Submitted on December 27, 2012 to . . ., pp. 1–20

Reproducing kernel Hilbert space method for solving Bratu's problem

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Abstract

In this paper, we use the reproducing kernel Hilbert space method (RKHSM) for solving a boundary value problem for the second order Bratu's differential equation. Convergence analysis of presented method is discussed. The numerical approximations to the exact solution are computed and compared with other existing methods. Our presented method produces more accurate results in comparison with those obtained by Adomian decomposition, Laplace decomposition, B-spline, Non-polynomial spline and Lie-group shooting methods. Our yardstick is absolute error. The comparison of the results with exact ones is made to confirm the validity and efficiency.

AMS Subject Classifications: 30E25, 34B15, 47B32, 46E22 and 74S30.
Keywords: Reproducing kernel method, series solutions, Bratu's problem, reproducing kernel space.

1 Introduction

We consider the classical Bratu's problem [1–3, 6–8, 12, 21, 24, 26, 31–33, 36]:

$$\begin{cases} u''(x) + \lambda \exp(u(x)) = 0, & 0 \leq x \leq 1, \\ u(0) = u(1) = 0. \end{cases} \tag{1.1}$$

The standard Bratu's problem (1.1) was used to model a combustion problem in a numerical slab [36].

The BVP (1.1) has the analytic solution given by (1.2),

$$u(x) = -2\ln\left[\frac{\cosh\left(\left(x-\frac{1}{2}\right)\frac{\theta}{2}\right)}{\cosh\left(\frac{\theta}{4}\right)}\right] \tag{1.2}$$

where θ is the solution of $\theta = \sqrt{2\lambda}\cosh(\theta/4)$ [36]. The Bratu's problem has no solution, one or two solutions when $\lambda > \lambda_c$, $\lambda = \lambda_c$ and $\lambda < \lambda_c$ respectively, where the critical value λ_c is given by $\lambda_c = 3,513830719$. Many authors have studied Bratu's problem (1.1) by analytical and numerical methods. Wazwaz [36] used the Adomian decomposition method for solving Bratu-type equations. Aregbesola [3] used the method of weighted residuals to show the existence and multiplicity of solutions to the Bratu's problem. Deeba et al. [12] used the Adomian decomposition method (ADM) for Troesch's and Bratu's problems. Khuri [24] introduced Laplace decomposition method (LDM) for solving Bratu's equation. Li and Liao [26] used the homotopy analysis method for solving strongly nonlinear problems. Syam and Hamdan [32] presented the Laplace Adomain decomposition method which produces an implicit equation in two variables for solving problem (1.1). Caglar et al. [8] have used the B-spline method for solving Bratu's problem and Jalilian [21] developed smooth approximate solutions of the one-dimensional Bratu's problem by using non-polynomial spline function. There are some other papers about this problem [1–3, 6–8, 12, 21, 24, 26, 31–33, 36].

In this paper, the RKHSM [4, 5, 9–11, 13–20, 22, 23, 25, 27–30, 34, 35, 37–44] will be used to investigate the Bratu's problem (1.1). The theory of reproducing kernels [4], was used for the first time at the beginning of the 20th century by S. Zaremba in his work on boundary value problems for harmonic and biharmonic functions. In recent years, a lot of attention has been devoted to the study of RKHSM to investigate various scientific models. The RKHSM which accurately computes the series solution is of great interest to applied sciences. The method provides the solution in a rapidly convergent series with components that can be elegantly computed. The book [11] provides excellent overviews of the existing reproducing kernel methods for solving various model problems such as integral and integro-differential equations.

The efficiency of the method was used by many authors to investigate several scientific applications. Geng and Cui [17] applied the RKHSM to handle the second-order boundary value problems. Yao and Cui [42] and Wang et al. [34] investigated a class of singular boundary value problems by this method and the obtained results were good. Zhou et al. [44] used the RKHSM effectively to solve second-order boundary value problems. In [28], the method was used to solve nonlinear infinite-delay-differential equations. Wang and Chao [35], Li and Cui [25], Zhou and Cui [43] independently employed the RKHSM to variable-coefficient partial differential equations. Geng and Cui [19], Du and Cui [15] investigated the approximate solution of the forced Duffing equation with integral boundary conditions by combining the homotopy perturbation method and the RKHSM. Lv and Cui [29] presented a new algorithm to solve linear fifth-order boundary value problems. Cui and Du [22] obtained the representation of the exact solution for the nonlinear Volterra-Fredholm integral equations by using the reproducing kernel Hilbert space method. Wu and Li [39] applied iterative reproducing kernel method to obtain the analytical approximate solution of a nonlinear oscillator with discontinuities. Recently, the method was applied to the fractional partial differential equations and multi-point boundary value problems [10, 20, 23, 39]. Yang at al. [40] used this method for solving the system of the linear Volterra integral equations with variable coefficients. A particular singular integral equation was solved by [13]. Barbieri and Meo [5] have studied evaluation of the integral terms in reproducing kernel methods. Third order three-point boundary value problems were considered by [38]. Chen and Chen [9] investigated the exact solution of system of linear operator equations in reproducing kernel spaces. For more details about RKHSM and the modified forms and its effectiveness, see [4, 5, 9–11, 13–20, 22, 23, 25, 27–30, 34, 35, 37–44] and the references therein.

The paper is organized as follows. Section 2 is devoted to several reproducing kernel spaces. The associated linear operator and the solution are presented in Section 3. Section 4 provides the main results, the exact and approximate solution of Bratu's problem (1.1) and an iterative method are developed in the reproducing kernel space. We have proved that the approximate solution converges to the exact solution uniformly. Some numerical experiments are illustrated in Section 5. Some conclusions are given in Section 6.

2 Preliminaries

2.1 Reproducing kernel spaces

In this section, we define some useful reproducing kernel spaces.

Definition 2.1 (Reproducing kernel function)**.** Let $E \neq \emptyset$. A function $K : E \times E \to \mathbb{C}$ is called a *reproducing kernel function* of the Hilbert space H if and only if

a) $K(\cdot, t) \in H$ for all $t \in E$,

b) $\langle \varphi, K(\cdot, t) \rangle = \varphi(t)$ for all $t \in E$ and all $\varphi \in H$.

The last condition is called "the reproducing property" as the value of the function φ at the point t is reproduced by the inner product of φ with $K(\cdot, t)$.

Definition 2.2. We define the space $W_2^3[0,1]$ by

$$W_2^3[0,1] = \{u \in AC[0,1]:\ u', u'' \in AC[0,1],\ u^{(3)} \in L^2[0,1],\ u(0) = u(1) = 0\}.$$

The third derivative of u exists almost everywhere since u'' is absolutely continuous. The inner product and the norm in $W_2^3[0,1]$ are defined by

$$\langle u, g\rangle_{W_2^3} = u(0)g(0) + u'(0)g'(0) + u(1)g(1) + \int_0^1 u^{(3)}(x)g^{(3)}(x)\mathrm{d}x, \quad u, g \in W_2^3[0,1]$$

and

$$\|u\|_{W_2^3} = \sqrt{\langle u, u\rangle_{W_2^3}}, \quad u \in W_2^3[0,1].$$

The space $W_2^3[0,1]$ is called a reproducing kernel space, as for each fixed $y \in [0,1]$ and any $u \in W_2^3[0,1]$, there exists a function R_y such that

$$u(y) = \langle u, R_y\rangle_{W_2^3}.$$

Definition 2.3. We define the space $W_2^1[0,1]$ by

$$W_2^1[0,1] = \{u \in AC[0,1]:\ u' \in L^2[0,1]\}.$$

The inner product and the norm in $W_2^1[0,1]$ are defined by

$$\langle u, g\rangle_{W_2^1} = \int_0^1 u(x)g(x) + u'(x)g'(x)\mathrm{d}x, \quad u, g \in G_2^1[0,1] \tag{2.1}$$

and

$$\|u\|_{W_2^1} = \sqrt{\langle u, u\rangle_{W_2^1}}, \quad u \in W_2^1[0,1]. \tag{2.2}$$

The space $W_2^1[0,1]$ is a reproducing kernel space, and its reproducing kernel function $T_x(y)$ is given by [11]

$$T_x(y) = \frac{1}{2\sinh(1)}\left[\cosh(x+y-1) + \cosh(|x-y|-1)\right] \tag{2.3}$$

Theorem 2.4. *The space $W_2^3[0,1]$ is a reproducing kernel space, and its reproducing kernel function R_y is given by*

$$R_y(x) = \begin{cases} \displaystyle\sum_{i=1}^{6} c_i(y)x^{i-1}, & x \le y, \\ \displaystyle\sum_{i=1}^{6} d_i(y)x^{i-1}, & x > y, \end{cases} \tag{2.4}$$

where

$$
\begin{aligned}
c_1(y) &= 0,\\
c_2(y) &= -y^2+y,\\
c_3(y) &= \frac{-1}{120}y^5+\frac{21}{20}y^2+\frac{1}{24}y^4-y-\frac{1}{12}y^3,\\
c_4(y) &= 0,\\
c_5(y) &= \frac{1}{24}y^2-\frac{1}{24}y,\\
c_6(y) &= -\frac{1}{120}y^2+\frac{1}{120},\\
d_1(y) &= \frac{1}{120}y^5,\\
d_2(y) &= -\frac{1}{24}y^4+y-y^2,\\
d_3(y) &= -\frac{1}{120}y^5+\frac{21}{20}y^2+\frac{1}{24}y^4-y\\
d_4(y) &= -\frac{1}{12}y^2,\\
d_5(y) &= \frac{1}{24}y^2,\\
d_6(y) &= -\frac{1}{120}y^2.
\end{aligned}
$$

Proof.

$$\langle u(x), R_y(x)\rangle_{W_2^3} = u(0)R_y(0)+u'(0)R_y'(0)+u(1)R_y(1)+\int_0^1 u^{(3)}(x)R_y^{(3)}(x)\mathrm{d}x, \tag{2.5}$$

Integrating this equation by parts for three times, we have

$$
\begin{aligned}
\langle u, R_y\rangle_{W_2^3} =& u(0)R_y(0)+u'(0)R_y'(0)+u(1)R_y(1)\\
&+u''(0)R_y''(0)+u''(1)R_y^{(3)}(1)-u''(0)R_y^{(3)}(0)\\
&-u'(1)R_y^{(4)}(1)+u'(0)R_y^{(4)}(0)+u(1)R_y^{(5)}(1)\\
&-u(0)R_y^{(5)}(0)-\int_0^1 u(x)R_y^{(6)}(x)\mathrm{d}x.
\end{aligned} \tag{2.6}
$$

Note that property of the reproducing kernel is

$$\langle u(x), R_y(x)\rangle_{W_2^3} = u(y) \tag{2.7}$$

If

$$
\begin{cases}
R_y^{(3)}(0)=0,\\
R_y'(0)+R_y^{(4)}(0)=0,\\
R_y^{(3)}(1)=0,\\
R_y^{(4)}(1)=0.
\end{cases} \tag{2.8}
$$

then (2.6) implies that

$$-R_y^{(6)}(x) = \delta(x-y).$$

When $x \neq y$,

$$R_y^{(6)}(x) = 0,$$

therefore

$$R_y(x) = \begin{cases} \sum_{i=1}^{6} c_i(y)x^{i-1}, & x \leq y, \\ \\ \sum_{i=1}^{6} d_i(y)x^{i-1}, & x > y, \end{cases} \tag{2.9}$$

Since

$$-R_y^{(6)}(x) = \delta(x-y),$$

we have

$$\partial^k R_{y^+}(y) = \partial^k R_{y^-}(y), \quad k = 0, 1, 2, 3, 4 \tag{2.10}$$

and

$$\partial^5 R_{y^+}(y) - \partial^5 R_{y^-}(y) = -1. \tag{2.11}$$

Due to $R_y(x) \in W_2^3[0,1]$, it follows that

$$R_y(0) = R_y(1) = 0, \tag{2.12}$$

from (2.8)–(2.12), the unknown coefficients $c_i(y)$ and $d_i(y)$ $(i = 1, 2, \ldots, 6)$ can be obtained. Thus $R_y(x)$ is given by

$$R_y(x) = \begin{cases} -xy^2 + xy - \frac{1}{120}x^2y^5 + \frac{21}{20}x^2y^2 + \frac{1}{24}x^2y^4y - \frac{1}{12}x^2y^3 \\ \frac{1}{24}x^4y^2 - \frac{1}{24}x^4y - \frac{1}{120}x^5y^2 + \frac{x^5}{120}, \quad x \leq y \\ \\ \frac{1}{120}y^5 - \frac{1}{24}xy^4 + xy - xy^2 - \frac{1}{120}x^2y^5 + \frac{21}{20}x^2y^2 + \frac{1}{24}x^2y^4 \\ -x^2y - \frac{1}{12}x^3y^2 + \frac{1}{24}x^4y^2 - \frac{1}{120}x^5y^2, \quad x > y \end{cases}$$

□

3 Solution representation in $W_2^3[0,1]$

In this section, the solution of equation (1.1) is considered in the reproducing kernel space $W_2^3[0,1]$. On defining the linear operator $L : W_2^3[0,1] \to W_2^1[0,1]$ as

$$Lu(x) = u''(x) \tag{3.1}$$

model problem (1.1) takes the form:

$$\begin{cases} Lu = f(x,u), \quad x \in [0,1], \\ u(0) = u(1) = 0 \end{cases} \tag{3.2}$$

where $f(x,u) = -\lambda \exp(u(x))$.

In Eq. (3.1) since $u(x)$ is sufficiently smooth we see that $L : W_2^3[0,1] \to W_2^1[0,1]$ is a bounded linear operator.

Theorem 3.1. *The linear operator L defined by* (3.1) *is a bounded linear operator.*

Proof. We only need to prove $\|Lu\|_{W_2^1}^2 \le M \|u\|_{W_2^3}^2$, where $M > 0$ is a positive constant. By (2.1) and (2.2), we have

$$\|Lu\|_{W_2^1}^2 = \langle Lu, Lu \rangle_{W_2^1} = \int_0^1 [Lu(x)]^2 + [Lu'(x)]^2 \, \mathrm{d}x.$$

By (2.7), we have

$$u(x) = \langle u(\cdot), R_x(\cdot) \rangle_{W_2^3},$$

and

$$Lu(x) = \langle u(\cdot), LR_x(\cdot) \rangle_{W_2^3},$$

so

$$|Lu(x)| \le \|u\|_{W_2^3} \|LR_x\|_{W_2^3} = M_1 \|u\|_{W_2^3},$$

where $M_1 > 0$ is a positive constant, thus

$$\int_0^1 [(Lu)(x)]^2 \, \mathrm{d}x \le M_1^2 \|u\|_{W_2^3}^2.$$

Since

$$(Lu)'(x) = \langle u(\cdot), (LR_x)'(\cdot) \rangle_{W_2^3},$$

then

$$|(Lu)'(x)| \le \|u\|_{W_2^3} \|(LR_x)'\|_{W_2^3} = M_2 \|u\|_{W_2^3},$$

where $M_2 > 0$ is a positive constant so, we have

$$[(Lu)'(t)]^2 \le M_2^2 \|u\|_{W_2^3}^2,$$

and

$$\int_0^1 \left[(Lu)'(x)\right]^2 \mathrm{d}x \leq M_2^2 \left\|u\right\|_{W_2^3}^2 ,$$

that is

$$\left\|Lu\right\|_{W_2^1}^2 \leq \int_0^1 \left(\left[(Lu)\,(x)\right]^2 + \left[(Lu)'(x)\right]^2\right) \mathrm{d}x \leq \left(M_1^2 + M_2^2\right) \left\|u\right\|_{W_2^3}^2 = M \left\|u\right\|_{W_2^3}^2 ,$$

where $M = M_1^2 + M_2^2 > 0$ is a positive constant. □

4 The structure of the solution and the main results

From (3.1) it is clear that $L : W_2^3[0,1] \to W_2^1[0,1]$ is a bounded linear operator. Put $\varphi_i(x) = T_{x_i}(x)$ and $\psi_i(x) = L^*\varphi_i(x)$, where L^* is conjugate operator of L. The orthonormal system $\left\{\widehat{\Psi}_i(x)\right\}_{i=1}^{\infty}$ of $W_2^3[0,1]$ can be derived from Gram-Schmidt orthogonalization process of $\{\psi_i(x)\}_{i=1}^{\infty}$,

$$\widehat{\psi}_i(x) = \sum_{k=1}^{i} \beta_{ik}\psi_k(x), \quad (\beta_{ii} > 0, \quad i = 1, 2, \ldots) \tag{4.1}$$

Theorem 4.1. *Let $\{x_i\}_{i=1}^{\infty}$ be dense in $[0,1]$ and $\psi_i(x) = L_y R_x(y)\big|_{y=x_i}$. Then the sequence $\{\psi_i(x)\}_{i=1}^{\infty}$ is a complete system in $W_2^3[0,1]$.*

Proof. We have

$$\psi_i(x) = (L^*\varphi_i)(x) = \langle (L^*\varphi_i)(y), R_x(y)\rangle = \langle (\varphi_i)(y), LyR_x(y)\rangle = L_y R_x(y)\big|_{y=x_i} .$$

The subscript y by the operator L indicates that the operator L applies to the function of y. Clearly, $\psi_i(x) \in W_2^3[0,1]$. For each fixed $u(x) \in W_2^3[0,1]$, let $\langle u(x), \psi_i(x)\rangle = 0$, $(i = 1, 2, \ldots)$, which means that,

$$\langle u(x), (L^*\varphi_i)(x)\rangle = \langle Lu(\cdot), \varphi_i(\cdot)\rangle = (Lu)(x_i) = 0.$$

Note that, $\{x_i\}_{i=1}^{\infty}$ is dense in $[0,1]$, hence, $(Lu)(x) = 0$. It follows that $u \equiv 0$ from the existence of L^{-1}. So the proof of Theorem (4.1) is complete. □

Theorem 4.2. *If $u(x)$ is the exact solution of (3.2), then*

$$u(x) = \sum_{i=1}^{\infty} \sum_{k=1}^{i} \beta_{ik} f(x_k, u_k) \widehat{\Psi}_i(x). \tag{4.2}$$

where $\{(x_i)\}_{i=1}^{\infty}$ is dense in $[0,1]$.

Proof. From the (4.1) and uniqueness of solution of (3.2) we have

$$\begin{aligned}
u(x) &= \sum_{i=1}^{\infty} \left\langle u(x), \widehat{\Psi}_i(x) \right\rangle_{W_2^3} \widehat{\Psi}_i(x) \\
&= \sum_{i=1}^{\infty} \sum_{k=1}^{i} \beta_{ik} \left\langle u(x), \Psi_k(x) \right\rangle_{W_2^3} \widehat{\Psi}_i(x) \\
&= \sum_{i=1}^{\infty} \sum_{k=1}^{i} \beta_{ik} \left\langle u(x), L^* \varphi_k(x) \right\rangle_{W_2^3} \widehat{\Psi}_i(x) \\
&= \sum_{i=1}^{\infty} \sum_{k=1}^{i} \beta_{ik} \left\langle Lu(x), \varphi_k(x) \right\rangle_{W_2^1} \widehat{\Psi}_i(x) \\
&= \sum_{i=1}^{\infty} \sum_{k=1}^{i} \beta_{ik} \left\langle f(x,u), T_{x_k} \right\rangle_{W_2^1} \widehat{\Psi}_i(x) \\
&= \sum_{i=1}^{\infty} \sum_{k=1}^{i} \beta_{ik} f(x_k, u_k) \widehat{\Psi}_i(x).
\end{aligned}$$

This completes the proof. □

Now the approximate solution $u_n(x)$ can be obtained from the n- term intercept of the exact solution u and

$$u_n(x) = \sum_{i=1}^{n} \sum_{k=1}^{i} \beta_{ik} f(x_k, u_k) \widehat{\Psi}_i(x). \tag{4.3}$$

Lemma 4.3. *If* $\|u_n - u\|_{W_2^3} \to 0$, $x_n \to x$, $(n \to \infty)$ *and* $f(x,u)$ *is continuous for* $x \in [0,1]$, *then [18]*

$$f(x_n, u_{n-1}(x_n)) \to f(x, u(x)) \quad as\ n \to \infty.$$

Theorem 4.4. *For any fixed* $u_0(x) \in W_2^3[0,1]$ *suppose the following conditions are satisfied:*

(i)

$$u_n(x) = \sum_{i=1}^{n} A_i \widehat{\psi}_i(x), \tag{4.4}$$

$$A_i = \sum_{k=1}^{i} \beta_{ik} f(x_k, u_{k-1}(x_k)), \tag{4.5}$$

(ii) $\|u_n\|_{W_2^3}$ *is bounded;*

(iii) $\{x_i\}_{i=1}^{\infty}$ *is dense in* $[0,1]$*;*

(iv) $f(x,u) \in W_2^1[0,1]$ *for any* $u(x) \in W_2^3[0,1]$.

Then $u_n(x)$ *in iterative formula* (4.4) *converges to the exact solution of* (4.2) *in* $W_2^3[0,1]$ *and*

$$u(x) = \sum_{i=1}^{\infty} A_i \widehat{\psi}_i(x),$$

where A_i *is given by (4.5).*

Proof. First, we will prove the convergence of $u_n(x)$. By (4.4), we have

$$u_{n+1}(x) = u_n(x) + A_{n+1}\widehat{\Psi}_{n+1}(x), \tag{4.6}$$

from the orthonormality of $\{\widehat{\Psi}_i\}_{i=1}^{\infty}$, it follows that

$$\|u_{n+1}\|^2 = \|u_n\|^2 + A_{n+1}^2 = \|u_{n-1}\|^2 + A_n^2 + A_{n+1}^2 = \ldots = \sum_{i=1}^{n+1} A_i^2, \tag{4.7}$$

from boundedness of $\|u_n\|_{W_2^3}$, we have

$$\sum_{i=1}^{\infty} A_i^2 < \infty,$$

i.e.,

$$\{A_i\} \in l^2 \quad (i = 1, 2, \ldots).$$

Let $m > n$, in view of $(u_m - u_{m-1}) \perp (u_{m-1} - u_{m-2}) \perp \ldots \perp (u_{n+1} - u_n)$, it follows that

$$\begin{aligned}
\|u_m - u_n\|_{W_2^3}^2 &= \|u_m - u_{m-1} + u_{m-1} - u_{m-2} + \ldots + u_{n+1} - u_n\|_{W_2^3}^2 \\
&\leq \|u_m - u_{m-1}\|_{W_2^3}^2 + \ldots + \|u_{n+1} - u_n\|_{W_2^3}^2 \\
&= \sum_{i=n+1}^{m} A_i^2 \to 0, \quad m, n \to \infty.
\end{aligned}$$

Considering the completeness of $W_2^3[0,1]$, there exists $u(x) \in W_2^3[0,1]$, such that

$$u_n(x) \to u(x) \quad as\ n \to \infty.$$

(ii) Second, we will prove $u(x)$ is the solution of (3.2). Taking limits in (4.4),

$$u(x) = \sum_{i=1}^{\infty} A_i \widehat{\psi}_i(x).$$

Since

$$(Lu)(x_j) = \sum_{i=1}^{\infty} A_i \left\langle L\widehat{\psi}_i(x), \varphi_j(x) \right\rangle_{W_2^1} = \sum_{i=1}^{\infty} A_i \left\langle \widehat{\psi}_i(x), L^* \varphi_j(x) \right\rangle_{W_2^3} = \sum_{i=1}^{\infty} A_i \left\langle \widehat{\psi}_i(x), \psi_j(x) \right\rangle_{W_2^3},$$

it follows that

$$\sum_{j=1}^{n} \beta_{nj} (Lu)(x_j) = \sum_{i=1}^{\infty} A_i \left\langle \widehat{\psi}_i(x), \sum_{j=1}^{n} \beta_{nj} \psi_j(x) \right\rangle_{W_2^3} = \sum_{i=1}^{\infty} A_i \left\langle \widehat{\psi}_i(x), \widehat{\psi}_n(x) \right\rangle_{W_2^3} = A_n.$$

If $n = 1$, then

$$Lu(x_1) = f(x_1, u_0(x_1)). \tag{4.8}$$

If $n = 2$, then

$$\beta_{21}(Lu)(x_1) + \beta_{22}(Lu)(x_2) = \beta_{21} f(x_1, u_0(x_1)) + \beta_{22} f(x_2, u_1(x_2)). \tag{4.9}$$

From (4.8) and (4.9), it is clear that

$$(Lu)(x_2) = f(x_2, u_1(x_2)).$$

Furthermore, it is easy to see by induction that

$$(Lu)(x_j) = f(x_j, u_{j-1}(x_j)). \tag{4.10}$$

Since $\{x_i\}_{i=1}^{\infty}$ is dense in $[0, 1]$, for any $y \in [0, 1]$, there exists subsequence $\{x_{n_j}\}$, such that $x_{n_j} \to y$, as $j \to \infty$. Hence, let $j \to \infty$ in (4.10), by the convergence of $u_n(x)$ and Lemma 4.3 we have,

$$(Lu)(y) = f(y, u(y)),$$

that is, $u(x)$ is the solution of (3.2) and

$$u(x) = \sum_{i=1}^{\infty} A_i \widehat{\psi}_i,$$

where A_i are given by (4.5). □

4.1 Convergence analysis

We assume that $\{x_i\}_{i=1}^{\infty}$ is dense in $[0,1]$. We discuss the convergence of the approximate solutions constructed in Section 4. Let $u(x)$ be the exact solution of (1.1), $u_n(x)$ be the n-term approximation solution of (1.1). We set

$$\|u\|_C = \max_{x\in[0,1]} |u(x)| .$$

Theorem 4.5. *If* $u \in W_2^3[0,1]$ *then*

$$\|u_n - u\|_{W_2^3} \to 0, \quad n \to \infty.$$

Moreover a sequence $\|u_n - u\|_{W_2^3}$ *is monotonically decreasing in* n.

Proof. From (4.2) and (4.3), it follows that

$$\|u_n - u\|_{W_2^3} = \left\| \sum_{i=n+1}^{\infty} \sum_{k=1}^{i} \beta_{ik} f(x_k, u_k) \widehat{\Psi}_i \right\|_{W_2^3} .$$

Thus

$$\|u_n - u\|_{W_2^3} \to 0, \quad n \to \infty.$$

In addition

$$\begin{aligned} \|u_n - u\|_{W_2^3}^2 &= \left\| \sum_{i=n+1}^{\infty} \sum_{k=1}^{i} \beta_{ik} f(x_k, u_k) \widehat{\Psi}_i \right\|_{W_2^3}^2 \\ &= \sum_{i=n+1}^{\infty} \left(\sum_{k=1}^{i} \beta_{ik} f(x_k, u_k) \widehat{\Psi}_i \right)^2 . \end{aligned}$$

Clearly, $\|u_n - u\|_{W_2^3}$ is monotonically decreasing in n. □

5 Numerical results

In this section, a numerical example is provided to show the accuracy of the present method for three specific values of λ which guarantees the existence of two locally unique solutions. In particular, having used $\lambda = 1, 2, 3.51$ we have constructed comparison tables to indicate the accuracy of the present method compared with existing results. All computations are performed by Maple 16. Results obtained by the method are compared (for the chosen values of λ) the exact solution, and with Adomian decomposition method (ADM) [36], Laplace decomposition method (LDM) [24], B-spline method [8], Non-polynomial spline method (NPSM) [21] and Lie-group shooting

x	Exact Solution	Approximate Solution	Absolute Error	Relative Error	Time
0.1	0.04984679002	0.04984679003	10^{-11}	$2.006147236 \times 10^{-10}$	0.593
0.2	0.08918993502	0.0891899350	2×10^{-11}	$2.242405491 \times 10^{-10}$	0.577
0.3	0.1176090956	0.1176090956	0.0	0.0	0.577
0.4	0.1347902526	0.1347902526	0.0	0.0	0.593
0.5	0.1405392142	0.1405392141	10^{-10}	$7.115451767 \times 10^{-10}$	0.546
0.6	0.1347902526	0.1347902526	0.0	0.0	0.593
0.7	0.1176090956	0.1176090956	0.0	0.0	0.577
0.8	0.08918993502	0.0891899350	2×10^{-11}	$2.242405491 \times 10^{-10}$	0.577
0.9	0.04984679002	0.04984679003	10^{-11}	$2.006147236 \times 10^{-10}$	0.593

Table 5.1: The numerical results of Example for boundary conditions at $\lambda = 1$.

method (LGSM) [1] of each values are found to be in good agreement with each others. The RKHSM does not require discretization of the variables, i.e., time and space, it is not effected by computation round off errors and one is not faced with necessity of large computer memory and time. The accuracy of the RKHSM for the Bratu's problem is controllable and absolute errors are small with present choice of λ (see Tables 5.1–5.6). The numerical results we obtained justify the advantage of this methodology. The numerical results are shown in Figure 5.1.

Example 5.1. We now consider the Bratu's problem

$$\begin{cases} u''(x) + \lambda \exp(u(x)) = 0, \quad 0 \le x \le 1, \\ u(0) = u(1) = 0. \end{cases} \tag{5.1}$$

Select the initial value $u_0(x) = 0$. Thus, if the method described above is applied to the (5.1), then we find the following Tables and Figure.

6 Conclusion

In this paper, we introduce an algorithm for solving the Bratu's problem with boundary conditions. For illustration purposes, an example was selected to show the computational accuracy. It may be concluded that, the RKHSM is very powerful and efficient

x	Exact Solution	Approximate Solution	Absolute Error	Relative Error	Time
0.1	0.1144107440	0.1144107440	0.0	0.0	0.656
0.2	0.2064191156	0.2064191157	10^{-10}	$4.844512569 \times 10^{-10}$	0.655
0.3	0.2738793116	0.2738793116	0.0	0.0	0.624
0.4	0.3150893646	0.3150893646	0.0	0.0	0.577
0.5	0.3289524216	0.3289524215	10^{-10}	$3.039953301 \times 10^{-10}$	0.624
0.6	0.3150893646	0.3150893646	0.0	0.0	0.577
0.7	0.2738793116	0.2738793116	0.0	0.0	0.624
0.8	0.2064191156	0.2064191157	10^{-10}	$4.844512569 \times 10^{-10}$	0.578
0.9	0.1144107440	0.1144107440	0.0	0.0	0.562

Table 5.2: The numerical results of Example for boundary conditions at $\lambda = 2$.

x	Exact Solution	Approximate Solution	Absolute Error	Relative Error	Time
0.1	0.3958056702	0.3958056762	6.0×10^{-9}	$1.515895413 \times 10^{-8}$	0.593
0.2	0.7390973562	0.7390973508	5.4×10^{-9}	$7.306209330 \times 10^{-9}$	0.718
0.3	1.008758182	1.008758182	0.0	0.0	0.702
0.4	1.182536568	1.182536573	5×10^{-9}	$4.228199056 \times 10^{-9}$	0.546
0.5	1.242742593	1.242742592	10^{-9}	$8.046718650 \times 10^{-10}$	0.624
0.6	1.182536568	1.182536573	5×10^{-9}	$4.228199056 \times 10^{-9}$	0.546
0.7	1.008758182	1.008758182	0.0	0.0	0.702
0.8	0.7390973562	0.7390973508	5.4×10^{-9}	$7.306209330 \times 10^{-9}$	0.718
0.9	0.3958056702	0.3958056762	6.0×10^{-9}	$1.515895413 \times 10^{-8}$	0.593

Table 5.3: The numerical results of Example for boundary conditions at $\lambda = 3.51$.

x	Laplace [24]	Decomposition [36]	B-spline [8]	RKHSM
0.1	$1.978800000003445e-6$	$2.685102500000001e-3$	$2.979700000002583e-6$	$1.0e-11$
0.2	$3.939399999999815e-6$	$2.021935000000003e-3$	$5.465999999995641e-6$	$2.0e-11$
0.3	$5.854800000010263e-6$	$1.523418999999915e-4$	$7.335699999999612e-6$	0.0
0.4	$7.703799999980721e-6$	$2.201747400000009e-3$	$8.496699999999136e-6$	0.0
0.5	$9.466499999999378e-6$	$3.015473299999988e-3$	$8.892100000018610e-6$	$1.0e-10$
0.6	$1.111169999998274e-5$	$2.201747400000009e-3$	$8.496699999999136e-6$	0.0
0.7	$1.257160000001090e-5$	$1.523418999999915e-4$	$7.335699999999612e-6$	0.0
0.8	$1.347531e-5$	$2.021935000000003e-3$	$5.465999999995641e-6$	$2.0e-11$
0.9	$1.196780000000536e-5$	$2.685102500000001e-3$	$2.979700000002583e-6$	$1.0e-11$

Table 5.4: The absolute errors for ADM, LDM, B-spline and RKHSM of Bratu's problem for the case $\lambda = 1$.

x	Laplace [24]	Decomposition [36]	B-spline [8]	RKHSM
0.1	$2.129029899999996e-3$	$1.521724399999999e-2$	$1.717889999999778e-5$	0.0
0.2	$4.209699400000017e-3$	$1.467511560000001e-2$	$3.259660000001774e-5$	$1.0e-10$
0.3	$6.186805800000028e-3$	$5.887811600000015e-3$	$4.489909999999542e-5$	0.0
0.4	$8.001913999999999e-3$	$3.246635400000031e-3$	$5.285839999996655e-5$	0.0
0.5	$9.599191999999979e-3$	$6.985078600000028e-3$	$5.561419999999817e-5$	$1.0e-10$
0.6	$1.092952429999999e-2$	$3.246635400000031e-3$	$5.285839999996655e-5$	0.0
0.7	$1.193342070000003e-2$	$5.887811600000015e-3$	$4.489909999999542e-5$	0.0
0.8	$1.237780840000000e-2$	$1.467511560000001e-2$	$3.259660000001774e-5$	$1.0e-10$
0.9	$1.087336550000000e-2$	$1.521724399999999e-2$	$1.717889999999778e-5$	0.0

Table 5.5: The absolute errors for ADM, LDM, B-spline and RKHSM of Bratu's problem for the case $\lambda = 2$.

x	B-spline [8]	NPSM [21]	LGSM [1]	RKHSM
0.1	$3.84172369550e-2$	$6.61e-6$	$4.45174e-5$	$6.0e-9$
0.2	$7.48135367780e-2$	$5.83e-6$	$7.12487e-5$	$5.4e-9$
0.3	$1.05827422823e-1$	$6.19e-6$	$7.30493e-5$	0.0
0.4	$1.27116880861e-1$	$6.89e-6$	$4.46877e-5$	$5.0e-9$
0.5	$1.34752877607e-1$	$7.31e-6$	$6.75722e-7$	$1.0e-9$
0.6	$1.27116880864e-1$	$6.89e-6$	$4.56074e-5$	$5.0e-9$
0.7	$1.05827422823e-1$	$6.19e-6$	$7.20013e-5$	0.0
0.8	$7.48135367760e-2$	$5.83e-6$	$7.05097e-5$	$5.4e-9$
0.9	$3.84172369530e-2$	$6.61e-6$	$4.41296e-5$	$6.0e-9$

Table 5.6: The absolute errors for B-spline method, Non-polynomial spline method (NPSM), Lie-group shooting method (LGSM) and RKHSM of Bratu's problem for the case $\lambda = 3.51$.

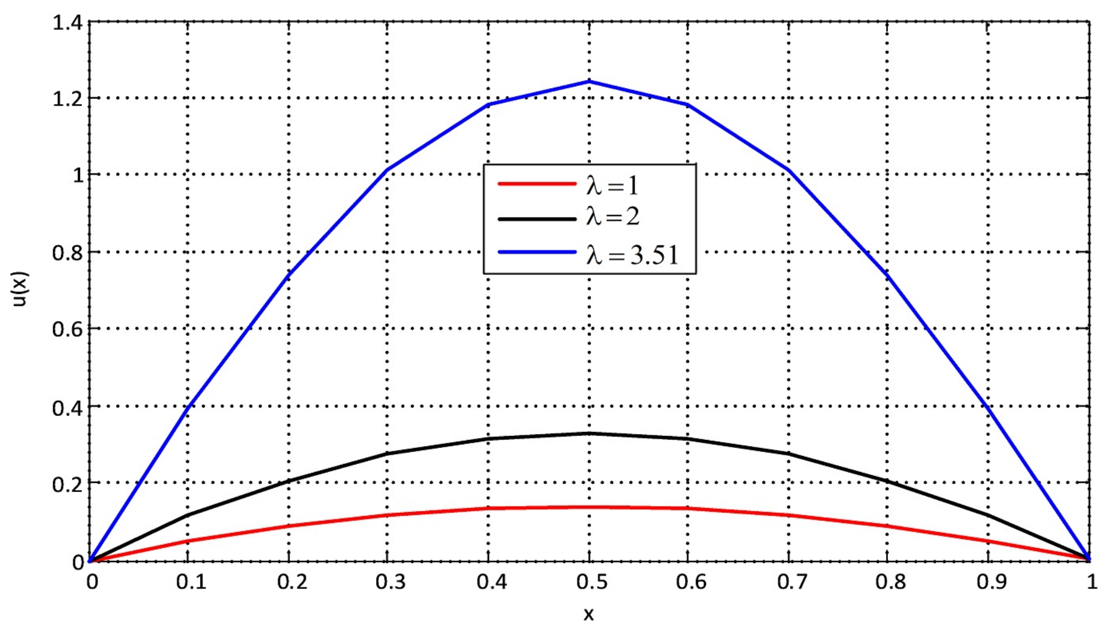

Figure 5.1: Plots of approximate solutions for different $\lambda's$.

in finding approximate solution for wide classes of problem. Solutions obtained by the present method are uniformly convergent. As shown in Tables 5.1–5.6 for the three computed cases the reproducing kernel Hilbert space method is more accurate than other methods. Clearly, the series solution methodology can be applied to much more complicated nonlinear differential equations and boundary value problems. However, if the problem becomes nonlinear, then the RKHSM does not require discretization or perturbation and it does not make closure approximation. Results of numerical example show that the present method is an accurate and reliable analytical method for Bratu's problem with boundary conditions. The present study has confirmed that the RKHSM offers significant advantages in terms of its straightforward applicability, its computational effectiveness and its accuracy to solve the strongly nonlinear equations.

References

[1] S. Abbasbandy, M. S. Hashemi, and Chein-Shan Liu. The Lie-group shooting method for solving the Bratu equation. *Commun. Nonlinear Sci. Numer. Simul.*, 16(11):4238–4249, 2011.

[2] Meltem Evrenosoglu Adiyaman and Sennur Somali. Taylor's decomposition on two points for one-dimensional Bratu problem. *Numer. Methods Partial Differential Equations*, 26(2):412–425, 2010.

[3] Y. A. S. Aregbesola and S. A. Odejide. A note on the Bratu problem. *Math. Sci. Res. J.*, 9(9):236–243, 2005.

[4] N. Aronszajn. Theory of reproducing kernels. *Trans. Amer. Math. Soc.*, 68:337–404, 1950.

[5] Ettore Barbieri and Michele Meo. A fast object-oriented Matlab implementation of the reproducing kernel particle method. *Comput. Mech.*, 49(5):581–602, 2012.

[6] B. Batiha. Numerical solution of Bratu-type equations by the variational iteration method. *Hacet. J. Math. Stat.*, 39(1):23–29, 2010.

[7] John P. Boyd. Chebyshev polynomial expansions for simultaneous approximation of two branches of a function with application to the one-dimensional Bratu equation. *Appl. Math. Comput.*, 143(2-3):189–200, 2003.

[8] Hikmet Caglar, Nazan Caglar, Mehmet Özer, Antonios Valarıstos, and Antonios N. Anagnostopoulos. B-spline method for solving Bratu's problem. *Int. J. Comput. Math.*, 87(8):1885–1891, 2010.

[9] Zhong Chen and Zhi-jie Chen. The exact solution of system of linear operator equations in reproducing kernel spaces. *Appl. Math. Comput.*, 203(1):56–61, 2008.

[10] Minggen Cui and Hong Du. Representation of exact solution for the nonlinear Volterra-Fredholm integral equations. *Appl. Math. Comput.*, 182(2):1795–1802, 2006.

[11] Minggen Cui and Yingzhen Lin. *Nonlinear numerical analysis in the reproducing kernel space*. Nova Science Publishers Inc., New York, 2009.

[12] Elias Deeba, S. A. Khuri, and Shishen Xie. An algorithm for solving boundary value problems. *J. Comput. Phys.*, 159(2):125–138, 2000.

[13] Hong Du and JiHong Shen. Reproducing kernel method of solving singular integral equation with cosecant kernel. *J. Math. Anal. Appl.*, 348(1):308–314, 2008.

[14] Juan Du and Minggen Cui. Constructive proof of existence for a class of fourth-order nonlinear BVPs. *Comput. Math. Appl.*, 59(2):903–911, 2010.

[15] Juan Du and Minggen Cui. Solving the forced Duffing equation with integral boundary conditions in the reproducing kernel space. *Int. J. Comput. Math.*, 87(9):2088–2100, 2010.

[16] F. Z. Geng. A numerical algorithm for nonlinear multi-point boundary value problems. *J. Comput. Appl. Math.*, 236(7):1789–1794, 2012.

[17] Fazhan Geng and Minggen Cui. Solving a nonlinear system of second order boundary value problems. *J. Math. Anal. Appl.*, 327(2):1167–1181, 2007.

[18] Fazhan Geng and Minggen Cui. Solving singular nonlinear second-order periodic boundary value problems in the reproducing kernel space. *Appl. Math. Comput.*, 192(2):389–398, 2007.

[19] Fazhan Geng and Minggen Cui. New method based on the HPM and RKHSM for solving forced Duffing equations with integral boundary conditions. *J. Comput. Appl. Math.*, 233(2):165–172, 2009.

[20] Fazhan Geng and Minggen Cui. A reproducing kernel method for solving nonlocal fractional boundary value problems. *Appl. Math. Lett.*, 25(5):818–823, 2012.

[21] R. Jalilian. Non-polynomial spline method for solving Bratu's problem. *Comput. Phys. Comm.*, 181(11):1868–1872, 2010.

[22] Wei Jiang and Minggen Cui. Constructive proof for existence of nonlinear two-point boundary value problems. *Appl. Math. Comput.*, 215(5):1937–1948, 2009.

[23] Wei Jiang and Yingzhen Lin. Representation of exact solution for the time-fractional telegraph equation in the reproducing kernel space. *Commun. Nonlinear Sci. Numer. Simul.*, 16(9):3639–3645, 2011.

[24] S. A. Khuri. A new approach to Bratu's problem. *Appl. Math. Comput.*, 147(1):131–136, 2004.

[25] Fuxiang Li and Minggen Cui. A best approximation for the solution of one-dimensional variable-coefficient Burgers' equation. *Numer. Methods Partial Differential Equations*, 25(6):1353–1365, 2009.

[26] Shuicai Li and Shi-Jun Liao. An analytic approach to solve multiple solutions of a strongly nonlinear problem. *Appl. Math. Comput.*, 169(2):854–865, 2005.

[27] Yingzhen Lin and Minggen Cui. A numerical solution to nonlinear multi-point boundary value problems in the reproducing kernel space. *Math. Methods Appl. Sci.*, 34(1):44–47, 2011.

[28] Xueqin Lü and Minggen Cui. Analytic solutions to a class of nonlinear infinite-delay-differential equations. *J. Math. Anal. Appl.*, 343(2):724–732, 2008.

[29] Xueqin Lv and Minggen Cui. An efficient computational method for linear fifth-order two-point boundary value problems. *J. Comput. Appl. Math.*, 234(5):1551–1558, 2010.

[30] Maryam Mohammadi and Reza Mokhtari. Solving the generalized regularized long wave equation on the basis of a reproducing kernel space. *J. Comput. Appl. Math.*, 235(14):4003–4014, 2011.

[31] Jalil Rashidinia and Reza Jalilian. Spline solution of two point boundary value problems. *Appl. Comput. Math.*, 9(2):258–266, 2010.

[32] Muhammed I. Syam and Abdelrahem Hamdan. An efficient method for solving Bratu equations. *Appl. Math. Comput.*, 176(2):704–713, 2006.

[33] S. G. Venkatesh, S. K. Ayyaswamy, and G. Hariharan. Haar wavelet method for solving initial and boundary value problems of Bratu-type. *Int. J. Comput. Math. Sci.*, 4(6):286–289, 2010.

[34] Wenyan Wang, Minggen Cui, and Bo Han. A new method for solving a class of singular two-point boundary value problems. *Appl. Math. Comput.*, 206(2):721–727, 2008.

[35] Yu-lan Wang and Lu Chao. Using reproducing kernel for solving a class of partial differential equation with variable-coefficients. *Appl. Math. Mech. (English Ed.)*, 29(1):129–137, 2008.

[36] Abdul-Majid Wazwaz. Adomian decomposition method for a reliable treatment of the Bratu-type equations. *Appl. Math. Comput.*, 166(3):652–663, 2005.

[37] B. Y. Wu and X. Y. Li. A new algorithm for a class of linear nonlocal boundary value problems based on the reproducing kernel method. *Appl. Math. Lett.*, 24(2):156–159, 2011.

[38] Boying Wu and Xiuying Li. Application of reproducing kernel method to third order three-point boundary value problems. *Appl. Math. Comput.*, 217(7):3425–3428, 2010.

[39] Boying Wu and Xiuying Li. Iterative reproducing kernel method for nonlinear oscillator with discontinuity. *Appl. Math. Lett.*, 23(10):1301–1304, 2010.

[40] Li-Hong Yang, Ji-Hong Shen, and Yue Wang. The reproducing kernel method for solving the system of the linear Volterra integral equations with variable coefficients. *J. Comput. Appl. Math.*, 236(9):2398–2405, 2012.

[41] Huanmin Yao. Reproducing kernel method for the solution of nonlinear hyperbolic telegraph equation with an integral condition. *Numer. Methods Partial Differential Equations*, 27(4):867–886, 2011.

[42] Huanmin Yao and Minggen Cui. A new algorithm for a class of singular boundary value problems. *Appl. Math. Comput.*, 186(2):1183–1191, 2007.

[43] Shiping Zhou and Minggen Cui. Approximate solution for a variable-coefficient semilinear heat equation with nonlocal boundary conditions. *Int. J. Comput. Math.*, 86(12):2248–2258, 2009.

[44] Yongfang Zhou, Yinzhen Lin, and Minggen Cui. An efficient computational method for second order boundary value problems of nonlinear differential equations. *Appl. Math. Comput.*, 194(2):354–365, 2007.