On Supercyclicity of Tuples of Operators

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Abstract

In this paper, we use a result of N. S. Feldman to show that there are no supercyclic subnormal tuples in infinite dimensions. Also, we investigate some spectral properties of hypercyclic tuples of operators. Besides, we prove that if T is a supercyclic ℓ-tuple of commuting $n \times n$ complex matrices, then $\ell \geq n$ and also there exists a supercyclic n-tuple of commuting diagonal $n \times n$ matrices. Furthermore, we see that if $T = (T_1, ..., T_n)$ is a supercyclic n-tuple of commuting $n \times n$ complex matrices, then T_j's are simultaneously diagonalizable.

Key words and phrases: supercyclicity, tuples, subnormal operators [1]

1. Introduction

An n-tuple of operators is a finite sequence of length n of commuting continuous linear operators $T_1, T_2, ..., T_n$ acting on a locally convex topological vector space X. Hypercyclic tuples of operators were introduced in [5, 6] and [8]. For an n-tuple $T = (T_1, T_2, ..., T_n)$ take $\mathcal{F}_T = \{T_1^{k_1} T_2^{k_2} ... T_n^{k_n} : k_i \geq 0, i = 1, 2, ..., n\}$. If there exists an element $x \in X$ such that $orb(T, x) = \{Sx : S \in \mathcal{F}_T\}$ is dense in X, then x is called a hypercyclic vector for T, and T is said to be a hypercyclic n-tuple of operators. A vector $x \in X$ is called a supercyclic vector for T if the set $\{Sx : S \in \mathcal{F}_T^p\}$ is dense in X, where $\mathcal{F}_T^p = \{\lambda S : S \in \mathcal{F}_T, \lambda \in \mathbb{C}\}$ and T is said to be a supercyclic n-tuple of operators [8]. These definitions generalize the hypercyclicity and supercyclicity of a single operator to a tuple of operators. Recently, the authors in [11] investigate the hypercyclicity of tuples of the adjoint of the weighted composition operator. The cyclicity of operators have received a good deal of attention in recent years. Reference [1] provides an overview of many results that are known.

In this paper, we first prove some elementary properties of a supercyclic tuple of operators. We then show that there are no supercyclic normal tuples in infinite

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dimensions. Also, we prove that if the Harte spectrum of a hypercyclic n-tuple of operators on a Banach space is nonempty then it intersects the complement of the unit polydisc of $\mathbb{C}^n$. Besides, we see that it is possible that the Harte spectrum of a hypercyclic n-tuple of operators does not intersect the closure of the unit polydisc of $\mathbb{C}^n$. Furthermore, we observe that the supercyclicity of an n-tuple of diagonal matrices on $\mathbb{C}^n$ may occur while there are no hypercyclic n-tuples of diagonalizable matrices on $\mathbb{C}^n$ [6]. Moreover, we show that if T is a supercyclic ℓ-tuple of commuting $n \times n$ complex matrices, then $\ell \geq n$ and if $T = (T_1, ..., T_n)$ is a supercyclic n-tuple of commuting $n \times n$ complex matrices, then T_j's are simultaneously diagonalizable.

2. Some properties of supercyclic tuple

We denote by X a separable Banach space with dimension greater than one. We give some basic results on supercyclicity of tuple of operators. The following Proposition follows directly from ([7], Theorem 1, pp. 348-349).

Proposition 1. *Suppose that $T = (T_1, T_2, ..., T_n)$ is an n-tuple of operators on X. If for any two nonempty open subsets U and V in X, there exist nonnegative integers $k_1, k_2, ..., k_n$ and $\lambda \in \mathbb{C}$ such that $\lambda T_1^{k_1} T_2^{k_2} ... T_n^{k_n}(U) \cap V \neq \emptyset$, then T is supercyclic.*

Remark 1. *Note that if T is an ℓ-tuple of commuting operators on X and T' is obtained from T by kicking out all non-dense range operators then either both T and T' are supercyclic or both are non-supercyclic. This allows to consider only tuples of dense range operators. Indeed, suppose that T_1 is not dense range and x is a supercyclic vector for T. Put*

$$G = \{\lambda T_1^{k_1} T_2^{k_2} ... T_\ell^{k_\ell} x : \lambda \in \mathbb{C}, k_1 > 0, k_i \geq 0, i = 2, 3, ..., \ell\}$$

and

$$H = \{\lambda T_2^{k_2} ... T_\ell^{k_\ell} x : \lambda \in \mathbb{C}, k_i \geq 0, i = 2, 3, ..., \ell\}$$

so $X = \overline{G \cup H}$ and $int(\overline{G}) = \emptyset$, hence $X = \overline{H}$.

Proposition 2. *Suppose that $T = (T_1, ..., T_n)$ is a supercyclic tuple on X. Then*
(1) the set of supercyclic vectors for the pair $T = (T_1, ..., T_n)$ is dense in X,
(2) if T_k is invertible for each k, then the tuple $(T_1^{-1}, ..., T_n^{-1})$ is also supercyclic on X;
(3) if $\mathcal{M}$ is an invariant subspace for T, then the quotient of T is supercyclic on $X/\mathcal{M}$.

Proof. (1) We prove it when $n = 2$. The proof in the general case is similar. Suppose that $J : X \to X$ is a linear and dense range map such that $T_1 J = J T_1$ and $T_2 J = J T_2$. Assume that x_0 is a supercyclic vector for (T_1, T_2). Since

$$T_1^{k_1} T_2^{k_2} J x_0 = J T_1^{k_1} T_2^{k_2} x_0$$

for all nonnegative integers k_1**,** k_2, we conclude that Jx_0 is also a supercyclic vector for (T_1, T_2). Let n_1 and n_2 be nonnegative integers and replace the operator J by $T_1^{n_1}T_2^{n_2}$. So $T_1^{n_1}T_2^{n_2}x_0$ is also a supercyclic vector for the pair (T_1, T_2); hence, the set of supercyclic vectors for the pair (T_1, T_2) is dense in X.

(2) Now, if U and V are two nonempty open subsets of X then there is a supercyclic vector x_0 for the pair (T_1, T_2) in U. So there is a nonzero scalar λ and nonnegative integers k_1 and k_2 such that $\lambda T_1^{k_1}T_2^{k_2}x_0 \in V$ which implies that $\lambda T_1^{k_1}T_2^{k_2}(U)\cap V$ is a nonempty set, therefore, $\frac{1}{\lambda}T_1^{-k_1}T_2^{-k_2}(V)\cap U$ is nonempty. Hence, Proposition 1 shows that the pair (T_1^{-1}, T_2^{-1}) is also supercyclic.

(3) Let x_0 be a supercyclic vector for T and $x \in X$. Since

$$\|\lambda T_1^{k_1}T_2^{k_2}...T_n^{k_n}x_0 - x + \mathcal{M}\| \leq \|\lambda T_1^{k_1}T_2^{k_2}...T_n^{k_n}x_0 - x\|$$

for every $\lambda \in \mathbb{C}$ and $k_i \geq 0, i = 1, 2, ..., n$, we conclude that $x_0 + \mathcal{M}$ is a supercyclic vector for the quotient of T on $X/\mathcal{M}$. □

The proof of the following Proposition is similar to the proof of supercyclicity criterion for a single operator due to H. N. Salas [9]. Also, this proposition and its corollary follow from the universality criterion of Bés and Peris [2].

Proposition 3. *(Supercyclicity Criterion) Suppose that* $(T_1, ..., T_n)$ *is an* n*-tuple of operators on the space* X*. Suppose further that there exist strictly increasing sequences of positive integers* $\{ki_j\}_j$*,* $i = 1, ..., n$ *, dense sets* Z *and* Y *in* X *and functions* $S_j : Y \longrightarrow X$ *such that*
(a) $\|T_1^{k1_j}...T_n^{kn_j}z\|\|S_jy\| \to 0$ *as* $j \to \infty$ *for any* $z \in Z$ *and* $y \in Y$;
(b) $T_1^{k1_j}...T_n^{kn_j}S_jy \to y$ *as* $j \to \infty$ *for each* $y \in Y$*.*
Then the pair $(T_1, ..., T_n)$ *is supercyclic.*

Corollary 1. *If* (T_1, T_2) *satisfies the supercyclicity criterion, then so is* $(T_1 \oplus T_1, T_2 \oplus T_2)$*, hence it is a supercyclic pair.*

Proposition 4. *Let* A *and* B *be, respectively, supercyclic and hypercyclic operators and let* C *be a dense range operator that commutes with* B*. If* $T_1 = A \oplus C$ *and* $T_2 = I \oplus B$ *then the pair* (T_1, T_2) *is supercyclic.*

Proof. Let x be a supercyclic vector for A and y be a hypercyclic vector for B . We claim that $x \oplus y$ is a supercyclic vector for the pair (T_1, T_2). To prove this, let U and V be two nonempty open sets.

There are $n \geq 0$ and a nonzero scalar λ such that $\lambda A^n x \in U$. Moreover, since C^n has a dense range, there exists z in X such that $C^n z \in V$ which implies that $z \in (C^n)^{-1}(V)$. But $\{\lambda B^k y : k \geq 0\}$ is dense in X, and so $\{\lambda B^k y : k \geq 0\} \cap (C^n)^{-1}(V) \neq \emptyset$ which implies the existence of $k \geq 0$ such that $\lambda B^k y \in (C^n)^{-1}(V)$. Therefore, $\lambda C^n B^k y \in V$. Thus, $\lambda T_1^n T_2^k (x \oplus y) = \lambda(A^n x \oplus C^n B^k y) \in U \times V$. Hence, by Proposition 1 the pair (T_1, T_2) is supercyclic. □

Corollary 2. *Take C to be the identity operator in Proposition* 4. *Then T_1 and T_2 are not supercyclic while (T_1, T_2) is.*

Recall that an operator S on a Hilbert space $\mathcal{H}$ is subnormal if there is a Hilbert space $\mathcal{K}$ containing $\mathcal{H}$ and a normal operator N on $\mathcal{K}$ such that $N\mathcal{H} \subseteq \mathcal{H}$ and $S = N|_{\mathcal{H}}$.

Proposition 5. *There is not any supercyclic n-tuple of subnormal operators with commuting normal extensions on a Hilbert space of infinite dimensional.*

Proof. Suppose that $\mathcal{H}$ is an infinite dimensional Hilbert space and $(A_1, A_2, ..., A_n)$ is a supercyclic subnormal n-tuple on $\mathcal{H}$. Let θ be an irrational multiple of π and a and b be two relatively prime integers greater than 1. Moreover, suppose that I is the identity operator. By Corollary 4.2 of [6] the set $\{\frac{a^{k_1}e^{ik_2\theta}}{b^{k_3}} : k_1, k_2, k_3 \geq 0\}$ is dense in $\mathbb{C}$. Thus, the supercyclicity of n-tuple $(A_1, A_2, ..., A_n)$ implies the hypercyclicity of $(n+3)$-tuple $(aI, \frac{1}{b}I, e^{i\theta}I, A_1, A_2, ..., A_n)$; but this tuple is a subnormal tuple and by Corollary 3.9 of [6] cannot be hypercyclic. □

We will see from Theorem 1 of Section 4 that there is an n-tuple of supercyclic normal operators on $\mathbb{C}^n$. As a direct consequence of the preceding proposition, we see that this result is not true on infinite dimensional Hilbert spaces.

Corollary 3. *If $n \geq 1$, then there is not any supercyclic normal n-tuple on an infinite dimensional Hilbert space $\mathcal{H}$.*

3. Spectral properties

Let $T = (T_1, ..., T_n)$ be an n-tuple of bounded operators on a Banach space X. The Harte spectrum of T is denoted by $\sigma(T)$; we note that $\lambda = (\lambda_1, ..., \lambda_n) \notin \sigma(T)$ if and only if there exist bounded operators $A_1, ..., A_n, B_1, ..., B_n$ on X such that

$$\sum_{i=1}^{n}(T_i - \lambda_i)A_i = \sum_{i=1}^{n} B_i(T_i - \lambda_i) = I;$$

the spectral radius of T is

$$r_2(T) = \max\{\|\lambda\|_2 : \lambda \in \sigma(T)\},$$

where $\|\lambda\|_2 = \left(\sum_{i=1}^{n} |\lambda_i|^2\right)^{\frac{1}{2}}$. Also, let

$$r_\infty(T) = \max\{\|\lambda\|_\infty : \lambda \in \sigma(T)\}$$

where

$$\|\lambda\|_\infty = \|(\lambda_1, ..., \lambda_n)\|_\infty = \max\{|\lambda_j| : 1 \leq j \leq n\}.$$

Denote the set of all nonnegative integers by $\mathbb{Z}_+$. For $\alpha = (\alpha_1, ..., \alpha_n) \in \mathbb{Z}_+^n$, define $|\alpha| = \alpha_1 + ... + \alpha_n$, $\alpha! = \alpha_1!...\alpha_n!$ and $T^\alpha = T_1^{\alpha_1}...T_n^{\alpha_n}$. If k is an integer, $k \geq |\alpha|$ then denote

$$\begin{pmatrix} k \\ \alpha \end{pmatrix} = \frac{k!}{\alpha!(k-|\alpha|)!}.$$

Remember that the unit polydisc in $\mathbb{C}^n$ is the set

$$D^n = \{(z_1, ..., z_n) : |z_j| < 1 \quad for \quad j = 1, ..., n\}$$

and the unit ball in $\mathbb{C}^n$ is the set

$$B_n = \{(z_1, ..., z_n) : \sum_{j=1}^{n} |z_j|^2 < 1\}.$$

Proposition 6. *Suppose that $T = (T_1, ..., T_n)$ is a hypercyclic n-tuple on a Banach space X, with nonempty Harte spectrum. Then $r_\infty(T) \geq 1$. Consequently, $\sigma(T) \cap (\mathbb{C}^n \setminus D^n)$ is nonempty. Furthermore, when all T_i are invertible $(1 \leq i \leq n)$, we have*

$$\min\{|\lambda_i| : 1 \leq i \leq n, \quad \lambda = (\lambda_1, ..., \lambda_n) \in \sigma(T)\} \leq 1.$$

Proof. It is well-known [12] that

$$r_\infty(T) = \lim_{k\to\infty} \|T^k\|_\infty^{\frac{1}{k}},$$

where $\|T^k\|_\infty = \max\{\|T_1^{k_1}...T_n^{k_n}\| : k_1+...+k_n = k\}$. It follows that $\{\|S\| : S \in \mathcal{F}_T\}$ is bounded if $r_\infty(T) < 1$. Thus $r_\infty(T) \geq 1$ if T is hypercyclic.

Furthermore, applying the above display to $T^{-1} = (T_1^{-1}, ..., T_n^{-1})$, we get

$$\min\{|\lambda_i| : 1 \leq i \leq n, \quad \lambda = (\lambda_1, ..., \lambda_n) \in \sigma(T)\} \leq 1.$$

□

It is known that the spectrum of every hypercyclic single operator meets $\overline{D^1}$. This fact does not hold for a tuple of operators as we are going to see in the following example.

Example 1. *Let T_1 be an invertible hypercyclic operator and for any $n \geq 2$ put*

$$T = (T_1, T_2, ..., T_{n-1}, 2I).$$

Since

$$\sigma(T) \subseteq \prod_{i=1}^{n-1} \sigma(T_i) \times \sigma(2I) = \{(z_1, z_2, ..., z_{n-1}, 2) : z_i \in \sigma(T_i), i = 1, ..., n-1\}$$

we see that $\sigma(T) \subseteq \mathbb{C}^n \setminus \overline{D^n}$. Moreover, the hypercyclicity of T_1 implies the hypercyclicity of T.

Remark 2. *Since $B_n \subseteq D^n$ and $r_2(T) \geq r_\infty(T)$, the preceding proposition and the above example are valid when we substitute D^n by B_n and $r_\infty(T)$ by $r_2(T)$.*

Recall that if there exists a nonzero vector x in X such that $(T_i - \lambda_i)x = 0$ for every $i = 1, 2, ..., n$, then $\lambda = (\lambda_1, ..., \lambda_n) \in \mathbb{C}^n$ is called a joint eigenvalue of T; furthermore, the joint point spectrum of T, denoted by $\sigma_p(T)$, is the set of all joint eigenvalues of T (see [3] and [4]). Now, we define the equal joint point spectrum of T, denoted by $\sigma_{ep}(T)$, to be a subset of $\sigma_p(T)$ such that $\lambda_1 = \lambda_2 = ... = \lambda_n$; i.e., the set all points $(\lambda, ..., \lambda) \in \mathbb{C}^n$ where there exists a nonzero vector x such that

$$(T_i - \lambda)x = 0 \quad for \quad every \quad i = 1, 2, ..., n.$$

Proposition 7. *If $T = (T_1, ..., T_n)$ is hypercyclic then $\sigma_{ep}(T^*) = \emptyset$.*

Proof. Assume that $(\lambda, ..., \lambda) \in \sigma_{ep}(T^*)$. So there exists a nonzero $x^* \in X^*$ such that $(T_i^* - \lambda)x^* = 0$ for $i = 1, 2, ..., n$. If $x \in X$ then

$$\langle x^*, T_1^{k_1} T_2^{k_2} ... T_n^{k_n} x\rangle = \langle (T_n^*)^{k_n} ... (T_1^*)^{k_1} x^*, x\rangle = \lambda^{k_1+...+k_n} \langle x^*, x\rangle$$

for all $k_1, ..., k_n \geq 0$.

Suppose that x is a hypercyclic vector for T. Since the linear map $x^* : X \to \mathbb{C}$ is continuous and onto, it maps the dense set $orb(T, x)$ onto a dense subset of $\mathbb{C}$. But the set $\{\lambda^{k_1+...+k_n} \langle x^*, x\rangle : k_i \geq 0, i = 1, 2, ..., n\}$ is not dense in $\mathbb{C}$. This is a contradiction. □

Lemma 1. *(Lemma 1.27 of [1]) Let $a, b, \lambda, \mu \in \mathbb{C}$. The set $\mathbb{C}.\{(a\lambda^n, b\mu^n) : n \in \mathbb{N}\}$ is not dense in $\mathbb{C}^2$.*

Proposition 8. *If $T = (T_1, T_2)$ is supercyclic then either $\sigma_{ep}(T^*) = \emptyset$ or $\sigma_{ep}(T^*) = \{(\lambda, \lambda)\}$ for some $\lambda \neq 0$.*

Proof. Suppose that there exists a vector x in X such that $\mathbb{C}.orb(T, x)$ is dense in X. If (λ, λ) and (μ, μ) are in $\sigma_{ep}(T^*)$ then there exist two nonzero vectors x^* and y^* in X^* such that $T_i^* x^* = \lambda x^*$ and $T_i^* y^* = \mu y^*$ for $i = 1, 2$. Assume that $\lambda \neq \mu$. Then the vectors x^* and y^* are linearly independent and it follows that the continuous linear map $L_{x^*,y^*} : X \to \mathbb{C}^2$ defined by $Ł_{x^*,y^*}(z) = (\langle x^*, z\rangle, \langle y^*, z\rangle)$ is onto. Putting $a = \langle x^*, x\rangle$ and $b = \langle y^*, x\rangle$, we observe that

$$\begin{aligned} L_{x^*,y^*}(\mathbb{C}.orb(T, x)) &= \mathbb{C}.\{(\langle x^*, T_1^n T_2^k x\rangle, \langle y^*, T_1^n T_2^k x\rangle) : k, n \geq 0\} \\ &= \mathbb{C}.\{(\langle (T_2^*)^k (T_1^*)^n x^*, x\rangle, \langle (T_2^*)^k (T_1^*)^n y^*, x\rangle) : k, n \geq 0\} \\ &= \mathbb{C}.\{(a\lambda^{n+k}, b\mu^{n+k}) : k, n \geq 0\} \end{aligned}$$

is dense in $\mathbb{C}^2$ which is absurd by Lemma 1. Therefore, $\sigma_{ep}(T^*)$ has at most one element and the corresponding eigenspace has dimension 1. Moreover, if $(0, 0) \in \sigma_{ep}(T^*)$ then there is a nonzero vector x^* in X^* such that $T_1^* x^* = T_2^* x^* = 0$. Hence, $x^*(\mathbb{C}.orb(T, x)) = \{0\}$ is dense in $\mathbb{C}$ which is a contradiction. □

4. Supercyclic matrix tuples

Let $L(\mathbb{C}^n)$ be the $\mathbb{C}$-algebra of $n \times n$ matrices with entries from $\mathbb{C}$. Recall that a subalgebra $\mathcal{A}$ of $L(\mathbb{C}^n)$ is called cyclic if there is $x \in \mathbb{C}^n$ such that $\{Sx : S \in \mathcal{A}\} = \mathbb{C}^n$. Also, we recall that a character on a complex algebra $\mathcal{A}$ is a nonzero algebra homomorphism from $\mathcal{A}$ to $\mathbb{C}$. The set of characters is always linearly independent in the space of linear functionals on $\mathcal{A}$. Moreover, the set of characters on a unital commutative Banach algebra is always nonempty. Hence an n-dimensional unital commutative complex algebra has at least 1 and at most n characters ([10]).

It is shown in ([6], Theorem 3.6) that there is no hypercyclic n-tuple of diagonalizable matrices on $\mathbb{C}^n$. In the following we prove that this result does not hold when hypercyclicity is replaced by supercyclicity. Note that there are not single supercyclic operators on finite dimensional spaces with dimension more than one.

Theorem 1. *If T is a supercyclic ℓ-tuple of $n \times n$ complex matrices, then $\ell \geq n$. There exists a supercyclic n-tuple of diagonal $n \times n$ matrices. Furthermore, if $T = (T_1, ..., T_n)$ is a supercyclic n-tuple of $n \times n$ complex matrices, then T_j's are simultaneously diagonalizable.*

Proof. First we prove that for each $n \geq 1$, there exists a supercyclic n-tuple of diagonal matrices on $\mathbb{C}^n$. The result obviously holds when $n = 1$; thus we may assume that $n \geq 2$. By Proposition 3.4 of [6] there exists a hypercyclic n-tuple $(A_1, A_2, ..., A_n)$ of diagonal matrices on $\mathbb{C}^{n-1}$. Assume that

$$A_i = \begin{bmatrix} a_{1i} & 0 & \cdots & 0 \\ 0 & a_{2i} & \cdots & 0 \\ & & \ddots & \\ 0 & 0 & \cdots & a_{(n-1)i} \end{bmatrix}$$

for $i = 1, 2, ..., n$ and suppose that $u = \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_{n-1} \end{bmatrix}$ is the corresponding hypercyclic vector. Then

$$\left\{ \begin{bmatrix} a_{11}^{k_1} a_{12}^{k_2} \cdots a_{1n}^{k_n} \alpha_1 \\ a_{21}^{k_1} a_{22}^{k_2} \cdots a_{2n}^{k_n} \alpha_2 \\ \vdots \\ a_{(n-1)1}^{k_1} a_{(n-2)2}^{k_2} \cdots a_{(n-1)n}^{k_n} \alpha_{n-1} \end{bmatrix} : k_i \geq 0, i = 1, 2, ..., n \right\}$$

is dense in $\mathbb{C}^{n-1}$. Since α_i is nonzero for every $1 \leq i \leq n-1$, by applying the invertible matrix

$$\begin{bmatrix} \alpha_1^{-1} & 0 & \cdots & 0 \\ 0 & \alpha_2^{-1} & \cdots & 0 \\ & & \ddots & \\ 0 & 0 & \cdots & \alpha_{n-1}^{-1} \end{bmatrix}$$

on the above dense set, we conclude that the set

$$F = \left\{ \begin{bmatrix} a_{11}^{k_1} a_{12}^{k_2} \cdots a_{1n}^{k_n} \\ a_{21}^{k_1} a_{22}^{k_2} \cdots a_{2n}^{k_n} \\ \vdots \\ a_{(n-1)1}^{k_1} a_{(n-1)2}^{k_2} \cdots a_{(n-1)n}^{k_n} \end{bmatrix} : k_i \geq 0, i = 1, 2, ..., n \right\}$$

is also dense in $\mathbb{C}^{n-1}$. Now, consider the n-tuple $(B_1, B_2, ..., B_n)$ of diagonal matrices on $\mathbb{C}^n$ as follows:

$$B_i = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & a_{1i} & 0 & \cdots & 0 \\ 0 & 0 & a_{2i} & \cdots & 0 \\ & & & \ddots & \\ 0 & 0 & 0 & \cdots & a_{(n-1)i} \end{bmatrix}$$

for $i = 1, 2, ..., n$. If $\nu = \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}$ then the density of F in $\mathbb{C}^{n-1}$ implies that

$$\{\lambda B_1^{k_1} B_2^{k_2} ... B_n^{k_n} \nu : \lambda \in \mathbb{C}, k_i \geq 0, i = 1, 2, ..., n\}$$

is dense in

$$E = \mathbb{C}^n - \{(0, \lambda_1, \lambda_2, ..., \lambda_{n-1}) : \lambda_i \in \mathbb{C}, i = 1, 2, ..., n-1\}.$$

But E is itself dense in $\mathbb{C}^n$, so we conclude that the vector ν is a supercyclic vector for the n-tuple $(B_1, B_2, ..., B_n)$.

Now, to prove the rest of the theorem, first note that by Remark 1 if T is an ℓ-tuple of $n \times n$ complex matrices and T' is obtained from T by kicking out all non-invertible matrices, then either both T and T' are supercyclic or both are non-supercyclic. This allows us to consider only tuples of invertible matrices.

So let $\mathcal{A} = \mathcal{A}_T$ be the (unital) algebra generated by invertible matrices T_j, $j = 1, ..., n$. Clearly, $\mathcal{A}$ is commutative. If T is supercyclic, then $\mathcal{A}$ is cyclic and so by Lemma 1.1 of [10], $dim\mathcal{A} = n$ and there is $x \in \mathbb{C}^n$ such that the mapping $S \mapsto Sx$ from $\mathcal{A}$ to $\mathbb{C}^n$ is a linear isomorphism. It follows that T is supercyclic if and only if $\mathcal{F}_T^p$ is dense in $\mathcal{A}$. Since invertible elements of a finite dimensional commutative Banach algebra $\mathcal{A}$ form a dense open subset, called $\mathcal{A}^\star$, the latter statement is equivalent to the density of $G_T = \mathcal{F}_T^p \setminus \{0\}$ in $\mathcal{A}^\star$. This density is the density of $\mathcal{F}_T$ (strictly speaking, of the set of corresponding cosets) in the factorgroup $\mathcal{A}^\star/\mathbb{C}^{'}I$ where $\mathbb{C}^{'} = \mathbb{C} \setminus \{0\}$. Also, the minimal cardinality of a supercyclic tuple of $n \times n$ complex matrices is exactly the minimal number of elements of the topological group $\mathcal{A}^\star/\mathbb{C}^{'}I$ generating a dense subsemigroup in $\mathcal{A}^\star/\mathbb{C}^{'}I$, where $\mathcal{A}$ is a unital cyclic commutative subalgebra of the algebra of $n \times n$ complex matrices. For the detailed description of this approach see [10].

On the other hand the exponential map, $\exp_{\mathcal{A}}(S) = e^S$ is a homomorphism from $(\mathcal{A}, +)$ onto $(\mathcal{A}^\star, \cdot)$, whose kernel is a subgroup of $(\mathcal{A}, +)$ generated by k linearly independent elements, where k is the number of characters of $\mathcal{A}$ (see Lemma 3.2 of [10]). Therefore, $\mathcal{A}^\star$ is isomorphic to $\mathbb{C}^n/\mathbb{Z}^k$ which is a topological group, isomorphic to the additive group $\mathbb{T}^k \times \mathbb{R}^{2n-k}$ where $\mathbb{T} = \mathbb{R}/\mathbb{Z}$. In the next step consider the exponential map $\exp_{\mathbb{C}}$ from $(\mathbb{C}, +)$ to $(\mathbb{C}^{'}, \cdot)$. So $\mathbb{C}^{'}$ is isomorphic to $\mathbb{C}/\mathbb{Z}$ which implies that $\mathcal{A}^\star/\mathbb{C}^{'}I$ is isomorphic to the additive group $\mathbb{T}^{k-1} \times \mathbb{R}^{2n-k-1}$. But by Lemma 2.1 of [10] the minimal m for which $\mathbb{T}^{k-1} \times \mathbb{R}^{2n-k-1}$ has an m-element subset generating a dense subsemigroup is $2n - k$. Now, the number of characters on $\mathcal{A}$ is always between 1 and n, thus $2n - k \geq n$, hence $\ell \geq n$. Also $k = n$ precisely when $\mathcal{A}$ can be obtained from the subalgebra of diagonal matrices by conjugation.

□

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