SOME CHARACTERIZATIONS OF 0-DISTRIBUTIVE SEMILATTICES

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Abstract. In this paper we discuss prime down-sets of a semilattice. We give a characterization of prime down-sets of a semilattice. We also give some characterizations of 0-distributive semilattices and a characterization of minimal prime ideals containing an ideal of a 0-distributive semilattice. Finally, we give a characterization of minimal prime ideals of a pseudocomplemented semilattice.

1. Introduction

Semilattices have been studied by many authors. The class of distributive semilattices is an important subclass of semilattices. We refer the readers to [4, 9, 10] for distributive semilattices. We also refer the monograph [5] for the background of distributive semilattices. The class of 0-distributive semilattices is a nice extension of the class of distributive semilattices. This extension is useful for the study of pseudocomplemented semilattices. For pseudocomplemented semilattices we refer the readers to [2, 3, 5, 6]. We also refer the readers to [7, 8] for 0-distributive semilattices (see [1, 11] for 0-distributive lattices). In this paper we study 0-distributive semilattices. By semilattice we mean meet-semilattice.

A semilattice $S$ with 0 is called 0-distributive if for any $a, b, c \in S$ such that $a \land b = 0 = a \land c$ implies $a \land d = 0$ for some $d \geq b, c$. The pentagonal lattice $P_5$ (see Figure 1) as a semilattice is 0-distributive but the diamond lattice $M_3$ (see Figure 1) as a semilattice is not 0-distributive. A semilattice $S$ is called directed above if for all $x, y \in S$ there exists $z \in S$ such that $z \geq x, y$. Every 0-distributive semilattice is directed above.

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Minimal prime ideals and maximal filters play an important role in semilattices. In Section 2, we introduce a notion of minimal prime down-set and maximal filters in semilattices. Here we give a characterization of minimal prime down-sets and maximal filters in semilattices.

Like as a distributive semilattice (or distributive lattice) Stone’s version separation theorem is not true for 0-distributive semilattice. For example, if we consider the pentagonal lattice $P_5$ (see Figure 1) as a 0-distributive semilattice, then $F = [c]$ is a filter and $I = (a)$, is an ideal such that $F \cap I = \emptyset$ but there is no prime filter containing $F$ and disjoint from $I$. In Section 3 we discuss Stone’s version separation theorem for 0-distributive semilattices. In this section we give some characterizations of 0-distributive semilattices.

In Section 4 we discuss the pseudocomplementation in semilattices. We close the paper with a characterization of a minimal prime ideals of a pseudocomplemented 0-distributive semilattice.

2. Prime down-sets and maximal Filters

Let $S$ be a semilattice. A non-empty subset $D$ of $S$ is called a down-set if $a \in D, b \in S$ with $b \leq a$ implies that $b \in D$. A down-set $D$ of $S$ is called a proper down-set if $D \neq S$. A prime down-set is a proper down-set $P$ of $S$ such that $a \land b \in P$ implies $a \in P$ or $b \in P$. A prime down-set $P$ is called minimal if there is a prime down-set $Q$ such that $Q \subseteq P$, then $P = Q$.

Theorem 2.1. Any prime down-set of a semilattice contains a minimal prime down-set.

Proof. Let $S$ be a semilattice with 0. Let $P$ be a prime down-set of $S$ and let $\mathcal{P}$ be the set of all prime down-sets contained in $P$. Then $\mathcal{P}$
is non-empty since \( P \in \mathcal{P} \). Let \( C \) be a chain in \( \mathcal{P} \) and let
\[
M := \bigcap\{X \mid X \in C\}.
\]
We claim that \( M \) is a prime down-set. Clearly \( M \) is non-empty as \( 0 \in M \). Let \( a \in M \) and \( b \leq a \). Then \( a \in X \) for all \( X \in C \). Hence \( b \in X \) for all \( X \in C \) as \( X \) is a down-set. Thus \( b \in M \). Now let \( x \land y \in M \) for some \( x, y \in S \). Then \( x \land y \in X \) for all \( X \in C \). Since \( X \) is a prime down-set for all \( X \in C \), we have either \( x \in X \) or \( y \in X \) for all \( X \in C \). This implies that either \( x \in M \) or \( y \in M \). Hence \( M \) is a prime down-set.

Thus by applying the dual form of Zorn’s Lemma to \( \mathcal{P} \), there is a minimal member of \( \mathcal{P} \).

Let \( S \) be a semilattice. A non-empty subset \( F \) of \( S \) is called a filter if
\begin{enumerate}
  \item \( a, b \in F \) implies \( a \land b \in F \)
  \item \( a \in S, b \in F \) with \( a \geq b \) implies \( a \in F \).
\end{enumerate}
A filter \( F \) of a semilattice \( S \) is called a proper filter if \( F \neq S \). A maximal filter \( F \) of \( S \) is a proper filter which is not contained in any other proper filter, that is, if there is a proper filter \( G \) such that \( F \subseteq G \), then \( F = G \).

Following result is due to [8].

**Lemma 2.2.** Let \( M \) be a proper filter of \( S \) with \( 0 \). Then \( M \) is maximal if and only if for all \( a \in S \setminus M \), there is some \( b \in M \) such that \( a \land b = 0 \).

Now we have the following result.

**Theorem 2.3.** Let \( F \) be a non-empty proper subset of a semilattice \( S \). Then \( F \) is a filter if and only if \( S \setminus F \) is a prime down-set.

Proof. Let \( F \) be a filter of a semilattice \( S \). Let \( x \in S \setminus F \) and \( y \leq x \). Then \( x \notin F \) and hence \( y \notin F \) as \( F \) is a filter. This implies \( y \in S \setminus F \). Thus \( S \setminus F \) is a down-set. Since \( F \) is a filter \( S \setminus F \neq S \). Thus \( S \setminus F \) is a proper down-set. To prove \( S \setminus F \) is a prime down-set, let \( a, b \in S \) such that \( a \land b \in S \setminus F \). Then \( a \land b \notin F \) and hence either \( a \notin F \) or \( b \notin F \) as \( F \) is filter. This implies either \( a \in S \setminus F \) or \( b \in S \setminus F \). Therefore, \( S \setminus F \) is a prime down-set.

Conversely, let \( S \setminus F \) be a prime down-set and \( x, y \in F \). Then clearly, \( x, y \notin S \setminus F \) and hence \( x \land y \notin S \setminus F \) as \( S \setminus F \) is a prime down-set. Thus \( x \land y \in F \). Suppose \( x \in F \) and \( x \leq y \). Then \( x \notin S \setminus F \). Since
$S \setminus F$ is a down-set, we have $y \notin S \setminus F$. Hence $y \in F$. This implies $F$ is a filter.

Theorem 2.4. Let $F$ be a non-empty subset of a semilattice $S$. Then $F$ is a maximal filter if and only if $S \setminus F$ is a minimal prime down-set.

Proof. Let $F$ be a maximal filter and $S \setminus F$ is not a minimal prime down-set. Then there exists a prime down-set $I$ such that $I \subseteq S \setminus F$ which implies $F \subseteq S \setminus I$ which contradict to the maximality of $F$. Hence $S \setminus F$ is minimal prime down-set.

Conversely, let $S \setminus F$ be a minimal prime down-set and $F$ is not a maximal filter. Thus there exists a proper filter $G$ such that $F \subseteq G$ which implies $S \setminus G \subseteq S \setminus F$ which contradict the minimality of $S \setminus F$. Hence $F$ is a maximal filter.

3. Minimal Prime ideals

Let $S$ be a semilattice. A down-set $I$ of $S$ is called an ideal if $a, b \in I$ implies the existence of $c \in I$ such that $a, b \leq c$. The set of all ideals of $S$ is denoted by $\mathcal{I}(S)$. An ideal $I$ of $S$ is called a proper ideal if $I \neq S$. A prime ideal $P$ is a proper ideal of $S$ such that $a \land b \in P$ implies either $a \in P$ or $b \in P$. A prime ideal $P$ is called minimal if there is a prime ideal $Q$ such that $Q \subseteq P$, then $P = Q$. A filter $F$ of $S$ is called a prime filter if $F \neq S$ and $S \setminus F$ is a prime ideal.

We shall often use the following lemma in this paper.

Lemma 3.1. Let $S$ be a directed above semilattice with 0. If $S$ is not 0-distributive, then the set

$$F := \{x \in S \mid x \geq a \land y \neq 0 \text{ for all } y \geq b, c\},$$

where $a, b, c \in S$ such that $a \land b = a \land c = 0$, is a proper filter.

Proof. Since $S$ is not 0-distributive, there are $p, q, r \in S$ such that $p \land q = p \land r = 0$ and $p \land d \neq 0$ for all $d \geq q, r$. Now we have $p \geq p \land d$. Thus $p \in F$. Hence $F$ is nonempty. Clearly $0 \notin F$. It is enough to show that $F$ is a filter. Let $x \in F$ and $z \geq x$. Then $x \geq a \land y$ for all $y \geq b, c$ and by transitivity $z \geq a \land y$ for all $y \geq b, c$. Hence $z \in F$. Again let $x, z \in F$. Then $x \geq a \land y$ and $z \geq a \land y$ for all $y \geq b, c$. Thus $x \land z \geq a \land y$ for all $y \geq b, c$. Hence $x \land z \in F$. This implies $F$ is a filter.
Now we have the following result.

**Theorem 3.2.** Every maximal filter of a 0-distributive semilattice is a prime filter.

Proof. Let $S$ be a 0-distributive semilattice. Again let $Q$ be a maximal filter of $S$. We shall show that $Q$ is prime. It is sufficient to show that $S \setminus Q$ is a prime ideal. By Theorem 2.4 we have $S \setminus Q$ is a minimal prime down-set. Now let $x, y \in S \setminus Q$. Then by Lemma 2.2 we have $a \land x = 0 = b \land y$ for some $a, b \in Q$. Let $c = a \land b$. Clearly $c \land x = 0 = c \land y$ and $c \in Q$. Hence by the 0-distributivity of $S$ there exists $z \in S$ such that $z \geq x, y$ and $c \land z = 0$. Hence $z \in S \setminus Q$. Thus $S \setminus Q$ is a prime ideal which implies $Q$ is prime. □

Let $A$ be non-empty subset of a semilattice $S$ with 0. Set

$$A^\perp := \{x \in S \mid a \land x = 0 \text{ for all } a \in A\}.$$  

Then $A^\perp$ is called the annihilator of $A$. If $A = S$ then $A^\perp = S^\perp = \{0\}$. For $a \in S$, the annihilator of $\{a\}$ is simply denoted by $a^\perp$ and hence $a^\perp = \{x \in S \mid a \land x = 0\}$. We can easily show that

$$A^\perp = \bigcap_{a \in A} a^\perp.$$

Let $S$ be a semilattice with 0. An ideal $I$ of $S$ is called an annihilator ideal if $I = A^\perp$ for some non-empty subset $A$ of $S$.

Our aim is to prove a Stone’s version separation theorem for 0-distributive semilattices. The following result due to [8, Theorem 7].

**Theorem 3.3.** Let $S$ be a semilattice with 0. Then $S$ is 0-distributive if and only if for any filter $F$ of $S$ such that $F \cap x^\perp = \emptyset (x \in S)$, there exists a prime filter containing $F$ and disjoint from $x^\perp$.

Our conjecture is:

**Conjecture 3.4.** Let $S$ be a directed above semilattice with 0. Then $S$ is 0-distributive if and only if for any filter $F$ and any annihilator ideal $I$ of $S$ such that $F \cap I = \emptyset$, there exists a prime filter containing $F$ and disjoint from $I$.

The necessary conditions of a directed above semilattice to be 0-distributive is given below, but unfortunately, we could not prove or disprove the condition is sufficient or not.
Theorem 3.5. Let $S$ be a directed above semilattice with 0. If for any filter $F$ and any annihilator ideal $I$ of $S$ such that $F \cap I = \emptyset$, there exists a prime filter containing $F$ and disjoint from $I$, then $S$ is 0-distributive.

Proof. Suppose the condition holds. If $S$ is not 0-distributive, then there are $a, b, c \in S$ such that $a \wedge b = 0 = a \wedge c$ and $a \wedge d \neq 0$ for all $d \geq b, c$ (such $d$ exists as $S$ is directed above). Let

$$F := \{x \in S \mid x \geq a \wedge y \text{ for all } y \geq b, c\}.$$ 

Then by Lemma 3.1, we have $F$ is a proper filter.

Let $I$ be an annihilator ideal such that $a \wedge d \notin I$ (such annihilator exists as $a \wedge d \notin S^\perp$). We shall show that $I \cap F = \emptyset$. If $x \in I \cap F$, then $x \geq a \wedge y$ for all $y \geq b, c$ which implies $a \wedge d \in I$, which is a contradiction. Hence $I \cap F = \emptyset$. Thus by the assumption, there is a prime filter $Q$ such that $F \subseteq Q$ and $I \cap Q = \emptyset$. This implies $a \in Q$ and $y \in Q$ for all $y \geq b, c$. We shall show that either $b \in Q$ or $c \in Q$. If $b, c \notin Q$ then $b, c \in S \setminus Q$. Since $Q$ is a prime filter, $S \setminus Q$ is a prime ideal. So, there is $e \in S \setminus Q$ such that $e \geq b, c$ which is a contradiction. Hence either $b \in Q$ or $c \in Q$. This implies, either $a \wedge b \in Q$ or $a \wedge c \in Q$. Hence $0 \in Q$, which contradicts the fact that $Q$ is a prime filter. Therefore, $a \wedge d = 0$ for some $d \geq b, c$ and hence $S$ is 0-distributive. □

Let $S$ be a semilattice. For $a \in S$, the ideal $(a]$ is called the ideal generated by $a$. It can be easily seen that $(a]^\perp = a^\perp$ for any $a \in S$. An ideal $I$ of $S$ is called an $\alpha$-ideal if $(i^+]^\perp \subseteq I$ for any $i \in I$.

Now we shall give some characterizations of 0-distributive semilattice. The following lemma is due to [1]

Lemma 3.6. Every proper filter of a semilattice with 0 is contained in a maximal filter.

We have the following result which is a generalization of [1, Theorem 3.1].

Theorem 3.7. Let $S$ be a semilattice with 0. Then the following statements (i)–(iv) are equivalent and any one of them implies (v) and (vi).

(i) $S$ is 0-distributive;
(ii) every maximal filter of $S$ is prime;
(iii) every minimal prime down-set of $S$ is a minimal prime ideal;
(iv) every proper filter of $S$ is disjoint from a minimal prime ideal;
(v) for each element $a \in S$ such that $a \neq 0$, there is a minimal prime ideal not containing $a$;
(vi) each element $a \in S$ such that $a \neq 0$ is contained in a prime filter.

Proof. (i)$\Rightarrow$(ii). This follows by the Lemma 3.2.

(ii)$\Rightarrow$(iii). Let $N$ be a minimal prime down-set. Then by Lemma 2.4 we have $S \setminus N$ is a maximal filter. Hence by (ii) $S \setminus N$ is a prime filter. Thus $N$ is a prime ideal.

(iii)$\Rightarrow$(iv). Let $F$ be a proper filter of $S$. By Lemma 3.6 there is a maximal filter $M$ such that $F \subseteq M$. Hence by Lemma 2.4 we have $S \setminus M$ is a minimal prime down-set. Thus by (iii) $S \setminus M$ is a minimal prime ideal. Clearly, $F \cap (S \setminus M) = \emptyset$.

(iv)$\Rightarrow$(i). Suppose $S$ is not 0-distributive. Then there are $a, b, c \in S$ such that $a \wedge b = a \wedge c = 0$ and $a \wedge d \neq 0$ for all $d \geq b, c$. Now set

$$F = \{ x \in S \mid x \geq a \wedge y \text{ for all } y \geq b, c \}.$$ Then by Lemma 3.1, we have $F$ is a proper filter and hence by (iv) there exists a prime ideal $Q$ such that $F \subseteq Q$. Thus $a \wedge p \notin Q$ for any $p \geq b, c$. This implies $a, p \notin Q$ for any $p \geq b, c$. Now $a \notin Q$ implies $b, c \in Q$. Then there is $m \geq b, c$ such that $m \in Q$ which is a contradiction. Therefore, $a \wedge d = 0$ for some $d \geq b, c$ and hence $S$ is 0-distributive.

(iv)$\Rightarrow$(v). Let $a \in S$ such that $a \neq 0$. Then $[a]$ is a proper filter. Then by (iv) $[a]$ is disjoint from a minimal prime ideal $N$ of $S$. Thus $a \notin N$.

(v)$\Rightarrow$(vi). Let $a \in S$ such that $a \neq 0$. Then by (v) there is a minimal prime ideal $P$ such that $a \notin P$ which implies $a \in S \setminus P$. By the definition of prime filter we have $S \setminus P$ is a prime filter.

Now we have following result which is a generalization of [1, Lemma 1.8].

**Lemma 3.8.** Let $A$ be a non-empty subset of a semilattice $S$ with 0. Then $A^\perp$ is the intersection of all the minimal prime down-set not containing $A$.

Proof. Let $S$ be a semilattice with 0 and $\emptyset \neq A \subseteq S$. Suppose

$$X := \bigcap \{ P \mid A \nsubseteq P \text{ and } P \text{ is a minimal prime down-set} \}$$
Let \( x \in A^\perp \). Then \( x \wedge y = 0 \) for all \( y \in A \). This implies there is \( z \notin P \) such that \( x \wedge z = 0 \in P \). As \( P \) is prime, we have \( x \in P \). Hence \( x \in X \).

Conversely let, \( x \in X \). If \( x \notin A^\perp \). Then \( x \wedge q \neq 0 \) for some \( q \in A \). Let \( D = \{ x \wedge q \} \). Then \( 0 \notin D \). Hence, \( D \neq S \). Then by Lemma 3.6 we have \( D \subseteq M \) for some maximal filter \( M \). Hence by Lemma 2.4 we have \( S \setminus M \) is a minimal prime down-set. Now \( x \notin S \setminus M \) as \( x \in D \) implies \( x \in M \). Moreover \( A \notin S \setminus M \) as \( q \in A \) but \( q \notin M \) implies \( q \notin S \setminus M \), which is a contradiction to \( x \in X \). Hence \( x \in A^\perp \). Thus the lemma is proved.

**Theorem 3.9.** Let \( S \) be a \( 0 \)-distributive semilattice. If \( A \) is a nonempty subset of \( S \) and \( F \) is a proper filter intersecting \( A \), there is a minimal prime ideal containing \( A^\perp \) and disjoint from \( F \).

Proof. Let \( S \) be a directed above semilattice with 0. Again let \( A \) be a nonempty subset of \( S \) and \( F \) be a proper filter such that \( F \cap A \neq \emptyset \). Then Lemma 2.3 \( S \setminus F \) is a prime down-set and by Lemma 2.1 \( N \subseteq S \setminus F \) for some minimal prime down-set \( N \). Clearly, \( N \cap F = \emptyset \). Also \( A \notin S \setminus F \) and so \( A \notin N \). By Lemma 3.8 \( A^\perp \subseteq N \). Since \( S \) is \( 0 \)-distributive, by theorem 3.7(iv) \( N \) is a minimal prime ideal.

4. **Pseudocomplementation for 0-distributive semilattices**

Let \( S \) be a semilattice with 0. An element \( d \in S \) is called the pseudocomplement of \( x \in S \), if \( x \wedge d = 0 \) and \( y \in S, x \wedge y = 0 \) implies \( y \leq d \). The pseudocomplement of \( x \) is denoted by \( x^* \). A semilattice \( S \) is called pseudocomplemented if each element of \( S \) has a pseudocomplement. The pseudocomplement of 0 is the largest element 1. Thus a pseudocomplemented semilattice contains both the smallest element and the largest element.

**Theorem 4.1.** Every pseudocomplemented semilattice is \( 0 \)-distributive but the converse is not true.

Proof. Let \( S \) be a pseudocomplemented semilattice. Suppose \( a, b, c \in S \) with \( a \wedge b = 0 = a \wedge c \). By the definition of pseudocomplemented, \( b \leq a^*, c \leq a^* \) and \( a \wedge a^* = 0 \). Thus \( S \) is a \( 0 \)-distributive semilattice.
To prove the converse is not true, consider the semilattice, $\mathcal{M}_2$ shown in the Figure 2, which is clearly 0-distributive but not pseudocomplemented as $a^*$ does not exist.

![Figure 2]

**Theorem 4.2.** Let $S$ be a pseudocomplemented semilattice and let $J$ be an ideal of $S$. Then a prime ideal $P$ containing $J$ is a minimal prime ideal containing $J$ if and only if for each $x \in P$ there is $y \in S \setminus P$ such that $x \land y \in J$.

Proof. Let $P$ be a prime ideal of $S$ containing $J$ such that the given condition holds. We shall show that $P$ is a minimal prime ideal containing $J$. Let $K$ be a prime ideal containing $J$ such that $K \subseteq P$. Let $x \in P$. Then there is $y \in S \setminus P$ such that $x \land y \in J$. Hence $x \land y \in K$ as $K$ containing $J$. Since $K$ is prime and $y \notin K$ implies $x \in K$. Hence $P \subseteq K$. Thus $K = P$. Therefore, $P$ is a minimal prime ideal containing $J$.

Conversely, let $P$ be a minimal prime ideal containing $J$. Let $x \in P$. Suppose for all $y \in S \setminus P$, $x \land y \notin J$. Set $D = (S \setminus P) \lor [x]$. We claim that $0 \notin D$. For if $0 \in D$, then $0 = q \land x$ for some $q \in S \setminus P$. Thus, $x \land q = 0 \in J$ which is a contradiction. Therefore, $0 \notin D$. Since $(0) = 1^+$ by Theorem 3.3, there is a prime filter $Q$ such that $D \subseteq Q$ and $0 \notin Q$. Let $M = S \setminus Q$. Then by the definition of prime filter of a semilattice, $M$ is a prime ideal. We claim that $M \cap D = \emptyset$. If $a \in M \cap D$, then $a \in M$ and hence $a \notin Q$. Thus $a \notin D$ which is a contradiction. Hence $M \cap D = \emptyset$. Therefore, $M \cap (S \setminus P) = \emptyset$ and hence $M \subseteq P$. Also $M \neq P$, because $x \in D$ implies $x \in Q$ and hence $x \notin M$. 

\[\blacksquare\]
but $x \in P$. This shows that $P$ is not minimal which is a contradiction. Hence the given condition holds.

We enclose the paper with the following useful characterization of minimal prime ideal.

**Theorem 4.3.** Let $S$ be a pseudocomplemented semilattice and let $P$ be a prime ideal of $S$. Then the followings are equivalent:

(i) $P$ is minimal.

(ii) $x \in P$ implies that $x^* \notin P$.

Proof. (i)$\Rightarrow$(ii). Let $P$ be a minimal prime ideal and let $x^* \in P$ for some $x \in P$. Set $D = (S \setminus P) \lor [x]$. We claim that $0 \notin D$. For if $0 \in D$, then $0 = q \land x$ for some $q \in S \setminus P$, which implies $q \leq x^* \in P$ which is a contradiction. Therefore, $0 \notin D$. Since $[0] = 1^+$ by Theorem 3.3, there is a prime filter $Q$ such that $D \subseteq Q$ and $0 \notin Q$. Let $M = S \setminus Q$. Then by the definition of prime filter of a semilattice, $M$ is a prime ideal. We claim that $M \cap D = \emptyset$. If $a \in M \cap D$, then $a \in M$ and hence $a \notin Q$. Thus $a \notin D$ which is a contradiction. Hence $M \cap D = \emptyset$. Therefore, $M \cap (S \setminus P) = \emptyset$ and hence $M \subseteq P$. Also $M \neq P$, because $x \in D$ implies $x \in Q$ and hence $x \notin M$ but $x \in P$. This shows that $P$ is not minimal which is a contradiction. Hence (ii) holds.

(ii)$\Rightarrow$(i). Let $P$ be a prime ideal of $S$ such that (ii) holds. We shall show that $P$ is a minimal prime ideal. Let $K$ be a prime ideal satisfying (ii) such that $K \subseteq P$. Let $x \in P$. Then $x \land x^* = 0 \in K$. Since $K$ is prime and $x^* \notin K$ implies $x \in K$. Hence $P \subseteq K$. Thus $K = P$. Therefore, $P$ is a minimal prime ideal.

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