb{R}^4_1 \mid \langle x, y\rangle = 0\ ,\ \forall y \in \Pi\}$$

where $\langle ., . \rangle$ is defined by (1) for $n = 4$ and $\nu = 1$. Since $G_y = SO(2) \subset diag(1, SO(3)) \times \{0\}$ up to conjugacy, so Π is a space-like subspace of $\mathbb{R}^4_1$ and hence $\Pi \cap \Pi^\perp = \{0\}$.
If x is a regular point then G_x is conjugate to some subgroup of G_y so if we assume that $G_x \subset G_y$, then G_y/G_x is diffeomorphic to S^1. So by considering the induced action of G_y on $\mathbb{R}^4_1$ one gets that $G_y(x)$ is diffeomorphic to S^1. Hence each point in $\Pi^\perp$ is a singular point and so $G(y) = \Pi^\perp$. This implies that $G_y = G_{y'}$ for each singular point y'. Hence for arbitrary $g \in G$, if $y' = gy$,

$$gG_yg^{-1} = G_{gy} = G_y$$

which shows that G_y is a normal Lie subgroup of G, hence $G = G_y \times F$. So

$$F \subset Z_G(SO(2)) \subset Z_{Iso_\circ(\mathbb{R}^4_1)}(SO(2))$$

and

$$Z_{Iso_\circ(\mathbb{R}^4_1)}(SO(2)) = \left\{ \left(\begin{bmatrix} SO_\circ(1,1) & \\ & SO(2) \end{bmatrix}, \begin{bmatrix} X \\ 0 \end{bmatrix} \right) \mid X \in R^2 \right\}.$$

Hence F is isomorphic to a Lie subgroup of $Iso_\circ(\mathbb{R}^2_1) = SO_\circ(1,1) \ltimes R^2$. If it is not abelian, then by Lemma 3.4 and Corollary 3.5 its action, and so the action of G on $\mathbb{R}^4_1$, is not proper. So F is a pure translation Lie subgroup of $Iso_\circ(\mathbb{R}^2_1)$ by Lemma 3.6. Thus

$$G = \left\{ \left(\begin{bmatrix} I_{2\times 2} & \\ & SO(2) \end{bmatrix}, \begin{bmatrix} X \\ 0 \end{bmatrix} \right) \mid X \in R^2 \right\}$$

up to conjugacy.

Case 2 : G_y is conjugate to $SO(3)$. We claim that $dimG_x > 0$ for each regular point $x \in \mathbb{R}^4_1$. If $dimG_x = 0$ then $dimG = dimG/G_x = 3$ and $dimG(y) = dimG/G_y = 0$, so $G(y) = \{y\}$ by the connectedness of G. Hence $G = G_y$ from which properness of the action implies that G must be compact, that is not possible by Theorem 3.1. Thus $dimG_x > 0$ for each regular point $x \in \mathbb{R}^4_1$ which implies that G_x is conjugate to $SO(2)$, hence $dimG = 4$. Let $G = S \ltimes L$ be the Levi decomposition of G. The Lie group S is a semisimple subgroup of G, so S fixes some point $x_\circ \in \mathbb{R}^4_1$ by Lemma 2.6 and by the properness of the action S must be compact. Since G_y, and so $SO(3)$, is the maximal compact Lie subgroup of G, hence S is conjugate to $SO(3)$, thus $dimL = 1$. Since G is not compact, L is isomorphic to the additive group R. We show that G is isomorphic to $SO(3) \times R$. Consider the adjoint action of $\mathfrak{so}(3)$ on the Lie algebra $\mathfrak{l}$ of L

$$ad|_{\mathfrak{so}(3)} : \mathfrak{so}(3) \longrightarrow Der(\mathfrak{l})$$

Since $ker(ad|_{\mathfrak{so}(3)})$ ia an ideal of the simple Lie algebra $\mathfrak{so}(3)$, so $ker(ad|_{\mathfrak{so}(3)}) = \mathfrak{so}(3)$ or $ker(ad|_{\mathfrak{so}(3)}) = 0$. On the other hand, $dim(Der(\mathfrak{l})) = 1$ shows that $ad|_{\mathfrak{so}(3)}$ is not one to one, thus $ker(ad|_{\mathfrak{so}(3)}) = \mathfrak{so}(3)$ which implies that $[\mathfrak{so}(3), \mathfrak{l}] = 0$. Hence G is isomorphic to $SO(3) \times R$. Since $SO(3)$ is the maximal compact subgroup of $Iso_\circ(\mathbb{R}^4_1)$ and

$$Z_{Iso_\circ(\mathbb{R}^4_1)}(SO(3)) = \left\{ \left(I_{4\times 4}, \begin{bmatrix} x \\ 0 \\ 0 \\ 0 \end{bmatrix} \right) \mid x \in R \right\}$$

so G is conjugate to

$$\left\{ \left(\begin{bmatrix} 1 & \\ & SO(3) \end{bmatrix}, \begin{bmatrix} x \\ 0 \end{bmatrix} \right) \mid x \in R \right\} \subset Iso(\mathbb{R}^4_1). \quad \square$$

Corollary 3.8 *Let $\mathbb{R}^4_1$ be of cohomogeneity one under the proper action of a connected and closed Lie subgroup $G \subset Iso(\mathbb{R}^4_1)$. If B is a singular orbit, then B is isometric to $\mathbb{R}^k_1$, $k = 1, 2$, and each principal orbit is isometric to $\mathbb{R}^k_1 \times S^{3-k}(r)$ for some $r > 0$, where k is fixed for all orbits. In particular B is a Lorentzian affine subspace.*

Proof : By Theorem 3.7 G is conjugate to either H_1 or H_2. If G is conjugate to H_1, there is $(A, a) \in SO_\circ(1,3) \ltimes \mathbb{R}^4_1$ such that

$$G = (A,a)H_1(A^{-1}, -A^{-1}a)$$

Denote by $\mathbb{R}^2_1$ the vector subspace

$$\left\{ \begin{bmatrix} x_1 \\ x_2 \\ 0 \\ 0 \end{bmatrix} \mid x_1, x_2 \in \mathbb{R} \right\} \subset \mathbb{R}^4_1$$

and consider the translation vector a as a point of $\mathbb{R}^4_1$, then

$$G(a) = (A,a)\mathbb{R}^2_1$$

Since $(A,a) \in Iso_\circ(\mathbb{R}^4_1)$, the orbit $G(a)$ is isometric to $\mathbb{R}^2_1$.

Choose $x \in \mathbb{R}^4_1$ and let $y = (A^{-1}, -A^{-1}a)x$. If y does not belong to the vector subspace $\mathbb{R}^2_1$ then

$$H_1(y) = S^1(r) \times \mathbb{R}^2_1 \qquad , \quad \text{where} \quad r = \sqrt{y_3^2 + y_4^2} \quad , \text{ where} \quad y = (y_1, y_2, y_3, y_4)^t$$

and

$$G(x) = (A,a)H_1(y) \ .$$

Hence $G(x)$ is isometric to $S^1(r) \times \mathbb{R}^2_1$.

If G is conjugate to H_2, by a similar discussion one gets the result. □

3.3 The case $\mathbb{R}^4_2$

Lemma 3.9 *Let $\mathbb{R}^4_2$ be of cohomogeneity one under the proper action of a connected and closed Lie subgroup $G \subset Iso(\mathbb{R}^4_2)$. Then G is solvable.*

Proof : Let $G = S \ltimes L$ be a Levi decomposition of G. We claim that S is trivial. If S is not trivial, then it fixes some point $x_\circ \in \mathbb{R}^4_2$ by Lemma 2.6. So by the properness of the action S is a compact subgroup of G. Since $SO(2) \times SO(2)$ is the maximal compact Lie subgroup of $Iso_\circ(\mathbb{R}^4_2)$, the Lie group S must be conjugate to some subgroup of $SO(2) \times SO(2)$ which is not possible obviously (there is no semisimple Lie group with dimension less than three). Hence S is trivial and G is solvable. □

Lemma 3.10 *Let $\varphi : E^0(2) \to SO(2,2) \ltimes R^4$ be a faithful Lie group representation. Suppose that $G = \varphi(E^0(2))$ and $\mathfrak{g}$ is the Lie algebra of G. If $p_1(G)$ is isomorphic to $SO(2)$, then $\mathfrak{g}$ is conjugate to the following Lie algebra.*

$$\left\{ \left(\begin{bmatrix} r(c_1t) & \\ & r(c_2t) \end{bmatrix}, \begin{bmatrix} A_1X + C_1T \\ A_2X + C_2T \end{bmatrix} \right) \mid T = \begin{bmatrix} t \\ t \end{bmatrix} \ , \ t \in \mathbb{R} \ , \ X \in \mathbb{R}^2 \right\} \tag{2}$$

where A_i's are fixed 2×2 real matrices such that $A_iA_i^t = \pm det(A_i)I_{2\times 2}$, C_i's are fixed 2×2 diagonal matrices, c_i's are fixed real numbers such that at least one of them must be equal to either 1 or -1, and at most one of them may be equal to zero, where $i = 1, 2$, and $r(t) = \left[\begin{smallmatrix} 0 & t \\ -t & 0 \end{smallmatrix} \right]$. Each A_i is either invertible or equal to the zero matrix, for $i = 1, 2$, and at least one of them is invertible. Furtherfore, if $c_i \neq \pm 1$ then $A_i = 0$, where $i = 1, 2$.

Proof : Let $\mathfrak{e}$ be the Lie algebra of $E^0(2)$ and $\mathfrak{so}(2,2) \oplus_\tau \mathbb{R}^4$ be the Lie algebra of $SO_\circ(2,2) \ltimes \mathbb{R}^4$, where $\tau : \mathfrak{so}(2,2) \to Der(\mathbb{R}^4)$ is the natural action of $\mathfrak{so}(2,2)$ on $\mathbb{R}^4$, i.e. $\tau(A)(X) = AX$. Suppose that $\Phi : \mathfrak{e} \to \mathfrak{so}(2,2) \oplus_\tau \mathbb{R}^4$ be the Lie algebra representation corresponding to φ (see Chapter 4 of [11]). Since $p_1(G)$ is isomorphic to $SO(2)$ and any maximal compact Lie subgroup of $SO_\circ(2,2) \ltimes R^4$ is conjugate to $SO(2) \times SO(2)$, so $p_1(G) \subset SO(2) \times SO(2)$, up to conjugacy. Hence without loss of generality we may assume that $p_1(\mathfrak{g}) \subset \mathfrak{so}(2) \oplus \mathfrak{so}(2)$. Since Φ is linear, so there exist fixed real numbers a_i , b_i , α_i , β_i , η_i , ξ_i, where $i = 1, 2, 3$, such that

$$\Phi\left(\begin{bmatrix} 0 & x_1 \\ -x_1 & 0 \end{bmatrix}, \begin{bmatrix} x_2 \\ x_3 \end{bmatrix} \right) = \left(\begin{bmatrix} 0 & \Sigma a_ix_i & 0 & 0 \\ -\Sigma a_ix_i & 0 & 0 & 0 \\ 0 & 0 & 0 & \Sigma b_ix_i \\ 0 & 0 & -\Sigma b_ix_i & 0 \end{bmatrix}, \begin{bmatrix} \Sigma\alpha_ix_i \\ \Sigma\beta_ix_i \\ \Sigma\eta_ix_i \\ \Sigma\xi_ix_i \end{bmatrix} \right) \tag{3}$$

By the fact that Φ preserves the bracket, one gets that $a_i = b_i = 0, \ i = 2, 3$, and the following relations hold.

$$\alpha_2 = a_1\beta_3 \quad , \quad \alpha_3 = -a_1\beta_2 \quad , \quad \beta_3 = a_1\alpha_2 \quad , \quad \beta_2 = -a_1\alpha_3 \tag{4}$$

and

$$\eta_2 = b_1\xi_3 \quad , \quad \eta_3 = -b_1\xi_2 \quad , \quad \xi_2 = -b_1\eta_3 \quad , \quad \xi_3 = b_1\eta_2 \ . \tag{5}$$

If $a_1 \neq \pm 1$ then (4) implies that $\alpha_i = \beta_i = 0$ and if $b_1 \neq \pm 1$ then (5) implies that $\eta_i = \xi_i = 0$, where $i = 2, 3$. Hence $a_1 \neq \pm 1$ and $b_1 \neq \pm 1$, simultaneously, imply that Φ, and so φ, is not faithful (see Theorem 2.21 of [11]), which is in contradict to the assumption. Thus at least one of them must be equal to either 1 or -1. On the other hand, If $a_1 = b_1 = 0$, then (4) and (5) imply that $\alpha_i = \beta_i = \eta_i = \xi_i = 0$, for $i = 2, 3$, which shows that Φ, and so φ, is not faithful, that is a contradiction. Thus at most one of a_1 or b_1 may be zero. To adjust the notation, assume that $c_1 := a_1$, $c_2 := b_1$ and

$$A_1 := \begin{bmatrix} \alpha_2 & \alpha_3 \\ \beta_2 & \beta_3 \end{bmatrix} \quad , \quad A_2 := \begin{bmatrix} \eta_2 & \eta_3 \\ \xi_2 & \xi_3 \end{bmatrix} \quad , \quad C_1 := \begin{bmatrix} \alpha_1 & 0 \\ 0 & \beta_1 \end{bmatrix} \quad , \quad C_2 := \begin{bmatrix} \eta_1 & 0 \\ 0 & \xi_1 \end{bmatrix}.$$

Therefore, if $c_i \neq \pm 1$ then $A_i = 0$ by (4) and (5). If $c_1 = 1$ then $A_1 = \begin{bmatrix} \alpha_2 & \alpha_3 \\ -\alpha_3 & \alpha_2 \end{bmatrix}$ by (4) and so $A_1 A_1^t = det(A_1) I_{2\times 2}$. If $c_1 = -1$ then $A_1 = \begin{bmatrix} \alpha_2 & \alpha_3 \\ \alpha_3 & -\alpha_2 \end{bmatrix}$ and so $A_1 A_1^t = -det(A_1) I_{2\times 2}$. Hence A_1 is invertible or it is equal to the zero matrix. A similar discussion about c_2 shows that $A_2 A_2^t = \pm det(A_2) I_{2\times 2}$.
If A_1 and A_2 are not invertible, then $A_1 = A_2 = 0$, and so Φ, hence φ, is not faithful, which is in contradict to the assumption. Thus at least one of A_1 or A_2 is invertible. □

In the proof of the following theorem, whenever we use $SO(2)\times SO(2)$ as a Lie subgroup of $SO_\circ(2,2) \ltimes \mathbb{R}^4$ it is meant that $SO(2) \times SO(2) = diag(SO(2), SO(2)) \times \{0\}$.

Theorem 3.11 *Let $\mathbb{R}^4_2$ be of cohomogeneity one under the proper action of a connected and closed Lie subgroup $G \subset Iso(\mathbb{R}^4_2)$. If there is a singular orbit, then G is isomorphic to one of the Lie groups $E^0(2) \times SO(2)$ or $SO(2) \times R^2$ or E_k, for some $k \in \mathbb{N}$, with the following representations in $SO(2,2) \ltimes R^4$, up to conjugacy.*

$$\begin{aligned}
K_1 &= \left\{ \left(\begin{bmatrix} H & \\ & SO(2) \end{bmatrix}, \begin{bmatrix} X \\ 0 \end{bmatrix} \right) | X \in R^2 \right\} \\
K_2 &= \left\{ \left(\begin{bmatrix} SO(2) & \\ & H \end{bmatrix}, \begin{bmatrix} 0 \\ X \end{bmatrix} \right) | X \in R^2 \right\} \\
K_3 &= \left\{ \left(\begin{bmatrix} R(kt) & \\ & R(ckt) \end{bmatrix}, \begin{bmatrix} X \\ 0 \end{bmatrix} \right) | \ t \in \mathbb{R} \ , \ X \in R^2 \right\} \\
K_4 &= \left\{ \left(\begin{bmatrix} R(ckt) & \\ & R(kt) \end{bmatrix}, \begin{bmatrix} 0 \\ X \end{bmatrix} \right) | \ t \in \mathbb{R} \ , \ X \in R^2 \right\}
\end{aligned}$$

where H is either $I_{2\times 2}$ or $SO(2)$, k is a fixed natural number, c is a fixed nonzero real number and $R(\theta) = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$.

Proof: Let $B = G(y)$ be the singular orbit, for some $y \in \mathbb{R}^4_2$. Since G_y is compact and each maximal compact Lie subgroup of $SO_\circ(2,2)$ is conjugate to $SO(2) \times SO(2)$, so $G_y \subset SO(2) \times SO(2)$ up to conjugacy, hence G_y is isomorphic to one of the following Lie groups

$$(i)\ \{I\} \quad , \quad (ii)\ SO(2) \times SO(2) \quad , \quad (iii)\ SO(2)$$

If $G_y = \{I\}$ then by Remark 2.3, up to conjugacy, $G_x \subsetneq G_y$ for each regular point $x \in \mathbb{R}^4_2$ which is not obviously possible. Hence $G_y \neq \{I\}$.

Case 1: If G_y is isomorphic (hence conjugate) to $SO(2) \times SO(2)$. For each regular point $x \in \mathbb{R}^4_2$, G_x is conjugate to a proper (compact) subgroup of G_y, so $dim G_x = 0$ or G_x is isomorphic to $SO(2)$. If $dim G_x = 0$ then $dim G = dim G/G_x = 3$ and by Lemma 3.9 G is solvable, so by Theorem 2.7 and Remark 2.8 G must be isomorphic to

$$\mathbb{T}^2 \times R = SO(2) \times SO(2) \times R$$

hence

$$R \subset Z_{Iso_\circ(\mathbb{R}^4_2)}(SO(2) \times SO(2)) = SO(2) \times SO(2)$$

which is not obviously possible. Thus G_x must be isomorphic to $SO(2)$. If G_x is isomorphic to $SO(2)$, then $dim G/G_x = 3$ implies that $dim G = 4$, so $dim G(y) = 2$, hence $G(y)$ is diffeomorphic to $\mathbb{R}^2$. Since G is solvable by Lemma 3.9 and its maximal compact Lie subgroup is isomorphic to $SO(2) \times SO(2)$, hence G is isomorphic to

$$SO(2) \times SO(2) \ltimes R^2$$

If $\mathfrak{g}$ is the Lie algebra of G, by rechoosing coordinates in $\mathfrak{g}$ we may assume that

$$\mathfrak{g} = \left\{ \left(\begin{bmatrix} 0 & t & 0 & 0 \\ -t & 0 & 0 & 0 \\ 0 & 0 & 0 & s \\ 0 & 0 & -s & 0 \end{bmatrix}, \begin{bmatrix} \alpha u \\ \beta v \\ a_1 u + a_2 v + a_3 t + a_4 s \\ b_1 u + b_2 v + b_3 t + b_4 s \end{bmatrix} \right) \mid s, t, u, v \in \mathbb{R} \right\}$$

where α, β, a_i and b_i are fixed real numbers. Closeness of $\mathfrak{g}$ under the bracket shows that $\alpha = 0$ if and only if $\beta = 0$. So if $\alpha \neq 0$, then closeness of $\mathfrak{g}$ under the bracket implies that $a_i = b_i = 0$, $1 \leqslant i \leqslant 3$, and so $\mathfrak{g}$ is conjugate to

$$\left\{ \left(\begin{bmatrix} 0 & t & 0 & 0 \\ -t & 0 & 0 & 0 \\ 0 & 0 & 0 & s \\ 0 & 0 & -s & 0 \end{bmatrix}, \begin{bmatrix} u \\ v \\ 0 \\ 0 \end{bmatrix} \right) \mid s, t, u, v \in \mathbb{R} \right\}$$

hence G is conjugate to

$$\left\{ \left(\begin{bmatrix} SO(2) & \\ & SO(2) \end{bmatrix}, \begin{bmatrix} X \\ 0 \end{bmatrix} \right) | X \in R^2 \right\}$$

and so G is isomorphic to $E^0(2) \times SO(2)$. If $\alpha = \beta = 0$, then by rechoosing coordinates one gets that $\mathfrak{g}$ is conjugate to

$$\left\{ \left(\begin{bmatrix} 0 & t & 0 & 0 \\ -t & 0 & 0 & 0 \\ 0 & 0 & 0 & s \\ 0 & 0 & -s & 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ u \\ v \end{bmatrix} \right) \mid s, t, u, v \in \mathbb{R} \right\}$$

hence G is conjugate to

$$\left\{ \left(\begin{bmatrix} SO(2) & \\ & SO(2) \end{bmatrix}, \begin{bmatrix} 0 \\ X \end{bmatrix} \right) |X \in R^2 \right\} .$$

Thus G is isomorphic to $SO(2) \times E^0(2)$.

Case 2: If G_y is isomorphic to $SO(2)$. For each regular point $x \in \mathbb{R}^4_2$, G_x is conjugate to a proper Lie subgroup of G_y , so $dimG_x = 0$, hence $dimG = dimG/G_x = 3$. Thus G is a three dimensional solvable Lie group such that its maximal compact Lie subgroup is isomorphic to $SO(2)$ by Theorem 3.1 and Lemma 3.9. Hence G is isomorphic to one of the following Lie groups by Theorem 2.7 and Remark 2.8

$$E_k \ (k \in \mathbb{N}) \ , \ SO(2) \times \mathbb{R}^2 \ , \ SO(2) \times Aff_\circ(\mathbb{R}) \ .$$

We claim that G is not isomorphic to $SO(2) \times Aff_\circ(\mathbb{R})$.
If G is isomorphic to $SO(2) \times Aff_\circ(\mathbb{R})$ then up to isomorphism,

$$Aff_\circ(\mathbb{R}) \subset Z_G(SO(2)) \subset Z_{Iso_\circ(\mathbb{R}^4_2)}(SO(2)) = SO(2) \times E^0(2)$$

which implies that $Aff_\circ(\mathbb{R})$ is isomorphic to a Lie subgroup of $E^0(2) = SO(2) \ltimes R^2$, that is not possible obviously. Hence we may consider the following subcases.

Case 2.1. If G is isomorphic to E_k for some $k \in \mathbb{N}$, then by Lemma 3.10, $\mathfrak{g}$ is conjugate to

$$\left\{ \left(\begin{bmatrix} r(c_1kt) & \\ & r(c_2kt) \end{bmatrix}, \begin{bmatrix} A_1X + C_1T \\ A_2X + C_2T \end{bmatrix} \right) \mid T = \begin{bmatrix} t \\ t \end{bmatrix} \ , \ t \in \mathbb{R} \ , \ X \in \mathbb{R}^2 \right\} \quad (6)$$

where c_i, C_i and A_i, for $i = 1, 2$, are defined in the proof of Lemma 3.10. It is known by the lemma that at least one of c_1 or c_2 must be equal to ± 1.

Case 2.1.1. Let $c_1 = 1$ and A_1 be invertible. In this case we show that G is conjugate to K_3. Without loss of generality assume that $\mathfrak{g}$ is equal to the Lie algebra defined by (6). We claim that $c_2 \neq 0$. If $c_2 = 0$ then $A_2 = 0$ by Lemma 3.10, and so by choosing a new coordinate, one gets that

$$a\mathfrak{g}a^{-1} = \left\{ \left(\begin{bmatrix} r(kt) & \\ & 0 \end{bmatrix}, \begin{bmatrix} X \\ C_2T \end{bmatrix} \right) \mid T = \begin{bmatrix} t \\ t \end{bmatrix} \ , \ t \in \mathbb{R} \ , \ X \in \mathbb{R}^2 \right\}$$

where $a = (I_{4\times4}, (-\beta_1/k, \alpha_1/k, 0, 0)^t)$. Hence G is conjugate to

$$\left\{ \left(\begin{bmatrix} R(kt) & \\ & I_{2\times2} \end{bmatrix}, \begin{bmatrix} Y \\ C_2T \end{bmatrix} \right) \mid T = \begin{bmatrix} t \\ t \end{bmatrix} \ , \ t \in \mathbb{R} \ , \ Y \in \mathbb{R}^2 \right\}$$

where $R(t) = \begin{bmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{bmatrix}$. If $C_2 = 0$ then there is no three dimensional orbit, which is in contradict to the cohomogeneity one assumption, and if $C_2 \neq 0$, then there is no two dimensional orbit, which is in contradict to the fact that $dimG(y) = 2$. Thus $c_2 \neq 0$ and so

$$a\mathfrak{g}a^{-1} = \left\{ \left(\begin{bmatrix} r(kt) & \\ & r(c_2kt) \end{bmatrix}, \begin{bmatrix} A_1X \\ A_2X \end{bmatrix} \right) \mid t \in \mathbb{R} \ , \ X \in \mathbb{R}^2 \right\}$$

where $a = (I_{4\times4}, (-\beta_1, \alpha_1, -\xi_1/kc_2, \eta_1/kc_2)^t)$.

We claim that $A_2 = 0$. If $A_2 \neq 0$ then it is invertible and c_2 is equal to 1 or -1, by Lemma 3.10. Since A_1 is invertible by assumption, so by choosing a new coordinate one may assume that

$$\mathfrak{g} = \left\{ \left(\begin{bmatrix} r(kt) & \\ & r(c_2kt) \end{bmatrix}, \begin{bmatrix} X \\ A_3X \end{bmatrix} \right) \mid t \in \mathbb{R} \ , \ X \in \mathbb{R}^2 \right\}$$

where $A_3 = A_2A_1^{-1}$, and so each orbit will be two dimensional which is in contradict to the cohomogeneity one assumption. Thus $A_2 = 0$ and so $\mathfrak{g}$ is conjugate to

$$\left\{ \left(\begin{bmatrix} r(kt) & \\ & r(c_2kt) \end{bmatrix}, \begin{bmatrix} X \\ 0 \end{bmatrix} \right) \mid t \in \mathbb{R} \ , \ X \in \mathbb{R}^2 \right\} .$$

Therefore G is conjugate to K_3.

If $c = -1$ and A_1 is invertible, then the same proof shows that G is conjugate to K_3.

Case 2.1.2. If $c_1 = 1$ and A_1 is not invertible. In this case we show that G is conjugate to K_4. Since A_1 is not invertible, $A_1 = 0$ by Lemma 3.10, and so A_2 is invertible. Hence c_2 is equal to 1 or -1 by Lemma 3.10. Thus by choosing a new coordinate, one gets that

$$a\mathfrak{g}a^{-1} = \left\{ \left(\begin{bmatrix} r(kt) & \\ & r(c_2kt) \end{bmatrix}, \begin{bmatrix} 0 \\ X \end{bmatrix} \right) \mid t \in \mathbb{R} \ , \ X \in \mathbb{R}^2 \right\}$$

where $a = (I_{4\times4}, (-\beta_1/k, \alpha_1/k, -\xi_1/c_2k, \eta_1/c_2k)^t)$. Thus G is conjugate to K_4 (note that c_2 is equal to either 1 or -1).

If $c_1 = -1$ and A_1 is not invertible, then the same discussion shows that G is conjugate to K_4.

Case 2.1.3. Let c_2 be equal to 1 or -1. If A_2 is invertible, then by a similar discussion to that of Case 2.1.1. one gets that G is conjugate to K_4. If A_2 is not invertible, then by a similar discussion to that of Case 2.1.2. one obtains that G is conjugate to K_4.

Case 2.2. G is isomorphic to $SO(2)\times R^2$. So

$$R^2 \subset Z_G(SO(2)) \subset Z_{Iso_\circ(\mathbb{R}^4_2)}(SO(2)) = SO(2)\times E^0(2)$$

hence R^2 is isomorphic to a Lie subgroup of $E^0(2) = SO(2)\ltimes R^2$. Thus, by Lemma 3.6, G is conjugate to either K_1 or K_2. $\square$

Corollary 3.12 *Let $\mathbb{R}^4_2$ be of cohomogeneity one under the proper action of a connected and closed Lie subgroup $G \subset Iso(\mathbb{R}^4_2)$. If B is a singular orbit, then B is isometric to either $\mathbb{R}^2$ or $\mathbb{R}^2_2$ and each principal orbit is isometric to either $\mathbb{R}^2\times S^1(r)$ or $\mathbb{R}^2_2\times S^1(r)$.*

Proof : By Theorem 3.11 the action of G is orbit equivalent to the action of one the Lie groups K_i, where $1\leqslant i\leqslant 4$. So by a similar proof of Corollary 3.8 one gets the result. $\square$

Remark 3.13 *The mapping $\mathbb{R}^n_\nu \to \mathbb{R}^n_{n-\nu}$, is defined by $(x_1,...,x_n)\mapsto(x_{\nu+1},...,x_n,x_1,...,x_\nu)$, is an anti isometry. So for a singular orbit B, if $\nu=4$ then $0\leqslant dimB\leqslant 3$ and B is a time-like affine subspace of $\mathbb{R}^4_4$ by Proposition 3.3, and if $\nu=3$ then B is anti-isometric to $\mathbb{R}^k_1$, where $k=1,2$, by Corollary 3.8. Thus we may sum up the results in the following Theorem.*

Theorem 3.14 *Let $\mathbb{R}^4_\nu$, $0\leqslant\nu\leqslant 4$, be of cohomogeneity one under the proper action of a connected and closed Lie subgroup $G\subset Iso(\mathbb{R}^4_\nu)$. If B is a singular orbit, then $dimB \geqslant min\{\nu, n-\nu\}$ and each orbit is a Riemannian, time-like or Lorentzian submanifold. In particular, there is neither degenerate nor light-like orbit.*

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