Best linear unbiased and invariant reconstructors for the past records

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Abstract

The best linear unbiased reconstructor and the best linear invariant reconstructor of the past upper records based on observed upper record values for the location–scale family of distributions are derived. The results are obtained in detail for two-parameter exponential distribution. A comparison study is performed in terms of mean squared reconstruction error and Pitman's measure of closeness criteria. Finally, a real example is given to illustrate the proposed procedures.

Keywords: Best linear unbiased estimator (BLUE), Best linear invariant estimator(BLIE), Pitman closeness, Location-scale family.
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1 Introduction

Let $\{X_i, i > 1\}$ be a sequence of independent and identically distributed (iid) continuous random variables. An observation X_j is called an upper record value if its value exceeds that of all previous observations, i. e., X_j is an upper record value if $X_j > X_i$ for every $i < j$. These type of data are of great importance in several real-life problems involving weather, industry, economic and sport data. For more details and applications of record values, see for example, Arnold *et al.* (1998). Ahsanullah and Shakil (2012) obtained some characterization results of Rayleigh distribution based on conditional expectation of record values.

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The prediction problem of the future records based on past records have been extensively studied in both frequentist and Bayesian approaches; Awad and Raqab (2000) considered the prediction problem of the future nth record value based on the first m $(m < n)$ observed record values from one-parameter exponential distribution. Raqab (2007) established different point predictors and prediction intervals for future records based on observed record values from two-parameter exponential distribution. Raqab and Balakrishnan (2008) discussed the problem of prediction of the future record values in nonparametric settings. Ahmadi and Doosparast (2006) investigated Bayesian estimation and prediction for some life distributions based on upper record values. MirMostafaee and Ahmadi (2011) obtained several point predictors such as the best unbiased predictor, the best invariant predictor and maximum likelihood predictor of future order statistics on the basis of observed record values from two-parameter exponential distribution.

Recently, some works have been done on the reconstruction problem. Klimczak and Rychlik (2005) obtained upper bounds for the expectations of increments of past order statistics and records based on observed order statistics and records, respectively. Balakrishnan *et al.* (2009) investigated the problem of reconstructing past upper records from the known values of future records in exponential and Pareto distributions. Razmkhah *et al.* (2010) obtained some point and interval reconstructors for the missing order statistics in exponential distribution. Asgharzadeh *et al.* (2012) studied the reconstruction of the past failure times based on left-censored samples for the proportional reversed hazard rate models. The main goal of this paper is to investigate the best linear unbiased reconstructor (BLUR) and the best linear invariant reconstructor (BLIR) on observed future record data from two-parameter exponential distribution. We also intend to present a comparison study of common reconstructors of past records from exponential distribution. With this in mind, we consider two criteria: mean squared reconstruction error (MSRE) and Pitman's measure of closeness (PMC). In recent years, many problems involving ordered data and Pitman closeness have been investigated. Pitman closeness of records to population quantiles was explored in Ahmadi and Balakrishnan (2009). Some interesting results regarding the sample median and Pitman closeness were provided by Iliopoulous and Balakrishnan (2010). Based on a Type-II right censored sample from one-parameter exponential distribution, Balakrishnan *et al.* (2012) compared the best linear unbiased predictor (BLUP) and the best linear invariant predictor (BLIP) of the censored order statistics in the one-sample case in terms of PMC.

The rest of this paper is organized as follows. Section 2 contains some preliminaries. In Sections 3 and 4, the BLUR and BLIR of the past upper records from two-parameter exponential distribution are investigated, respectively. These two reconstructors and two others where obtained by Balakrishnan *et al.* (2009) are compared in terms of MSRE and

PMC criteria in Section 5. A real example is given in Section 6.

2 Preliminaries

Let $R_1, ..., R_n$ be the first n upper record values coming from a X-sequence of iid continuous random variables with cumulative distribution function (cdf) $F(x;\theta)$ and probability distribution function (pdf) $f(x;\theta)$, where $\theta \in \Theta$ may be a vector of parameters and Θ is the parameter space. Suppose we have failed to observe the first m records, namely, we have observed $\mathbf{R} = (R_{m+1}, R_{m+2}, ..., R_n), \quad m+1 < n$. In this case, the joint pdf of $\mathbf{R}$ can be expressed as follows (see, for example, Arnold *et al.*, 1998, pp. 10–11)

$$f_{\mathbf{R}}(\mathbf{r};\theta) = \frac{\big(H(r_{m+1};\theta)\big)^m}{m!} f(r_{m+1};\theta) \prod_{i=m+1}^{n-1} \frac{f(r_{i+1};\theta)}{\bar{F}(r_i;\theta)}, \quad r_{m+1} < \ldots < r_n, \tag{1}$$

where $\bar{F}(x;\theta) = 1 - F(x;\theta)$ is the survival function of X, $H(x;\theta) = -\log \bar{F}(x;\theta)$, $\mathbf{r} = (r_{m+1}, \ldots, r_n)$ is the observed value of $\mathbf{R}$. From (1) and using the Markovian property of record values, it can be shown that the conditional pdf of R_l given the observed data set $\mathbf{R}$ is just the conditional pdf of R_l given R_{m+1} and is given by

$$\begin{aligned} f_{R_l|\mathbf{R}}(r_l|\mathbf{r};\theta) &= f_{R_l|R_{m+1}}(r_l|r_{m+1};\theta) \\ &= \frac{\big(H(r_l;\theta)\big)^{l-1}\big(H(r_{m+1};\theta) - H(r_l;\theta)\big)^{m-l}}{B(l, m-l+1)\big(H(r_{m+1};\theta)\big)^m} \frac{f(r_l;\theta)}{\bar{F}(r_l;\theta)}, \quad r_l < r_{m+1}, \end{aligned} \tag{2}$$

where $B(.,.)$ is the complete beta function.
From (2), the unbiased reconstructor of R_l, $l = 1, ..., m$, is the conditional mean of R_l given R_{m+1}, i.e.,

$$R_l^* = E(R_l|\mathbf{R}) = E(R_l|R_{m+1}), \tag{3}$$

and hence it depends only on R_{m+1}. If the parameters in (3) are unknown, they have to be estimated in this conditional expectation.

A random variable X is said to have a two-parameter exponential distribution, denoted by $\text{Exp}(\mu, \sigma)$, if its cdf is

$$F(x;\theta) = 1 - e^{-\frac{x-\mu}{\sigma}}, \quad x \geq \mu, \quad \sigma > 0, \tag{4}$$

where $\theta = (\mu, \sigma)$, μ and σ are the location and scale parameters, respectively. The exponential distribution is a very commonly used distribution in reliability engineering and survival analysis. We refer the reader to Balakrishnan and Basu (1995) for some research based on exponential distribution. In what follows, we shall study the behavior of reconstructors of the past records for $\text{Exp}(\mu, \sigma)$ in detail. The first result for the case of $\text{Exp}(\mu, \sigma)$ is that the unbiased reconstructor of R_l based on $\mathbf{R}$ is given by

$$R_l^*(\mu) = \mu + \frac{l}{m+1}(R_{m+1} - \mu), \tag{5}$$

it does not depend on the scale parameter. When μ is unknown, we can substitute it with its estimator based on $\mathbf{R} = (R_{m+1}, R_{m+2}, ..., R_n)$.

3 The best linear unbiased reconstructor

In this section, first we focus our attention to derive the BLUR in general for the location–scale families. Let $\mathbf{X} = (X_{m+1}, ..., X_n)'$ be an observed $(n - m)$-dimensional random vector from the cdf $F(x; \mu, \sigma) = F_0((x-\mu)/\sigma)$, where μ and σ are the location and scale parameters, respectively, and F_0 does not depend on μ and σ. Also, let Y be an unknown past observation from the same distribution. Reconstructing the outcome Y based on the observed value of $\mathbf{X}$ is the main goal of this section. Toward this end, let us assume that $a_0 = E((Y-\mu)/\sigma), a_i = E((Xi-\mu)/\sigma), w_i = Cov((Y-\mu)/\sigma, (X_i-\mu)/\sigma)$ and $v_{i,j} = Cov((X_i-\mu)/\sigma, (X_j-\mu)/\sigma)$, for $i, j = m+1, ..., n$. Denote by $\mathbf{1}$ the vector of 1's, $\mathbf{a} = (a_{m+1}, ..., a_n)^T$, $\mathbf{w}^T = (w_{m+1}, ..., w_n)$ and $\mathbf{V} = (v_{i,j}), i, j = m+1, ..., n$. To obtain the BLUR of Y based on $\mathbf{X}$, we will have to minimize the variance of $(Y - \mathbf{c}\mathbf{X}^T)$ subject to the condition of unbiasedness where $\mathbf{c} = (c_{m+1}, ..., c_n)$ is an arbitrary vector of real values. Then, following along the same lines as in the prediction problem, the BLUR of Y is given by (see, Goldberger (1962) in the context of prediction)

$$\hat{Y} = \hat{\mu}(\mathbf{X}) + \hat{\sigma}(\mathbf{X})a_0 + \mathbf{w}^T\mathbf{V}^{-1}[\mathbf{X}^T - \hat{\mu}(\mathbf{X})\mathbf{1} - \hat{\sigma}(\mathbf{X})\mathbf{a}], \tag{6}$$

where $\hat{\mu}(\mathbf{X})$ and $\hat{\sigma}(\mathbf{X})$ are the BLUEs of μ and σ on the basis of $\mathbf{X}$, respectively, and $\mathbf{V}^{-1}$ is the inverse matrix of $\mathbf{V}$.

Now, assume that the observed data are the upper record values from the two-parameter exponential distribution. Using (6), the BLUR of R_l based on $\mathbf{R}$ is given by

$$\hat{R}_l = \hat{\mu} + l\hat{\sigma} + \omega^T\Sigma^{-1}[\mathbf{R}^T - \hat{\mu}\mathbf{1} - \hat{\sigma}\alpha], \tag{7}$$

where $\hat{\mu}$ and $\hat{\sigma}$ are the BLUEs of μ and σ based on $\mathbf{R}$, respectively, α is the vector of means of the corresponding record values taken from $\text{Exp}(0, 1)$, $\sigma^2\Sigma$ is the covariance matrix of $\mathbf{R}$ and

$$\omega^T = 1/\sigma^2(Cov(R_{m+1}, R_l), Cov(R_{m+2}, R_l), ..., Cov(R_n, R_l)).$$

By doing some algebraic calculations it can be shown that

$$\omega^T = (l, l, ..., l)$$

and

$$\omega^T \Sigma^{-1} = (\frac{l}{m+1}, 0, 0, ..., 0). \tag{8}$$

Substituting (8) into (7), we get

$$\hat{R}_l = \hat{\mu} + l\hat{\sigma}. \tag{9}$$

Furthermore, the BLUEs of μ and σ on the basis of $\mathbf{R}$ are given by

$$\hat{\mu} = \frac{nR_{m+1} - (m+1)R_n}{n-(m+1)} \quad \text{and} \quad \hat{\sigma} = \frac{R_n - R_{m+1}}{n-(m+1)}, \tag{10}$$

respectively (see, Arnold *et al.*, 1998). Therefore, by substituting (10) in (9), it is deduced that

$$\hat{R}_l = \frac{(n-l)R_{m+1} - ((m+1)-l)R_n}{n-(m+1)} \tag{11}$$

is the BLUR of R_l on the basis of the data set $\mathbf{R}$. The MSRE of $\hat{R}_l$ is computed to be

$$\text{MSRE}(\hat{R}_l) = E[\hat{R}_l - R_l]^2 = \sigma^2 \frac{(n-l)((m+1)-l)}{n-(m+1)}. \tag{12}$$

It is easy to show that MSRE$(\hat{R}_l)$ is increasing in m and decreasing with respect to n and l, if the other arguments are fixed. Moreover, $\hat{R}_l$ and MSRE$(\hat{R}_l)$ converge to $R_{m+1} - \hat{\sigma}$ and $\sigma^2(n-m)/(n-(m+1))$, respectively, when l tends to m.

Remark 1 *It may be noted that by substituting the BLUE of μ into the unbiased reconstructor of R_l in* (5)*, the same result will be obtained for $\hat{R}_l$ as in* (11)*, i.e., $R_l^*(\hat{\mu}) = \hat{R}_l$.*

4 The best linear invariant reconstructor

Same as in previous section, first we present the results for BLIR in general for location–scale families. Let $\mathbf{X} = (X_{m+1}, ..., X_n)'$ and Y be as defined in Section 3. Then, follow the results of Mann (1969), in the context of prediction, the BLIR of Y based on the data set $\mathbf{X}$ can be derived as

$$\widetilde{Y} = \hat{Y} - (\frac{c_1}{1+c_2})\hat{\sigma}(\mathbf{X}), \tag{13}$$

where $\hat{Y}$ is the BLUR of Y, $c_2 = \sigma^{-2}Var(\hat{\sigma}(\mathbf{X}))$ and

$$c_1 = 1/\sigma^2 Cov(\hat{\sigma}(\mathbf{X}), (1 - \mathbf{w}^T\mathbf{V}^{-1}\mathbf{1})\hat{\mu}(\mathbf{X}) + (a_0 - \mathbf{w}^T\mathbf{V}^{-1}\mathbf{a})\hat{\sigma}(\mathbf{X})),$$

with $\mathbf{V}$, $\mathbf{W}$, $\mathbf{a}$, $\hat{\mu}(\mathbf{X})$, $\hat{\sigma}(\mathbf{X})$ and a_0 being as defined in Section 2.
When the upper record statistics $\mathbf{R} = (R_{m+1}, R_{m+2}, ..., R_n)$ from the Exp(μ, σ) distribution are observed, using (13), the BLIR of R_l on the basis of $\mathbf{R}$ is as follows

$$\widetilde{R}_l = \hat{R}_l - (\frac{V_1}{1+V_2})\hat{\sigma}, \tag{14}$$

where $V_2 = \sigma^{-2}Var(\hat{\sigma})$ and $V_1 = \sigma^{-2}\big[Cov(\hat{\sigma}, (1 - \omega^T\Sigma^{-1}\mathbf{1})\hat{\mu} + (l - \omega^T\Sigma^{-1}\alpha)\hat{\sigma})\big]$ in which ω, Σ^{-1} and α are as defined in Section 2, $\hat{\mu}$ and $\hat{\sigma}$ are the BLUEs of μ and σ, respectively, and $\hat{R}_l$ is the BLUR of R_l. By performing some algebraic calculations, we obtain

$$\widetilde{R}_l = \frac{\big(n+1-l\big)R_{m+1} - \big(m+1-l\big)R_n}{n-m}. \tag{15}$$

Moreover, it can be shown that the MSRE of $\widetilde{R}_l$ is given by

$$\text{MSRE}(\widetilde{R}_l) = E[\widetilde{R}_l - R_l]^2 = \sigma^2\frac{\big(n+1-l\big)\big(m+1-l\big)}{n-m}, \tag{16}$$

which is increasing with respect to m and decreasing with respect to n and l, if the other arguments are fixed. Also, $\widetilde{R}_l$ and MSRE$(\widetilde{R}_l)$ converge to $R_{m+1} - \widetilde{\sigma}$ and $\sigma^2(1 + 1/(n-m))$, respectively, when l tends to m.

Remark 2 *It may be noted that the BLIEs of μ and σ for a two-parameter exponential distribution are given by (see, Arnold et al., 1998)*

$$\widetilde{\mu} = \frac{(n+1)R_{m+1} - (m+1)R_n}{n-m} \quad \text{and} \quad \widetilde{\sigma} = \frac{R_n - R_{m+1}}{n-m}, \tag{17}$$

respectively. By plugging the BLIE of μ into (5)*, we arrive at the same result for $\widetilde{R}_l$ as given in* (15)*, i.e., $R_l^*(\widetilde{\mu}) = \widetilde{R}_l$.*

5 Comparison results

In Sections 3 and 4, we obtained two reconstructors (BLUR and BLIR) for R_l based on $\mathbf{R}$ from a two-parameter exponential distribution. Balakrishnan *et al.* (2009) also considered the same plan and obtained two other reconstructors for R_l namely maximum likelihood reconstructor (MLR), denoted by $\hat{R}_{l,M}$, and conditional median reconstructor (CMR), denoted by $\hat{R}_{l,C}$. For a comparison study, let us restate their results here. The MLR of R_l is

$$\hat{R}_{l,M} = \frac{(n-l+1)R_{m+1} - (m-l)R_n}{n-m+1} \tag{18}$$

with

$$MSRE(\hat{R}_{l,M}) = \sigma^2\left\{\frac{(n+1-l)(m+1-l)}{n-m+1} + \frac{(n+1+m-2l)(n+1-l)}{(n-m+1)^2}\right\}. \tag{19}$$

The CMR of R_l is

$$\hat{R}_{l,C} = \frac{m(R_n - R_{m+1})M_{l,m} + nR_{m+1} - mR_n}{n-m} \tag{20}$$

with

$$\begin{aligned} MSRE(\hat{R}_{l,C}) &= \sigma^2\left\{(m+1-l)(m+2-l) + \left(\frac{m(1-M_{l,m})}{n-m}\right)^2 (n-m-1)\right. \\ &\left.\times(n-m) - 2\left(\frac{m(1-M_{l,m})}{n-m}\right)(m+1-l)(n-m-1)\right\}, \end{aligned} \tag{21}$$

where $M_{l,m}$ stands for the median of beta distribution with parameters l and $m-l+1$. In this section, we intend to compare the four reconstructors based on MSRE and PMC criteria.

5.1 Comparison based on MSRE

Using (12) and (16), we have

$$\text{MSRE}(\widetilde{R}_l) = \text{MSRE}(\hat{R}_l) - \sigma^2\frac{\big((m+1)-l\big)^2}{(n-(m+1))(n-m)}. \tag{22}$$

From (22), it is obvious that BLIR is better than BLUR in the sense of MSRE. Moreover, from (16) and (19), we have

$$\text{MSRE}(\widetilde{R}_l) = \text{MSRE}(\hat{R}_{l,M}) - \sigma^2\left(\frac{n+1-l}{n-m+1}\right)^2(1-\frac{1}{n-m}), \tag{23}$$

which implies that BLIR is better than MLR. Furthermore, using (12) and (19), we get

$$\begin{aligned} \text{MSRE}(\hat{R}_l) &= \text{MSRE}(\hat{R}_{l,M}) - \sigma^2\frac{n+1+m-2l}{(n-m+1)^2(n-m-1)} \\ &\times\Big((n-m)^2 - (n+m-2(l-1))\Big). \end{aligned} \tag{24}$$

Using (24) it is deduced that BLUR is better than MLR when $(n-m)^2 > (n+m-2(l-1))$.

In order to complete the comparison study, we have computed the numerical values of MSREs of four reconstructors for $n = 10$ and all selected values of l and m. The results are presented in Table 1.

Table 1. Numerical values of MSREs for $n = 10$.

	m							
l	1	2	3	4	5	6	7	8
1	2.0000[1]	3.5802	5.6250	8.3673	12.2222	18.0000	27.5000	45.5556
	1.3333[2]	2.6490	4.6140	7.0042	10.3351	15.3214	23.6237	40.2204
	1.1250[3]	2.5714	4.5000	7.2000	11.2500	18.0000	31.5000	72.0000
	1.1111[4]	2.5000	4.2857	6.6667	10.0000	15.0000	23.3333	40.0000
2		2.0000	3.6562	5.8776	9.0000	13.6800	21.3750	36.0000
		1.1250	2.7857	4.7456	7.4591	11.5082	18.2395	31.6855
		1.1429	2.6667	4.8000	8.0000	13.3333	24.0000	56.0000
		1.1250	2.5714	4.5000	7.2000	11.2500	18.0000	31.5000
3			2.0000	3.7551	6.2222	9.9200	16.0000	27.5556
			1.2673	2.8408	5.0000	8.2094	13.5341	24.1591
			1.1667	2.8000	5.2500	9.3333	17.5000	42.0000
			1.1429	2.6667	4.8000	8.0000	13.3333	24.0000
4				2.0000	3.8889	6.7200	11.3750	20.2222
				1.2768	2.9486	5.4168	9.5000	17.6358
				1.2000	3.0000	6.0000	12.0000	30.0000
				1.1667	2.8000	5.2500	9.3333	17.5000
5					2.0000	4.0800	7.5000	14.0000
					1.2995	3.1281	6.1357	12.1146
					1.2500	3.3333	7.5000	20.0000
					1.2000	3.0000	6.0000	12.0000
6						2.0000	4.3750	8.8889
						1.3395	3.4403	7.5950
						1.3333	4.0000	12.0000
						1.2500	3.3333	7.5000
7							2.0000	4.8889
							1.4104	4.0764
							1.5000	6.0000
							1.3333	4.0000

[1], [2], [3] and [4] stand for the σ^{-2}MSRE of $\hat{R}_{l,M}$, $\hat{R}_{l,C}$, $\hat{R}_l$ and $\widetilde{R}_l$, respectively.

For $n = 10$ from Table 1, by empirical evidence, it is observed that in the sense of MSRE, BLIR is more accurate reconstructor than others. Furthermore, in the most cases the MSRE of $\hat{R}_{l,M}$ is greater than others.

5.2 Comparison based on PMC

Here, we use the PMC criterion to compare the reconstructors $\hat{R}_l$, $\widetilde{R}_l$, $\hat{R}_{l,M}$ and $\hat{R}_{l,C}$. The measure is based on the probabilities of the relative closeness of competing estimators to an unknown parameter. First, we recall two definitions. If T_1 and T_2 are two estimators of a common parameter θ, PMC of T_1 relative to T_2 is defined by

$$PMC(T_1, T_2|\theta) = P(|T_1 - \theta| < |T_2 - \theta|), \quad \forall \ \theta \in \Omega, \tag{25}$$

where Ω is the parameter space. The estimator T_1 is called Pitman–closer (with respect to θ) than T_2 if and only if

$$PMC(T_1, T_2|\theta) \geq 1/2, \quad \forall \ \theta \in \Omega,$$

with strict inequality holding for at least one θ.
Based on a Type-II right censored sample $(X_{1:n}, X_{2:n}, ..., X_{r:n})$ from $\text{Exp}(0, \sigma)$ distribution, Balakrishnan *et al.* (2011) compared BLUE and BLIE of σ in terms of PMC criterion. Also,

Balakrishnan *et al.* (2012) compared BLUP and BLIP of future order statistics under Type-II censoring for $\text{Exp}(0,\sigma)$ distribution in terms of PMC criterion. Now, we compare each two reconstructors of R_l based on PMC criterion.

(i) PMC of $\hat{R}_l$ relative to $\widetilde{R}_l$ (BLUR and BLIR): From (11), (15) and (25), we have

$$\begin{aligned}
\pi_{\hat{R}_l,\widetilde{R}_l}(l,m;n) &= P(|\hat{R}_l - R_l| < |\widetilde{R}_l - R_l|) \\
&= P\Big(\Big|R_{m+1} - \frac{m+1-l}{n-m-1}(R_n - R_{m+1}) - R_l\Big| < \Big|R_{m+1} - \frac{m+1-l}{n-m} \\
&\quad \times (R_n - R_{m+1}) - R_l\Big|\Big) \\
&= P\bigg(\frac{m+1-l}{(n-m-1)^2}(R_n - R_{m+1}) - \frac{2}{n-m-1}(R_{m+1} - R_l) < \frac{m+1-l}{(n-m)^2} \\
&\quad \times (R_n - R_{m+1}) - \frac{2}{n-m}(R_{m+1} - R_l)\bigg) \\
&= P\bigg(\frac{R_n - R_{m+1}}{R_{m+1} - R_l} < \frac{2}{(m+1-l)}\bigg(\frac{1}{n-m-1} + \frac{1}{n-m}\bigg)^{-1}\bigg).
\end{aligned}$$

It can be shown that $(R_n - R_{m+1})/(R_{m+1} - R_l)$ has the F-distribution with $2(n-m-1)$ and $2(m+1-l)$ degrees of freedom. So, we express $\pi_{\hat{R}_l,\widetilde{R}_l}(l,m;n)$ as

$$\pi_{\hat{R}_l,\widetilde{R}_l}(l,m;n) = F_{2(n-m-1),2(m+1-l)}\left(\frac{2(n-m)(n-m-1)}{(m+1-l)(2n-2m-1)}\right), \tag{26}$$

where $F_{a,b}(x)$ stands for the cdf of F-distribution with a and b degrees of freedom at x.

(ii) PMC of $\hat{R}_l$ relative to $\hat{R}_{l,M}$ (BLUR and MLR): From (11), (18) and (25), we have

$$\begin{aligned}
\pi_{\hat{R}_l,\hat{R}_{l,M}}(l,m;n) &= P(|\hat{R}_l - R_l| < |\hat{R}_{l,M} - R_l|) \\
&= P\Big(\Big|R_{m+1} - \frac{m+1-l}{n-m-1}(R_n - R_{m+1}) - R_l\Big| < \Big|R_{m+1} - \frac{m-l}{n-m+1} \\
&\quad \times (R_n - R_{m+1}) - R_l\Big|\Big) \\
&= F_{2(n-m-1),2(m+1-l)}\bigg(2\bigg(\frac{m+1-l}{n-m-1} + \frac{m-l}{n-m+1}\bigg)^{-1}\bigg).
\end{aligned} \tag{27}$$

(iii) PMC of $\hat{R}_l$ relative to $\hat{R}_{l,C}$ (BLUR and CMR): From (11), (20) and (25), we have

$$\begin{aligned}
\pi_{\hat{R}_l,\hat{R}_{l,C}}(l,m;n) &= P(|\hat{R}_l - R_l| < |\hat{R}_{l,C} - R_l|) \\
&= P\Bigg\{(\Big|R_{m+1} - \frac{m+1-l}{n-m-1}(R_n - R_{m+1}) - R_l\Big| < \\
&\quad \Big|R_{m+1} - \frac{m(1-M_{l,m})}{n-m}(R_n - R_{m+1}) - R_l\Big|\Bigg\} \\
&= F_{2(n-m-1),2(m+1-l)}\left(2\left(\frac{m+1-l}{n-m-1} + \frac{m(1-M_{l,m})}{n-m}\right)^{-1}\right). \qquad (28)
\end{aligned}$$

(iv) PMC of $\widetilde{R}_l$ relative to $\hat{R}_{l,M}$ (BLIR and MLR): Similarly, we have

$$\begin{aligned}
\pi_{\widetilde{R}_l,\hat{R}_{l,M}}(l,m;n) &= P(|\widetilde{R}_l - R_l| < |\hat{R}_{l,M} - R_l|) \\
&= P\Big(\Big|R_{m+1} - \frac{m+1-l}{n-m}(R_n - R_{m+1}) - R_l\Big| < \Big|R_{m+1} - \frac{m-l}{n-m+1} \\
&\quad \times(R_n - R_{m+1}) - R_l\Big|\Big) \\
&= F_{2(n-m-1),2(m+1-l)}\left(\left(\frac{m+1-l}{n-m} + \frac{m-l}{n-m+1}\right)^{-1}\right). \qquad (29)
\end{aligned}$$

(v) PMC of $\widetilde{R}_l$ relative to $\hat{R}_{l,C}$ (BLIR and CMR):

$$\pi_{\widetilde{R}_l,\hat{R}_{l,C}}(l,m;n) = F_{2(n-m-1),2(m+1-l)}\left(2\left(\frac{m+1-l}{n-m} + \frac{m(1-M_{l,m})}{n-m}\right)^{-1}\right). \qquad (30)$$

(vi) PMC of $\hat{R}_{l,M}$ relative to $\hat{R}_{l,C}$ (MLR and CMR):

$$\pi_{\hat{R}_{l,M},\hat{R}_{l,C}}(l,m;n) = F_{2(n-m-1),2(m+1-l)}\left(2\left(\frac{m-l}{n-m+1} + \frac{m(1-M_{l,m})}{n-m}\right)^{-1}\right). \qquad (31)$$

From the above results, one can examine the performance of any two reconstructors in terms of PMC criterion by comparing the values of median of the F-distribution with $2(n-m-1)$ and $2(m+1-l)$ degrees of freedom and corresponding values in (26)–(31), but, numerical computations are needed. We have computed the PMC probabilities in (26)–(31) for $n = 5, 10, 15, 20$ and all selected values of l and m. The results for $n = 10$ are presented in Tables 2–7, respectively, which examine how much a reconstructor is better than the other. The numerical results for $n = 5, 15$ and 20 are available with the authors, and will be sent on request to interested readers.

Table 2. Values of $\pi_{\hat{R}_l,\widetilde{R}_l}(l,m;n)$ for $n = 10$.

	m							
l	1	2	3	4	5	6	7	8
1	0.8894	0.8938	0.8219	0.6631	0.4423	0.2539	0.1554	0.1521
2		0.8758	0.8661	0.7603	0.5574	0.3337	0.1950	0.1713
3			0.8583	0.8271	0.6749	0.4377	0.2498	0.1960
4				0.8352	0.7703	0.5623	0.3267	0.2288
5					0.8034	0.6868	0.4332	0.2740
6						0.7570	0.5680	0.3393
7							0.6849	0.4375
8								0.5714

Table 3. Values of $\pi_{\hat{R}_l,\hat{R}_{l,M}}(l,m;n)$ for $n = 10$.

	m							
l	1	2	3	4	5	6	7	8
1	0.9396	0.9351	0.8812	0.7492	0.5346	0.3215	0.1946	0.1741
2		0.9314	0.9167	0.8345	0.6531	0.4154	0.2434	0.1965
3			0.9206	0.8898	0.7646	0.5318	0.3104	0.2254
4				0.9057	0.8482	0.6620	0.4023	0.2638
5					0.8842	0.7811	0.5248	0.3170
6						0.8503	0.6700	0.3942
7							0.7901	0.5100
8								0.6667

Table 4. Values of $\pi_{\hat{R}_l,\hat{R}_{l,C}}(l,m;n)$ for $n = 10$.

	m							
l	1	2	3	4	5	6	7	8
1	0.9141	0.9097	0.8519	0.7012	0.4763	0.2739	0.1646	0.1560
2		0.8758	0.8893	0.7919	0.5924	0.3581	0.2064	0.1759
3			0.8814	0.8526	0.7081	0.4666	0.2646	0.2016
4				0.8601	0.7991	0.5943	0.3461	0.2358
5					0.8308	0.7184	0.4583	0.2832
6						0.7876	0.5985	0.3521
7							0.7182	0.4566
8								0.6002

Table 5. Values of $\pi_{\widetilde{R}_l,\hat{R}_{l,M}}(l,m;n)$ for $n = 10$.

	m							
l	1	2	3	4	5	6	7	8
1	0.9461	0.9447	0.9030	0.7974	0.6142	0.4118	0.2786	0.2663
2		0.9397	0.9305	0.8675	0.7218	0.5116	0.3402	0.2985
3			0.9315	0.9104	0.8157	0.6255	0.4199	0.3390
4				0.9207	0.8808	0.7410	0.5212	0.3912
5					0.9060	0.8356	0.6421	0.4602
6						0.8847	0.7658	0.5528
7							0.8521	0.6735
8								0.8000

Table 6. Values of $\pi_{\widetilde{R}_l,\hat{R}_{l,C}}(l,m;n)$ for $n=10$.

	m							
l	1	2	3	4	5	6	7	8
1	0.9204	0.9204	0.8752	0.7496	0.5487	0.3482	0.2290	0.2264
2		0.8835	0.9041	0.8264	0.6577	0.4394	0.2813	0.2536
3			0.8918	0.8745	0.7589	0.5491	0.3508	0.2882
4				0.8741	0.8326	0.6675	0.4422	0.3331
5					0.8509	0.7714	0.5572	0.3931
6						0.8187	0.6836	0.4750
7							0.7717	0.5860
8								0.7062

Table 7. Values of $\pi_{\hat{R}_{l,M},\hat{R}_{l,C}}(l,m;n)$ for $n=10$.

	m							
l	1	2	3	4	5	6	7	8
1	0.9727	0.9576	0.9278	0.8348	0.6584	0.4475	0.3002	0.2786
2		0.9397	0.9498	0.8957	0.7618	0.5508	0.3659	0.3125
3			0.9569	0.9326	0.8489	0.6663	0.4509	0.3557
4				0.9486	0.9077	0.7796	0.5578	0.4118
5					0.9378	0.8691	0.6829	0.4862
6						0.9224	0.8070	0.5863
7							0.8985	0.7167
8								0.8576

From our computations results and by empirical evidence, we observe that:

1. The BLIR (CMR) is Pitman–closer than BLUR (BLUR) for the following situations:

   (i) when $n=10,20$, $m=n/2+a$ and $l=1,2,...,2a+1,(a=0,1,...,n/2-2)$;

   (ii) when $n=5,15$, $m=(n+1)/2+a$ and $l=1,2,...,2a+2,(a=0,1,...,(n+1)/2-3)$;

   otherwise, the BLUR (BLUR) is Pitman–closer than the BLIR (CMR) specially for $l=m$. [See for example Table 2 (Table 4) when $n=10$].

2. The MLR (CMR) is Pitman–closer than BLUR (BLIR) in the following situations:

   (i) when $n=10,20$, $m=n/2+a$ and $l=1,2,...,2a,(a=1,...,n/2-2)$;

   (ii) when $n=5,15$, $m=(n+1)/2+a$ and $l=1,2,...,2a+1,(a=0,1,...,(n+1)/2-3)$;

   otherwise, the BLUR (BLIR) is Pitman–closer than the MLR (CMR) specially for $l=m$. [See for example Table 3 (Table 6) when $n=10$].

3. The MLR (CMR) is Pitman–closer than BLIR (MLR) for the following situations:

   (i) when $n=10,20$, $m=n/2+a$ and $l=1,2,...,2a-1,(a=1,...,n/2-2)$;

   (ii) when $n=5,15$, $m=(n+1)/2+a$ and $l=1,2,...,2a,(a=1,...,(n+1)/2-3)$;

   otherwise, the BLUR (MLR) is Pitman–closer than the BLIR (CMR) specially for $l=m$. [See for example Table 5 (Table 7) when $n=10$].

6 A real example

To illustrate the performance of the proposed reconstructors, we use a real data set concerning the times (in minutes) between 48 consecutive telephone calls to a company's switchboard which is presented by Castillo *et al.* (2005) where the authors found that the exponential distribution gave an adequate fit. Table 8 contains the corresponding data.

Table 8. Times (in minutes) between 48 consecutive calls.

1.34	0.14	0.33	1.68	1.86	1.31	0.83	0.33
2.20	0.62	3.20	1.38	0.96	0.28	0.44	0.59
0.25	0.51	1.61	1.85	0.47	0.41	1.46	0.09
2.18	0.07	0.02	0.64	0.28	0.68	1.07	3.25
0.59	2.39	0.27	0.34	2.18	0.41	1.08	0.57
0.35	0.69	0.25	0.57	1.90	0.56	0.09	0.28

From the data of Table 8, six upper records extracted which are 1.34, 1.68, 1.86, 2.20, 3.20 and 3.25. Assuming that only $R_{m+1}, ..., R_6$ have been observed, we show that how one can reconstruct the missing record values and compare the results with the exact values. With this in mind, we compute different point reconstructors presented in the paper for the lost record value R_l $(1 \leq l \leq m)$. Using (11), (15), (18) and (20) the values of the BLUR, BLIR, MLR and CMR are obtained, respectively. The results are presented in Table 9 for $m = 3$ and 4 and $1 \leq l \leq m$.

Table 9. The numerical values of point reconstructors.

m	l	Exact value	$\hat{R}_l$	$\widetilde{R}_l$	$\hat{R}_{l,M}$	$\hat{R}_{l,C}$
3	1	1.34	0.625	1.150	1.675	1.366
	2	1.68	1.150	1.500	1.938	1.675
	3	1.86	1.675	1.850	2.200	1.983
4	1	1.34	3.000	3.100	3.150	3.116
	2	1.68	3.050	3.125	3.167	3.139
	3	1.86	3.100	3.150	3.183	3.161
	4	2.20	3.150	3.175	3.200	3.184

For $m = 3$, the observed records are $R_4 = 2.20, R_5 = 3.20$ and $R_6 = 3.25$. Based on these observations we have reconstructed the first three records. From Table 9, it is observed that for $m = 3$ the point reconstructors are reasonably close to the exact values. When $m = 4$, it is assumed that two records $R_5 = 3.20$ and $R_6 = 3.25$ are observed. In this case, the point reconstructors are not close to the exact values but not too far. It should be mentiond that one example does not tell us much more.

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