IDEALS IN THE TERNARY SEMIRING OF NON-POSITIVE INTEGERS

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Abstract. Characterizations of prime ideals, semiprime ideals, irreducible k-ideals and irreducible principal T-ideals in the ternary semiring of non-positive integers are investigated.

1. Introduction

Generalizing the notion of ternary ring introduced by Lister [15], Dutta and Kar [4] introduced the notion of ternary semiring. A non-empty set S together with a binary operation called addition $(+)$ and a ternary operation called ternary multiplication $(\cdot)$ is called ternary semiring if it satisfies the following conditions for all $a, b, c, d, e \in S$:

1. $(a+b)+c = a+(b+c)$;
2. $a+b = b+a$;
3. $(a \cdot b \cdot c) \cdot d \cdot e = a \cdot (b \cdot c \cdot d) \cdot e = a \cdot b \cdot (c \cdot d \cdot e)$;
4. there exists $0 \in S$ such that $a+0 = a = 0+a$, $a \cdot b \cdot 0 = a \cdot 0 \cdot b = 0 \cdot a \cdot b = 0$;
5. $(a+b) \cdot c \cdot d = a \cdot c \cdot d + b \cdot c \cdot d$;
6. $a \cdot (b+c) \cdot d = a \cdot b \cdot d + a \cdot c \cdot d$;
7. $a \cdot b \cdot (c+d) = a \cdot b \cdot c + a \cdot b \cdot d$.

Clearly, every semiring is a ternary semiring. Denote the sets of all non-positive, negative, and positive integers respectively by $\mathbb{Z}_0^-$, $\mathbb{Z}^-$, and $\mathbb{N}$. The set $\mathbb{Z}_0^-$ is a ternary semiring under usual addition and ternary multiplication of non-positive integers but it is not a semiring.

If there exists an element e in a ternary semiring S such that $eex = exe = xee = x$ for all $x \in S$, then e is called the identity element of S. A ternary semiring S is said to be commutative if $abc = acb = cab$ for all $a, b, c \in S$. The ternary semiring $(\mathbb{Z}_0^-, +, \cdot)$ is commutative with identity element -1. A non-empty subset I of a commutative ternary semiring S is called an ideal of S if the following conditions are satisfied:

1. $a, b \in I$ implies $a+b \in I$;
2. $a \in I, r, s \in S$ implies $rsa \in I$.

An ideal I of a ternary semiring S is called a k-ideal (= subtractive ideal) if a, $a+b \in I$, $b \in S$, then $b \in I$. If S is a commutative ternary semiring with identity element, then a proper ideal I of S is called i) prime if $abc \in I$, $a, b, c \in S$ implies $a \in I$ or $b \in I$ or $c \in I$; ii) semiprime if $a^3 \in I$, $a \in S$ implies $a \in I$. Clearly, every prime ideal is a semiprime ideal. The concept of irreducible ideals in a ternary semirings can be defined

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on the similar lines as in semirings and rings. If $n \in (\mathbb{Z}_0^-, +, \cdot)$ and $n \leq -2$, then n can be written as

$$\begin{aligned} n &= (-p_1)^{r_1}(-p_2)^{r_2}\cdots(-p_k)^{r_k}(-1)^{r_1}(-1)^{r_2}\cdots(-1)^{r_k}(-1) \\ &= (-p_1)^{r_1}(-p_2)^{r_2}\cdots(-p_k)^{r_k}(-1)^{(\sum_{i=1}^{k} r_i)+1} \end{aligned}$$

where $p_1,\ p_2,\ \cdots,\ p_k \in \mathbb{N}$ are pairwise distinct primes and $r_i, k \in \mathbb{N}$. An ideal I of $(\mathbb{Z}_0^-, +, \cdot)$ is said to be generated by a subset $A = \{a_1, a_2, \cdots, a_n\}$ of $\mathbb{Z}_0^-$ if for every $x \in I$, there exist $\alpha_i, \beta_i \in \mathbb{Z}_0^-$ such that $x = \sum_{i=1}^{n} \alpha_i\beta_i a_i$. If $A = \{a\}$, then $\mathbb{Z}_0^-\mathbb{Z}_0^- a$ is called a principal ideal generated by a. For $a_1, a_2, \cdots, a_k \in \mathbb{Z}_0^-$, we denote i) $\langle a_1, a_2, \cdots, a_k\rangle$ = the ideal generated by $a_1, a_2, \cdots, a_k$ in the ternary semiring $\mathbb{Z}_0^-$; ii) $(a_1, a_2, \cdots, a_k)$ = g.c.d. of $a_1, a_2, \cdots, a_k$. Two elements $a_1, a_2 \in \mathbb{Z}_0^-$ are said to be relatively prime if $(a_1, a_2) = 1$. For $n \in \mathbb{Z}^-$, we denote $I_n = \{r \in \mathbb{Z}^- : r \leq n\} \cup \{0\}$. Clearly I_n is an ideal in the ternary semiring $\mathbb{Z}_0^-$. An ideal I of $\mathbb{Z}_0^-$ is called i) a T-ideal if $I_n \subseteq I$ for some $n \in \mathbb{Z}^-$; ii) a principal T-ideal if $I = \langle a\rangle \cup I_n$ for some $a \in \mathbb{Z}_0^-$ and $n \in \mathbb{Z}^-$. Further i) for $n \in \mathbb{Z}^-$, we denote $n+1$ as the immediate successor of n in $\mathbb{Z}_0^-$; ii) for $n \in \mathbb{Z}^- - \{-1\}$, we denote $n+2$ as the immediate successor of $n+1$ in $\mathbb{Z}_0^-$. For example, $-5 = (-6)+1$ is the immediate successor of -6 and $-4 = (-6)+2$ is the immediate successor of $(-6)+1$ $(= -5)$.

Dutta and Kar [6, 7] have characterized respectively the prime k-ideals and semiprime k-ideals of the ternary semiring of non-positive integers. Some works on ternary semirings may be found in [2, 5, 8, 9]. Theory of ideals in the semiring of non-negative integers is recently studied by Gupta and Chaudhari [11] and by Chaudhari and Ingale [3]. Theory of ideals in the ternary semiring of non-positive integers is studied by Kar [13].

In this paper, we obtain characterizations of prime ideals, semiprime ideals, irreducible k-ideals and irreducible principal T-ideals in the ternary semiring of non-positive integers. In Section 2, we obtain characterizations of prime ideals, semiprime ideals and irreducible k-ideals in the ternary semiring of non-positive integers. In section 3, we obtain characterization of irreducible principal T-ideals in the ternary semiring of non-positive integers.

The following results will be used to prove our results.

Lemma 1.1. [13, Lemma 3.12] *Let $I = \langle a_1, a_2, \cdots, a_n\rangle \subseteq \mathbb{Z}_0^-$. If $(a_1, a_2, \cdots, a_n) = d$, then there exists a largest $t \in \mathbb{Z}_0^-$ such that $(-1)(-d)r \in I$ for all $r \leq t$.*

Lemma 1.2. [13, Lemma 3.3] *If $a, b \in \mathbb{Z}_0^-$ are relatively prime, then there exist $p, q \in \mathbb{Z}_0^-$ such that $(-1)qa = (-1)pb + (-1)$ or $(-1)pb = (-1)qa + (-1)$.*

Theorem 1.3. [13] *Every ideal of $\mathbb{Z}_0^-$ is finitely generated.*

Theorem 1.4. [6, Theorem 5.5] *An ideal I of $\mathbb{Z}_0^-$ is a k-ideal if and only if I is a principal ideal.*

Theorem 1.5. [2, Theorem 3.8] *An ideal I of $\mathbb{Z}_0^-$ is semiprime if and only if $a^n \in I$ where n is an odd natural number implies $a \in I$.*

2. Prime ideals, semiprime ideals and irreducible k-ideals in $\mathbb{Z}_0^-$

In this section, we characterize prime ideals, semiprime ideals and irreducible k-ideals in the ternary semiring $\mathbb{Z}_0^-$. We give a short and elementary proof of [13, Lemma 3.4]. This lemma will be used in the proof of subsequent theorem.

Lemma 2.1. *Let $a, b \in \mathbb{Z}_0^-$, $b < a < -1$ and let a, b be relatively prime. Then there exists $m \in \mathbb{Z}_0^-$ such that $t \in \langle a, b \rangle$ for all $t \leq m$.*

Proof. By Lemma 1.2, there exist $p, q \in \mathbb{Z}_0^-$ such that $(-1)qa = (-1)pb+(-1)$ or $(-1)pb = (-1)qa + (-1)$. Without loss of generality assume that $(-1)qa = (-1)pb + (-1)$. Clearly $p, q \neq 0$. Let us write $m = (-1)paqa \in \langle a, b \rangle$. Let $t = m + r$ where $r \leq 0$. If $r = 0$, then $t = m \in \langle a, b \rangle$. If $a < r < 0$, then

$$\begin{aligned} t &= m + r = (-1)paqa + r = pa(-1)qa + r \\ &= pa((-1)pb + (-1)) + r = -(pa + r)pb + (-1)pa + rpb + r \\ &= -(pa + r)pb + (-1)pa + rqa \in \langle a, b \rangle. \end{aligned}$$

If $r \leq a$, then by the division algorithm $r = (-1)au + v$ where $u, v \in \mathbb{Z}_0^-$ and $a < v \leq 0$. Then $t = m + v + (-1)au \in \langle a, b \rangle$. ∎

Now the following theorem gives a characterization of non-zero prime ideals in the ternary semiring $\mathbb{Z}_0^-$:

Theorem 2.2. *A non-zero ideal I of the ternary semiring $\mathbb{Z}_0^-$ is prime if and only if $I = \langle -p \rangle$ for some prime number $p \in \mathbb{N}$ or $I = \langle -2, -3 \rangle$.*

Proof. Let I be a prime ideal. By Theorem 1.3, I is a finitely generated ideal. If I is a principal ideal say $I = \langle m \rangle$, $m < -1$, then let

$$m = (-1)^{(\sum_{i=1}^{k} r_i)+1}(-p_1)^{r_1}(-p_2)^{r_2} \cdots (-p_k)^{r_k}$$

where $p_1, p_2, \cdots, p_k \in \mathbb{N}$ are pairwise distinct primes and $r_i, k \in \mathbb{N}$. If $k \geq 2$, then $(-1)ab = m \in I$ where $a = (-1)^{r_1+1}(-p_1)^{r_1}$, $b = (-1)^{(\sum_{i=2}^{k} r_i)+1}(-p_2)^{r_2} \cdots (-p_k)^{r_k}$. Since I is a prime ideal, we have $-1 \in I$ or $a \in I$ or $b \in I$, a contradiction. So $k = 1$ and hence $m = (-1)^{r_1+1}(-p_1)^{r_1}$. Again if $r_1 \geq 2$, then $(-1)^{r_1+1}(-p_1)^{r_1} \in I$. Since I is a prime ideal, we have $-1 \in I$ or $-p_1 \in I$, a contradiction. So $r_1 = 1$ and hence $I = \langle -p \rangle$.

Now assume that I is not a principal ideal. Take $I = \langle a_1, a_2, \cdots, a_n \rangle$ where $a_n < a_{n-1} < \cdots < a_1 < -1$, a_i does not divide a_j for all $i < j$, $j = 2, 3, \cdots, n$, $n \geq 2$. By using the procedure as in the above part, we have $a_1 = -p$ for some prime number $p \in \mathbb{N}$. Then a_1, a_2 are relatively prime. By Lemma 2.1, there exists $m \in \mathbb{Z}_0^-$ such that

$$t \in \langle a_1, a_2 \rangle \subseteq I \quad \text{for all} \quad t \leq m. \tag{2.1}$$

If $a_1 < -2$, then by (2.1), choose a smallest j such that $(-1)^{j+1}(-2)^j \in I$, $j > 1$. Since I is a prime ideal, $-1 \in I$ or $-2 \in I$, a contradiction. Hence $a_1 = -2$. If $a_2 < -3$, then by (2.1), choose a smallest s such that $(-1)^{s+1}(-3)^s \in I$, $s > 1$. Since I is a prime ideal, $-1 \in I$ or $-3 \in I$, a contradiction. Hence $a_2 = -3$. So $I = \langle -2, -3 \rangle$. The converse is trivial. ∎

From Theorem 1.4 and Theorem 2.2, we have the following corollary in which characterization of non-zero prime k-ideals in the ternary semiring $\mathbb{Z}_0^-$ is obtained. This corollary shows that the [6, Lemma 5.9] is not true for the ideal $\{0\}$ where $\{0\}$ is a prime k-ideal.

Corollary 2.3. *A non-zero k-ideal I of the ternary semiring $\mathbb{Z}_0^-$ is prime if and only if $I = \langle -p \rangle$ for some prime number $p \in \mathbb{N}$.*

Now the following theorem gives a characterization of non-zero semiprime ideals in the ternary semiring $\mathbb{Z}_0^-$:

Theorem 2.4. *A non-zero ideal I of the ternary semiring $\mathbb{Z}_0^-$ is semiprime if and only if $I = \langle m \rangle$ where $m = (-1)^{k+1}(-p_1)(-p_2)\cdots(-p_k)$, $p_1, p_2, \cdots, p_k \in \mathbb{N}$ are pairwise distinct primes or $I = \langle -2, -3 \rangle$.*

Proof. Let I be a semiprime ideal. By Theorem 1.3, I is a finitely generated ideal. If I is a principal ideal say $I = \langle m \rangle$, $m < -1$, then let

$$m = (-1)^{(\sum_{i=1}^{k} r_i)+1}(-p_1)^{r_1}(-p_2)^{r_2}\cdots(-p_k)^{r_k}$$

where $p_1, p_2, \cdots, p_k \in \mathbb{N}$ are pairwise distinct primes and $r_i, k \in \mathbb{N}$. If $r_i \geq 2$ for some i, then $(\frac{m}{(-1)(-p_i)})^3 \in I$ but $\frac{m}{(-1)(-p_i)} \notin I$, a contradiction. Hence each $r_i = 1$. Now assume that I is not a principal ideal. Take $I = \langle a_1, a_2, \cdots, a_n \rangle$ where $a_n < a_{n-1} < \cdots < a_1 < -1$, a_i does not divide a_j for all $i < j$, $j = 2, 3, \cdots, n$ and $n \geq 2$. Let $d = (a_1, a_2, \cdots, a_n)$. If $-d < -1$, then let

$$-d = (-1)^{(\sum_{i=1}^{k} r_i)+1}(-p_1)^{r_1}(-p_2)^{r_2}\cdots(-p_k)^{r_k}$$

where $p_1, p_2, \cdots, p_k \in \mathbb{N}$ are pairwise distinct primes and $r_i \geq 1$ for all i. If $a_1 < -d$, then by Lemma 1.1, choose a smallest odd $t \in \mathbb{N}$ such that $(-d)^t \in I$. By Theorem 1.5, $-d \in I$, a contradiction as $a_1 < -d$. If $-d = a_1$, then $a_1 \mid a_2$, a contradiction. Hence $-d = -1$. If $a_1 < -2$, then by Lemma 1.1, choose a smallest odd j such that $(-2)^j \in I$, $j > 1$. By Theorem 1.5, $-2 \in I$, a contradiction as $a_1 < -2$. Hence $a_1 = -2$. If $a_2 < -3$, then by Lemma 1.1, choose a smallest s such that $(-3)^s \in I$, $s > 1$. By Theorem 1.5, $-3 \in I$, a contradiction as $a_2 < -3$. Hence $a_2 = -3$. Now $\langle -2, -3 \rangle \subseteq I$ implies $I = \langle -2, -3 \rangle$.

Conversely, If $I = \langle m \rangle$ where $m = (-1)^{k+1}(-p_1)(-p_2)\cdots(-p_k)$, $p_1, p_2, \cdots, p_k \in \mathbb{N}$ are pairwise distinct primes and $a^3 \in I$, then clearly $m|a^3$ implies $m|a$. Hence $a \in \langle m \rangle = I$. If $I = \langle -2, -3 \rangle$, then by Theorem 2.2, I is a prime ideal and hence I is a semiprime ideal. ∎

From Theorem 1.4 and Theorem 2.4, we have the following corollary in which characterization of non-zero semiprime k-ideals in the ternary semiring $\mathbb{Z}_0^-$ is obtained. This corollary shows that [7, Theorem 5.5] is not true for the ideal $\{0\}$ where $\{0\}$ is a semiprime k-ideal.

Corollary 2.5. *A non-zero k-ideal I of the ternary semiring $\mathbb{Z}_0^-$ is semiprime if and only if $I = \langle (-1)^{k+1}(-p_1)(-p_2)\cdots(-p_k) \rangle$ where $p_1, p_2, \cdots, p_k \in \mathbb{N}$ are pairwise distinct primes.*

Now the following theorem gives a characterization of non-zero irreducible k-ideals in the ternary semiring $\mathbb{Z}_0^-$:

Theorem 2.6. *A non-zero proper ideal I in the semiring $\mathbb{Z}_0^-$ is an irreducible k-ideal if and only if $I = \langle (-1)^{n+1}(-p)^n \rangle$ for some prime number $p \in \mathbb{N}$ and for some $n \in \mathbb{N}$.*

Proof. Let I be an irreducible k-ideal of $\mathbb{Z}_0^-$. By Theorem 1.4, $I = \langle m \rangle$ for some $m < -1$. Since I is an irreducible ideal, $I = \langle (-1)^{n+1}(-p)^n \rangle$ for some prime number $p \in \mathbb{N}$ and for some $n \in \mathbb{N}$. Conversely suppose that $I = \langle (-1)^{n+1}(-p)^n \rangle$ for some prime number $p \in \mathbb{N}$

and for some $n \in \mathbb{N}$. By Theorem 1.4, I is a k-ideal. If I is not an irreducible ideal, then there exist ideals A, B of $\mathbb{Z}_0^-$ such that $I = A \cap B$ and $I \neq A$, $I \neq B$. Let $a \in A$, $b \in B$ be such that $a, b \notin I$. If $a = -1$ or $b = -1$, then $A = \mathbb{Z}_0^-$ or $B = \mathbb{Z}_0^-$ and hence $I = B$ or $I = A$, a contradiction. Suppose that $a < -1$, $b < -1$. Let

$$a = (-1)^{(\sum_{i=1}^{k} \alpha_i)+\alpha+1}(-p_1)^{\alpha_1}(-p_2)^{\alpha_2}\cdots(-p_k)^{\alpha_k}(-p)^{\alpha},$$
$$b = (-1)^{(\sum_{i=1}^{k} \beta_i)+\beta+1}(-p_1)^{\beta_1}(-p_2)^{\beta_2}\cdots(-p_k)^{\beta_k}(-p)^{\beta}$$

where $p_1, p_2, \cdots, p_k, p$ are pairwise distinct primes and $\alpha_i, \alpha, \beta_i, \beta \geq 0$. Now $a, b \notin I$ implies α, $\beta < n$. Denote $l = (-1)^{(\sum_{i=1}^{k} \lambda_i)+\lambda+1}\ (-p_1)^{\lambda_1}(-p_2)^{\lambda_2}\cdots(-p_k)^{\lambda_k}(-p)^{\lambda}$ where $\lambda_i = \max\{\alpha_i, \beta_i\}$, $\lambda = \max\{\alpha, \beta\}$. Then $a \in A$, $b \in B$ implies $l \in A \cap B = I = \langle(-1)^{n+1}(-p)^n\rangle$. So $(-1)^{n+1}(-p)^n | l$. Hence $n \leq \lambda$, a contradiction. So I is an irreducible ideal. ∎

Corollary 2.7. *Let I be a non-zero proper ideal in the ternary semiring $\mathbb{Z}_0^-$. Then the following statements are equivalent:*

1) *I is a prime k-ideal;*
2) *$I = \langle -p\rangle$ for some prime $p \in \mathbb{N}$;*
3) *I is an irreducible and semiprime k-ideal.*

Proof. (1) ⇒ (2) Follows from Corollary 2.3.

(2) ⇒ (3) By Corollary 2.3, I is a prime k-ideal and hence I is a semiprime ideal. Clearly every prime ideal is an irreducible ideal and hence I is an irreducible ideal.

(3) ⇒ (1) Follows from Theorem 2.6, Corollary 2.5 and Corollary 2.3. ∎

3. Irreducible principal T-ideals in $\mathbb{Z}_0^-$

In this section, we characterize irreducible principal T-ideals in the ternary semiring $\mathbb{Z}_0^-$. In general, the union of two ideals in a commutative ternary semiring S may not be an ideal of S. But for any ideal I_n in the ternary semiring $\mathbb{Z}_0^-$, we have the following lemma:

Lemma 3.1. *If A is an ideal of the ternary semiring $\mathbb{Z}_0^-$, then $A \cup I_n$ is an ideal of $\mathbb{Z}_0^-$.*

Theorem 3.2. *I_n is an irreducible ideal if and only if $n \geq -3$.*

Proof. Let I_n be an irreducible ideal. Suppose that $n \leq -4$. Denote $A = \langle n+1\rangle \cup I_n$ and $B = \langle n+2\rangle \cup I_n$. By Lemma 3.1, A, B are ideals of $\mathbb{Z}_0^-$ such that $I_n \neq A$ and $I_n \neq B$. Clearly, $I_n = A \cap B$. Hence I_n is a reducible ideal, a contradiction. So $n \geq -3$. Conversely suppose that $n \geq -3$. Clearly $I_{-1} = \mathbb{Z}_0^-$ and $I_{-2} = \langle -2, -3\rangle$ are irreducible ideals. Now if $I_{-3} = A \cap B$ and $I_{-3} \subset A$, then $A = \mathbb{Z}_0^-$ or $A = I_{-2}$ and hence $B = I_{-3}$. ∎

Corollary 3.3. *A principal T-ideal $I = \langle m\rangle \cup I_n$ is irreducible in $\mathbb{Z}_0^-$ for $n \geq -3$ and for every $m \in \mathbb{Z}_0^-$.*

Proof. Clearly $\langle m\rangle \subseteq I_n$ for all $m \leq n$. So $I = I_{-1}$ or I_{-2} or I_{-3} for $n \geq -3$ and every $m \in \mathbb{Z}_0^-$. By Theorem 3.2, I is an irreducible ideal. ∎

Theorem 3.4. *Every ideal $I \supseteq \langle -2\rangle$ of $\mathbb{Z}_0^-$ is irreducible.*

Proof. Let $A \neq I \neq B$ be ideals of $\mathbb{Z}_0^-$ such that $I = A \cap B$. Then there are $a \in A$ and $b \in B$ such that $a, b \notin I \supseteq \langle -2 \rangle$. Hence a and b are odd negative integers. We may assume that $a \geq b$ and therefore $b = a + (-1)(-2)r$ for some $r \in \mathbb{Z}_0^-$. Since $(-1)(-2)r \in \langle -2 \rangle \subseteq I \subseteq A$, we get $b \in A$ and therefore $b \in A \cap B = I$, a contradiction. Hence either $I = A$ or $I = B$. ∎

Now we prove the following lemmas which will be used in the subsequent theorems.

Lemma 3.5. *If $a, b \in \mathbb{Z}_0^-$ and $n \in \mathbb{Z}^-$ are such that $a + b \leq n$, then $A = \langle a \rangle \cup \langle b \rangle \cup I_n$ is an ideal of $\mathbb{Z}_0^-$.*

Proof. Let $x, y \in A$. If x and y satisfy at least one of the following: i) $x = 0$ or $y = 0$; ii) $x \leq n$ or $y \leq n$; iii) $x, y \in \langle a \rangle$ or $x, y \in \langle b \rangle$, then clearly $x + y \in A$. Now without loss of generality assume that $n < x < 0$, $n < y < 0$ and $x \in \langle a \rangle$, $y \in \langle b \rangle$. Then $x + y = (-1)ra + (-1)tb \leq a + b \leq n$ for some $r, t \in \mathbb{Z}^-$. Hence $x + y \in A$. If $\alpha, \beta \in \mathbb{Z}_0^-$ and $x \in A$, then clearly $\alpha\beta x \in A$. Hence A is an ideal of $\mathbb{Z}_0^-$. ∎

Lemma 3.6. *Let I be an ideal in the ternary semiring $\mathbb{Z}_0^-$ such that $I \supseteq I_n$ where $n \in \mathbb{Z}^-$. If $b, c \in \mathbb{Z}_0^-$ are such that $(-1)(-2)b \leq n$, $b \leq c$ and c does not divide b, then $I = (\langle b \rangle \cup I) \cap (\langle c \rangle \cup I)$.*

Proof. Clearly $(-1)(-2)b \leq n$ implies

$$(-1)rb \in I_n \subseteq I \quad \text{for all} \quad r \leq -2. \tag{3.1}$$

Let $x \in (\langle b \rangle \cup I) \cap (\langle c \rangle \cup I)$. If $x \notin I$, then clearly $x = (-1)rb = (-1)tc$ for some $r, t \in \mathbb{Z}^-$. By (3.1), $r = -1$ and hence $b = (-1)tc$ i.e. $c \mid b$, a contradiction. Hence $x \in I$. So $(\langle b \rangle \cup I) \cap (\langle c \rangle \cup I) \subseteq I$. Clearly $I \subseteq (\langle b \rangle \cup I) \cap (\langle c \rangle \cup I)$. Hence $I = (\langle b \rangle \cup I) \cap (\langle c \rangle \cup I)$. ∎

The following two theorems are essential to obtain the characterization of the irreducible principal T-ideals in the ternary semiring $\mathbb{Z}_0^-$.

Theorem 3.7. *Let $I = \langle -3 \rangle \cup I_n$ be a principal T-ideal in $\mathbb{Z}_0^-$. Then I is an irreducible ideal if and only if $n > -6$.*

Proof. Let I be an irreducible ideal of $\mathbb{Z}_0^-$. Let if possible $n \leq -6$. Choose smallest $k \in \mathbb{Z}^-$ such that $n < (-1)(-3)k$. Then $-3 \leq n - (-1)(-3)k \leq -1$.

If $n - (-1)(-3)k = -1$, then denote $A = \langle (-1)(-3)k + 1 \rangle \cup I$ and $B = \langle (-1)(-3)k + 2 \rangle \cup I$. Now $(-3) + ((-1)(-3)k + 1) \leq (-3) + ((-1)(-3)k + 2) = (-1) + (-1)(-3)k = n$. Hence by Lemma 3.5, A and B are ideals of $\mathbb{Z}_0^-$. Since $n < (-1)(-3)k$, $A \neq I$ and $B \neq I$. Now $(-1)(-3)k + 1 = n + 2$, $n \leq -6$ implies $(-1)(-2)((-1)(-3)k + 1) = (-1)(-2)(n + 2) = n + (n + 4) \leq n$. If $((-1)(-3)k + 2) \mid ((-1)(-3)k + 1)$, then $(-1)(-3)k + 2 = -1$. Hence $k = -1$ and so $n = -4$, a contradiction. Now $((-1)(-3)k + 2)$ does not divide $((-1)(-3)k + 1)$. Hence by Lemma 3.6, $I = (\langle (-1)(-3)k + 1 \rangle \cup I) \cap (\langle (-1)(-3)k + 2 \rangle \cup I) = A \cap B$.

If $n - (-1)(-3)k = -2$, then denote $A = \langle (-1)(-3)k - 1 \rangle \cup I$ and $B = \langle (-1)(-3)k + 1 \rangle \cup I$. Now $(-3) + ((-1)(-3)k - 1) \leq (-3) + ((-1)(-3)k + 1) = (-2) + (-1)(-3)k = n$. Hence by Lemma 3.5, A and B are ideals of $\mathbb{Z}_0^-$. Now $A \neq I$, $B \neq I$ as $n < n + 1 = (-1)(-3)k - 1 < (-1)(-3)k + 1$. Now $n - (-1)(-3)k = -2$ and $n \leq -6$ implies $k \leq -2$. Hence $((-1)(-3)k + 1)$ does not divide $((-1)(-3)k - 1)$. Clearly $(-1)(-2)((-1)(-3)k - 1) \leq n$. Hence by Lemma 3.6, $I = (\langle (-1)(-3)k - 1 \rangle \cup I) \cap (\langle (-1)(-3)k + 1 \rangle \cup I) = A \cap B$.

If $n-(-1)(-3)k=-3$, then denote $A=\langle(-1)(-3)k-2\rangle\cup I$ and $B=\langle(-1)(-3)k-1\rangle\cup I$. Now $(-3)+((-1)(-3)k-2)\leq(-3)+((-1)(-3)k-1)=((-3)+(-1)(-3)k)-1=n-1\leq n$. Hence by Lemma 3.5, A and B are ideals of $\mathbb{Z}_0^-$. Now $A\neq I$ and $B\neq I$ as $(-1)(-3)k-1>(-1)(-3)k-2=n+1>n$. Clearly $(-1)(-2)((-1)(-3)k-2)\leq n$ and $((-1)(-3)k-1)$ does not divide $((-1)(-3)k-2)$. Hence by Lemma 3.6, $I=(\langle(-1)(-3)k-2\rangle\cup I)\cap(\langle(-1)(-3)k-1\rangle\cup I)=A\cap B$. Thus in any case, I is not an irreducible ideal of $\mathbb{Z}_0^-$, a contradiction. Hence $n>-6$.

Conversely, suppose that $n>-6$. If $n\geq-4$, then $I=I_{-1}$ or I_{-2} or I_{-3} which are irreducible ideals. Suppose that $n=-5$. If I is not an irreducible ideal, then there exist ideals A, B of $\mathbb{Z}_0^-$ such that $I=A\cap B$ and $I\neq A$, $I\neq B$. Choose $a\in A$, $b\in B$ such that $a,b\notin I$. Without loss of generality assume that $a\geq b$. Clearly $a=-1$ or -2 or -4 and $b=-1$ or -2 or -4 and hence $b=(-1)at$ for some $t\in\mathbb{Z}_0^-$. Now $b\in A\cap B=I$, a contradiction. So I is an irreducible ideal. ∎

Theorem 3.8. *If $a<-3$, $n<-3$ and $I=\langle a\rangle\cup I_n$ a principal T-ideal in $\mathbb{Z}_0^-$, then I is not an irreducible ideal.*

Proof. If $a\leq n$, then $I=I_n$ and so by Theorem 3.2, I is not an irreducible ideal. Suppose that $a>n$. Choose smallest $k\in\mathbb{Z}^-$ such that

$$(-1)ak>n. \tag{3.2}$$

Then $n-(-1)ak\leq-1$. If $n-(-1)ak=-1$, then denote $A=\langle(-1)ak+1\rangle\cup I$ and $B=\langle(-1)ak+2\rangle\cup I$. If $a+((-1)ak+2)>n$, then $a+2>n-(-1)ak=-1$ which is impossible as $a<-3$. Hence $a+((-1)ak+2)\leq n$ and so $a+((-1)ak+1)\leq n$. By Lemma 3.5, A and B are ideals of $\mathbb{Z}_0^-$. Since $n<(-1)ak$, $(-1)ak+1\notin I_n$ and $(-1)ak+2\notin I_n$. Hence $(-1)ak+1\notin I$ and $(-1)ak+2\notin I$. So $A\neq I$ and $B\neq I$. If $(-1)(-2)((-1)ak+1)>n$, then $(-2)ak+2>n$. Hence $(-1)ak+2>n-(-1)ak=-1$. So $(-1)ak=-2$ which is impossible as $a<-3$ and $k\in\mathbb{Z}^-$. Hence $(-1)(-2)((-1)ak+1)\leq n$. Now $((-1)ak+2)$ does not divide $((-1)ak+1)$ as $a<-3$, $k\in\mathbb{Z}^-$. Hence by Lemma 3.6, $I=(\langle(-1)ak+1\rangle\cup I)\cap(\langle(-1)ak+2\rangle\cup I)=A\cap B$.

If $n-(-1)ak=-2$, then denote $A=\langle(-1)ak-1\rangle\cup I$ and $B=\langle(-1)ak+1\rangle\cup I$. If $a+((-1)ak+1)>n$, then $a+1>n-(-1)ak=-2$, a contradiction as $a<-3$. Hence $a+((-1)ak+1)\leq n$ and so $a+((-1)ak-1)\leq n$. By Lemma 3.5, A and B are ideals of $\mathbb{Z}_0^-$. Since $n-(-1)ak=-2$, $(-1)ak-1\notin I_n$ and $(-1)ak+1\notin I_n$. Hence $(-1)ak-1\notin I$ and $(-1)ak+1\notin I$. So $A\neq I$ and $B\neq I$. By using (3.2), $(-1)(-2)((-1)ak+1)\leq n$. Now $((-1)ak+1)$ does not divide $((-1)ak-1)$ as $a<-3$. Hence by Lemma 3.6, $I=(\langle(-1)ak-1\rangle\cup I)\cap(\langle(-1)ak+1\rangle\cup I)=A\cap B$.

If $n-(-1)ak\leq-3$, then denote $A=\langle(-1)ak-2\rangle\cup I$ and $B=\langle(-1)ak-1\rangle\cup I$. By using (3.2), $a+((-1)ak-2)\leq a+((-1)ak-1)\leq n$. By Lemma 3.5, A and B are ideals of $\mathbb{Z}_0^-$. Since $n-(-1)ak\leq-3$, $(-1)ak-2\notin I_n$ and $(-1)ak-1\notin I_n$. Hence $(-1)ak-2\notin I$ and $(-1)ak-1\notin I$. So $A\neq I$ and $B\neq I$. By using (3.2), $(-1)(-2)((-1)ak-1)\leq n$. Also $((-1)ak-1)$ does not divide $((-1)ak-2)$. Hence by Lemma 3.6, $I=(\langle(-1)ak-2\rangle\cup I)\cap(\langle(-1)ak-1\rangle\cup I)=A\cap B$. Thus, I is not an irreducible ideal of $\mathbb{Z}_0^-$. ∎

The following theorem gives the characterization of irreducible principal T-ideals in the ternary semiring $\mathbb{Z}_0^-$.

Theorem 3.9. *A principal T-ideal $I = \langle a \rangle \cup I_n$ is irreducible in $\mathbb{Z}_0^-$ if and only if any one of the following conditions holds:*

1) $a = 0$, $n \geq -3$;
2) $a = -1$, for any n;
3) $a = -2$, for any n;
4) $a = -3$, $n > -6$;
5) $a \leq -4$, $n \geq -3$.

Proof. Follows from Theorem 3.2, Theorem 3.4, Theorem 3.7, Theorem 3.8 and Corollary 3.3. ∎

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