Existence of positive solutions for a three-point boundary value problem with fractional q-differences

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Abstract

In this paper, we consider the following nonlinear q-fractional three-point boundary value problem

$$
\begin{aligned}
&(D_q^\alpha u)(t) + f(t, u(t)) = 0, \quad 0 < t < 1, \quad 2 < \alpha < 3, \\
&u(0) = (D_q u)(0) = 0, \quad (D_q u)(1) = \beta (D_q u)(\eta),
\end{aligned}
$$

where $0 < \beta\eta^{\alpha-2} < 1$. By the properties of the Green function and the lower and upper solution method, some new existence to the above boundary value problem are established. As applications, examples are presented to illustrate the main results.

Keywords: Fractional q-difference equations; Lower and upper solution method; Positive solution

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1. Introduction

Recently, an increasing interest in studying the existence of solutions for boundary value problems of fractional order functional differential equations has been observed [1–11]. Fractional differential equations describe many phenomena in various fields of science and engineering such as physics, mechanics, chemistry, control, engineering, etc. For an extensive collection of such results, we refer the readers to the monographs by Samko et al [12], Podlubny [13] and Kilbas et al [14].

On the other hand, the q-difference calculus or quantum calculus is an old subject that was first developed by Jackson [15, 16]. It is rich in history and in applications as the reader can confirm in the paper [17].

The origin of the fractional q-difference calculus can be traced back to the works by Al-Salam [18] and Agarwal [19]. More recently, maybe due to the explosion in research within

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the fractional differential calculus setting, new developments in this theory of fractional q-difference calculus were made, e.g., q-analogues of the integral and differential fractional operators properties such as the q-Laplace transform, q-Taylor's formula [20, 21], just to mention some.

Recently, some works consider the existence of positive solutions for nonlinear q-fractional boundary value problem (see [22, 23]). As is well-known, the aim of finding positive solutions to boundary value problems is of main importance in various fields of applied mathematics (see the book [24] and references therein). In addition, since q-calculus has a tremendous potential for applications [17], we find it pertinent to investigate such a demand. To the authors' knowledge, no one has studied the existence of positive solutions for nonlinear q-fractional three-point boundary value problem (1.1) and (1.2) by using the lower and upper solution method.

In this paper, we deal with the following three-point boundary value problem

$$(D_q^\alpha u)(t) + f(t, u(t)) = 0, \quad 0 < t < 1, \quad 2 < \alpha < 3, \tag{1.1}$$

$$u(0) = (D_q u)(0) = 0, \quad (D_q u)(1) = \beta (D_q u)(\eta), \tag{1.2}$$

where $0 < \beta\eta^{\alpha-2} < 1$, $0 < q < 1$. We will prove the existence of a positive solution for the boundary value problems (1.1)-(1.2) by using the lower and upper solution method. This work is motivated by papers [22, 23, 25].

2. Preliminaries

We need the following definitions and lemmas that will be used to prove our main results. Let $q \in (0, 1)$ and define

$$[a]_q = \frac{1 - q^a}{1 - q}, \quad a \in \mathbb{R}.$$

The q-analogue of the power function $(a - b)^n$ with $\mathbb{N}_0$ is

$$(a - b)^0 = 1, \quad (a - b)^n = \prod_{k=0}^{n-1} (a - bq^k), \quad n \in \mathbb{N}, \quad a, b \in \mathbb{R}.$$

More generally, if $\alpha \in \mathbb{R}$, then

$$(a - b)^{(\alpha)} = a^\alpha \prod_{n=0}^{\infty} \frac{a - bq^n}{a - bq^{\alpha+n}}.$$

Note that, if $b = 0$ then $a^{(\alpha)} = a^\alpha$. The q-gamma function is defined by

$$\Gamma_q(x) = \frac{(1 - q)^{(x-1)}}{(1 - q)^{x-1}}, \quad x \in \mathbb{R} \setminus \{0, -1, -2, \ldots\},$$

and satisfies $\Gamma_q(x + 1) = [x]\Gamma_q(x)$.

The q-derivative of a function f is here defined by

$$(D_q f)(x) = \frac{f(x) - f(qx)}{(1 - q)x}, \quad (D_q f)(0) = \lim_{x \to 0} (D_q f)(x),$$

and q-derivatives of higher order by

$$(D_q^0 f)(x) = f(x) \quad \text{and} \quad (D_q^n f)(x) = D_q(D_q^{n-1} f)(x), \quad n \in \mathbb{N}.$$

The q-integral of a function f defined in the interval $[0, b]$ is given by

$$(I_q f)(x) = \int_0^x f(t) d_q t = x(1-q) \sum_{n=0}^{\infty} f(xq^n) q^n, \quad x \in [0, b].$$

If $a \in [0, b]$ and f is defined in the interval $[0, b]$, its integral from a to b is defined by

$$\int_a^b f(t) d_q t = \int_0^b f(t) d_q t - \int_0^a f(t) d_q t.$$

Similarly as done for derivatives, an operator I_q^n can be defined, namely,

$$(I_q^0 f)(x) = f(x) \quad \text{and} \quad (I_q^n f)(x) = I_q(I_q^{n-1} f)(x), \quad n \in \mathbb{N}.$$

The fundamental theorem of calculus applies to these operators I_q and D_q, i.e.,

$$(D_q I_q f)(x) = f(x),$$

and if f is continuous at $x = 0$, then

$$(I_q D_q f)(x) = f(x) - f(0).$$

Basic properties of the two operators can be found in the book [26]. We now point out three formulas that will be used later (${}_iD_q$ denotes the derivative with respect to variable i)

$$[a(t-s)]^{(\alpha)} = a^{\alpha}(t-s)^{(\alpha)}, \tag{2.1}$$

$${}_tD_q(t-s)^{(\alpha)} = [\alpha]_q(t-s)^{(\alpha-1)}, \tag{2.2}$$

$$\left({}_xD_q \int_0^x f(x,t) d_q t\right)(x) = \int_0^x {}_xD_q f(x,t) d_q t + f(qx, x). \tag{2.3}$$

Remark 2.1. [22] We note that if $\alpha > 0$ and $a \leq b \leq t$, then $(t-a)^{(\alpha)} \geq (t-b)^{(\alpha)}$.

The following definition was considered first in [19].

Definition 2.1. Let $\alpha \geq 0$ and f be a function defined on $[0, 1]$. The fractional q-integral of the Riemann-Liouville type is $(I_q^0 f)(x) = f(x)$ and

$$(I_q^{\alpha} f)(x) = \frac{1}{\Gamma_q(\alpha)} \int_0^x (x - qt)^{(\alpha-1)} f(t) d_q t, \quad \alpha > 0, \quad x \in [0, 1].$$

Definition 2.2 ([20])**.** The fractional q-derivative of the Riemann-Liouville type of order $\alpha \geq 0$ is defined by $(D_q^0 f)(x) = f(x)$ and

$$(D_q^{\alpha} f)(x) = (D_q^m I_q^{m-\alpha} f)(x), \quad \alpha > 0,$$

where m is the smallest integer greater than or equal to α.

Next, we list some properties that are already known in the literature. Its proof can be found in [19, 20]

Lemma 2.1. *Let $\alpha, \beta \geq 0$ and f be a function defined on $[0,1]$. Then the next formulas hold:*
(1) $(I_q^\beta I_q^\alpha f)(x) = (I_q^{\alpha+\beta} f)(x)$,
(2) $(D_q^\alpha I_q^\alpha f)(x) = f(x)$.

Lemma 2.2 ([22])**.** *Let $\alpha > 0$ and p be a positive integer. Then the following equality holds:*

$$(I_q^\alpha D_q^p f)(x) = (D_q^p I_q^\alpha f)(x) - \sum_{k=0}^{p-1} \frac{x^{\alpha-p+k}}{\Gamma_q(\alpha+k-p+1)} (D_q^k f)(0).$$

3. Related lemmas

Lemma 3.1. *Let $0 < \eta < 1$ and $\beta \neq \frac{1}{\eta^{\alpha-2}}$. If $h \in C[0,1]$, then the boundary value problem*

$$(D_q^\alpha u)(t) + h(t) = 0, \quad 0 < t < 1, \quad 2 < \alpha < 3, \tag{3.1}$$

$$u(0) = (D_q u)(0) = 0, \quad (D_q u)(1) = \beta (D_q u)(\eta) \tag{3.2}$$

has a unique solution

$$u(t) = \int_0^1 G(t, qs) h(s) d_q s + \frac{\beta t^{\alpha-1}}{[\alpha-1]_q (1-\beta\eta^{\alpha-2})} \int_0^1 H(\eta, qs) h(s) d_q s, \tag{3.3}$$

where

$$G(t,s) = \frac{1}{\Gamma_q(\alpha)} \begin{cases} (1-s)^{(\alpha-2)} t^{\alpha-1} - (t-s)^{(\alpha-1)}, & 0 \leq s \leq t \leq 1, \\ (1-s)^{(\alpha-2)} t^{\alpha-1}, & 0 \leq t \leq s \leq 1, \end{cases} \tag{3.4}$$

$$H(t,s) = {}_t D_q G(s,t) = \frac{[\alpha-1]_q}{\Gamma_q(\alpha)} \begin{cases} (1-s)^{(\alpha-2)} t^{\alpha-2} - (t-s)^{(\alpha-2)}, & 0 \leq s \leq t \leq 1, \\ (1-s)^{(\alpha-2)} t^{\alpha-2}, & 0 \leq t \leq s \leq 1. \end{cases} \tag{3.5}$$

Proof. Consider $p = 3$. In view of Lemma 2.1 and Lemma 2.2, from (3.1) we see that

$$(I_q^\alpha D_q^3 I_q^{3-\alpha} u)(t) = -(I_q^\alpha h)(t)$$

and

$$u(t) = c_1 t^{\alpha-1} + c_2 t^{\alpha-2} + c_3 t^{\alpha-3} - \int_0^t \frac{(t-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} h(s) d_q s. \tag{3.6}$$

From (3.2), we know that $c_3 = 0$. Differentiating both sides of (3.6) one obtain, with the help of (2.1) and (2.2)

$$(D_q y)(x) = [\alpha-1]_q c_1 t^{\alpha-2} + [\alpha-2]_q c_2 t^{\alpha-3} - \frac{[\alpha-1]_q}{\Gamma_q(\alpha)} \int_0^t (t-qs)^{(\alpha-2)} h(s) d_q s.$$

Using the boundary condition (3.2), we have $c_2 = 0$ and

$$c_1 = \frac{1}{\Gamma_q(\alpha)(1-\beta\eta^{\alpha-2})} \left[\int_0^1 (1-qs)^{(\alpha-2)} h(s) d_q s - \int_0^\eta \beta (\eta-qs)^{(\alpha-2)} h(s) d_q s \right].$$

Therefore, the unique solution of boundary value problem (3.1)-(3.2) is

$$
\begin{aligned}
u(t) &= -\int_0^t \frac{(t-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} h(s)d_qs \\
&\quad + \frac{t^{\alpha-1}}{\Gamma_q(\alpha)(1-\beta\eta^{\alpha-2})}\left[\int_0^1 (1-qs)^{(\alpha-2)}h(s)d_qs - \int_0^\eta \beta(\eta-qs)^{(\alpha-2)}h(s)d_qs\right] \\
&= -\int_0^t \frac{(t-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} h(s)d_qs - \frac{\beta t^{\alpha-1}}{\Gamma_q(\alpha)(1-\beta\eta^{\alpha-2})}\int_0^\eta (\eta-qs)^{(\alpha-2)}h(s)d_qs \\
&\quad + \left(\frac{t^{\alpha-1}}{\Gamma_q(\alpha)} + \frac{\beta\eta^{\alpha-2}t^{\alpha-1}}{\Gamma_q(\alpha)(1-\beta\eta^{\alpha-2})}\right)\int_0^1 (1-qs)^{(\alpha-2)}h(s)d_qs \\
&= \frac{1}{\Gamma_q(\alpha)}\int_0^t ((1-qs)^{(\alpha-2)}t^{\alpha-1} - (t-qs)^{(\alpha-1)})h(s)d_qs \\
&\quad + \frac{1}{\Gamma_q(\alpha)}\int_t^1 (1-qs)^{(\alpha-2)}t^{\alpha-1}h(s)d_qs + \frac{\beta\eta^{\alpha-2}t^{\alpha-1}}{\Gamma_q(\alpha)(1-\beta\eta^{\alpha-2})}\int_0^1 (1-qs)^{(\alpha-2)}h(s)d_qs \\
&\quad - \frac{\beta t^{\alpha-1}}{\Gamma_q(\alpha)(1-\beta\eta^{\alpha-2})}\int_0^\eta (\eta-qs)^{(\alpha-2)}h(s)d_qs \\
&= \int_0^1 G(t,qs)h(s)d_qs + \frac{\beta t^{\alpha-1}}{[\alpha-1]_q(1-\beta\eta^{\alpha-2})}\int_0^1 H(\eta,qs)h(s)d_qs.
\end{aligned}
$$

The proof is complete. □

Lemma 3.2. *G defined by (3.4) has the following properties:*
(1) G is a continuous function and $G(t,qs) \geq 0$;
(2) $G(t,qs)$ is strictly increasing in the first variable.

Proof. The continuity of G is easily checked. On the other hand, let

$$h_1(t,s) = (1-s)^{(\alpha-2)}t^{\alpha-1} - (t-s)^{(\alpha-1)}, \quad 0 \leq s \leq t \leq 1,$$

and

$$h_2(t,s) = (1-s)^{(\alpha-2)}t^{\alpha-1}, \quad 0 \leq t \leq s \leq 1.$$

It is obvious that $h_2(t,qs) \geq 0$. Now, $h_1(0,qs) = 0$ and, in view of Remark 2.1, for $t \neq 0$

$$
\begin{aligned}
h_1(t,qs) &= (1-qs)^{(\alpha-2)}t^{\alpha-1} - (1-q\frac{s}{t})^{(\alpha-1)}t^{\alpha-1} \\
&\geq t^{\alpha-1}\left[(1-qs)^{(\alpha-2)} - (1-qs)^{(\alpha-1)}\right] \geq 0.
\end{aligned}
$$

Then we conclude that $G(t,qs) \geq 0$ for all $(t,s) \in [0,1]\times[0,1]$. This concludes the proof of Lemma 3.2 (1).

Next, for fixed $s \in [0,1]$,

$$
\begin{aligned}
{}_tD_qh_1(t,qs) &= (1-qs)^{(\alpha-2)}[\alpha-1]_qt^{\alpha-2} - [\alpha-1]_q(t-qs)^{(\alpha-2)} \\
&= (1-qs)^{(\alpha-2)}[\alpha-1]_qt^{\alpha-2} - [\alpha-1]_q(1-q\frac{s}{t})^{(\alpha-2)}t^{\alpha-2} \\
&\geq (1-qs)^{(\alpha-2)}[\alpha-1]_qt^{\alpha-2} - [\alpha-1]_q(1-qs)^{(\alpha-2)}t^{\alpha-2} = 0.
\end{aligned}
$$

This implies that $h_1(t,qs)$ is an increasing function of t. Obviously, $h_2(t,qs)$ is increasing in t. Therefore $G(t,qs)$ is an increasing function of t for fixed $s \in [0,1]$. The proof is complete. □

Remark 3.1. Obviously, following the proof Lemma 3.2 (1) we have $H(\eta, qs) \geq 0$. By Lemma 3.1 and 3.2, we have $u(t) \geq 0$ if $1 - \beta\eta^{\alpha-2} > 0$ and $h(t) \geq 0$ on $t \in [0, 1]$.

4. Single positive solution of the boundary value problems (1.1)-(1.2)

In this section, we establish the existence of single positive solution for boundary value problem (1.1) and (1.2) by lower and upper solution method. We assume that $f : [0, 1] \times [0, +\infty) \to [0, +\infty)$ is continuous in this section.

Lemma 4.1. *If $u(t) \in C[0, 1]$ and is a positive solution of (1.1) and (1.2), then $m\rho(t) \leq u(t) \leq M\rho(t)$, where*

$$\rho(t) := \int_0^1 G(t, qs)d_qs + \frac{\beta t^{\alpha-1}}{[\alpha-1]_q(1-\beta\eta^{\alpha-2})} \int_0^1 H(\eta, qs)d_qs$$

and m, M are two constants.

Proof. Since $u(t) \in C[0, 1]$, there exists $M' > 0$ so that $|u(t)| \leq M'$ for $t \in [0, 1]$. Set

$$m := \min_{(t,u)\in[0,1]\times[0,M']} f(t, u), \quad M := \max_{(t,u)\in[0,1]\times[0,M']} f(t, u).$$

By view of Lemma 3.1, we have

$$\begin{aligned} m\rho(t) \leq u(t) &= \int_0^1 G(t, qs)f(s, u(s))d_qs \\ &+ \frac{\beta t^{\alpha-1}}{[\alpha-1]_q(1-\beta\eta^{\alpha-2})} \int_0^1 H(\eta, qs)f(s, u(s))d_qs \leq M\rho(t). \end{aligned}$$

Thus we finished the proof of Lemma 4.1. □

Now we introduce the following two definitions about the upper and lower solutions of fractional boundary value problem (1.1) and (1.2).

Definition 4.1. A function $\theta(t)$ is called a lower solution of fractional boundary value problem (1.1) and (1.2) if $\theta(t) \in C[0, 1]$ and $\theta(t)$ satisfies

$$\begin{aligned} &-(D_q^\alpha\theta)(t) \leq f(t, \theta(t)), \quad 0 < t < 1, \quad 2 < \alpha \leq 3, \\ &\theta(0) \leq 0, \quad (D_q\theta)(0) \leq 0, \quad (D_q\theta)(1) \leq \beta(D_q\theta)(\eta). \end{aligned}$$

Definition 4.2. A function $\gamma(t)$ is called an upper solution of fractional boundary value problem (1.1) and (1.2) if $\gamma(t) \in C[0, 1]$ and $\gamma(t)$ satisfies

$$\begin{aligned} &-(D_q^\alpha\gamma)(t) \geq f(t, \gamma(t)), \quad 0 < t < 1, \quad 2 < \alpha \leq 3, \\ &\gamma(0) \geq 0, \quad (D_q\gamma)(0) \geq 0, \quad (D_q\gamma)(1) \geq \beta(D_q\gamma)(\eta). \end{aligned}$$

The main result of this paper is the following.

Theorem 4.1. *The fractional boundary value problem (1.1) and (1.2) has a positive and strictly increasing solution $u(t)$ if the following conditions are satisfied:*

(i) $f : [0,1] \times [0,+\infty) \to [0,+\infty)$ *is continuous and nondecreasing with respect to the second variable and* $f(t,0) \neq 0$ *for* $t \in Z \subset [0,1]$ *with* $\mu(Z) > 0$ *(*μ *denotes the Lebesgue measure);*

(ii) *There exists a positive constant* $\mu < 1$ *such that*

$$k^\mu f(t,u) \leq f(t,ku), \quad \forall\ 0 \leq k \leq 1.$$

Proof. At first, we will prove that the functions $\theta(t) = k_1 g(t)$, $\gamma(t) = k_2 g(t)$ are lower and upper solutions of (1.1) and (1.2), respectively, where $0 < k_1 \leq \min\left\{\frac{1}{a_2}, (a_1)^{\frac{\mu}{1-\mu}}\right\}$, $k_2 \geq \max\left\{\frac{1}{a_1}, (a_2)^{\frac{\mu}{1-\mu}}\right\}$ and

$$a_1 = \min\left\{1, \inf_{t\in[0,1]} f(t,\rho(t))\right\} > 0, \quad a_2 = \max\left\{1, \sup_{t\in[0,1]} f(t,\rho(t))\right\}$$

and

$$g(t) = \int_0^1 G(t,qs) f(s,\rho(s)) d_q s + \frac{\beta t^{\alpha-1}}{[\alpha-1]_q(1-\beta\eta^{\alpha-2})} \int_0^1 H(\eta,qs) f(s,\rho(s)) d_q s.$$

By view of Lemma 3.1, we know that $g(t)$ is a positive solution of the following equations

$$\begin{aligned} &(D_q^\alpha u)(t) + f(t,\rho(t)) = 0, \quad 0 < t < 1, \quad 2 < \alpha < 3, \\ &u(0) = (D_q u)(0) = 0, \quad (D_q u)(1) = \beta (D_q u)(\eta). \end{aligned}$$

From the conclusion of Lemma 4.1, we know that

$$a_1 \rho(t) \leq g(t) \leq a_2 \rho(t), \quad \forall\ t \in [0,1].$$

Thus, by virtue of the assumption of the Theorem 4.1, it follows that

$$\begin{aligned} &k_1 a_1 \leq \frac{\theta(t)}{\rho(t)} \leq k_1 a_2 \leq 1, \quad \frac{1}{k_2 a_2} \leq \frac{\rho(t)}{\gamma(t)} \leq \frac{1}{k_2 a_1} \leq 1, \\ &(k_1 a_1)^\mu \geq k_1, \ (k_2 a_2)^\mu \leq k_2. \end{aligned}$$

Therefore, we have

$$f(t,\theta(t)) = f\left(t, \frac{\theta(t)}{\rho(t)}\rho(t)\right) \geq \left(\frac{\theta(t)}{\rho(t)}\right)^\mu f(t,\rho(t)) \geq (k_1 a_1)^\mu f(t,\rho(t)) \geq k_1 f(t,\rho(t)),$$

$$k_2 f(t,\rho(t)) = k_2 f\left(t, \frac{\rho(t)}{\gamma(t)}\gamma(t)\right) \geq k_2 \left(\frac{\rho(t)}{\gamma(t)}\right)^\mu f(t,\gamma(t)) \geq k_2 (k_2 a_2)^{-\mu} f(t,\gamma(t)) \geq f(t,\gamma(t)).$$

It implies that

$$\begin{aligned} &-(D_q^\alpha \theta)(t) = k_1 f(t,\rho(t)) \leq f(t,\theta(t)), \quad 0 < t < 1, \quad 2 < \alpha < 3, \\ &-(D_q^\alpha \gamma)(t) = k_2 f(t,\rho(t)) \geq f(t,\gamma(t)), \quad 0 < t < 1, \quad 2 < \alpha < 3. \end{aligned}$$

Obviously, $\theta(t) = k_1 g(t)$, $\gamma(t) = k_2 g(t)$ satisfies the boundary conditions (1.2). So, $\alpha(t) = k_1 g(t)$, $\beta(t) = k_2 g(t)$ are lower and upper solutions of (1.1) and (1.2) respectively.

Next, we will prove that the fractional boundary value problem

$$(D_q^\alpha u)(t) + g(t, u(t)) = 0, \quad 0 < t < 1, \quad 2 < \alpha < 3, \tag{4.1}$$

$$u(0) = (D_q u)(0) = 0, \quad (D_q u)(1) = \beta (D_q u)(\eta), \tag{4.2}$$

has a solution, where

$$g(t, u(t)) = \begin{cases} f(t, \theta(t)), & if\ u(t) \le \theta(t), \\ f(t, u(t)), & if\ \theta(t) \le u(t) \le \gamma(t), \\ f(t, \gamma(t)), & if\ \gamma(t) \le u(t). \end{cases}$$

Thus, we consider the operator $A : C[0,1] \to C[0,1]$ define as

$$Au(t) \;=\; \int_0^1 G(t, qs) g(s, u(s)) d_q s + \frac{\beta t^{\alpha-1}}{[\alpha-1]_q (1 - \beta \eta^{\alpha-2})} \int_0^1 H(\eta, qs) g(s, u(s)) d_q s,$$

where $G(t, qs)$ and $H(\eta, qs)$ are defined in Lemma 3.1. Since the function $f(t, u)$ in nondecreasing in u, this shows that, for any $u \in C[0,1]$,

$$f(t, \theta(t)) \le g(t, u(t)) \le f(t, \gamma(t)) \quad \text{for } t \in [0,1].$$

The operator $A : C[0,1] \to C[0,1]$ is continuous in view of continuity of $G(t,s)$ and $g(t, u(t))$. By means of Arzela-Ascoli theorem, A is a compact operator. Therefore, from Leray-Schauder fixed point theorem, the operator A has a fixed point, i.e., fractional boundary value problem (4.1)-(4.2) has a solution.

Finally, we will prove that fractional boundary value problem (1.1) and (1.2) has a positive solution.

Suppose $u^*(t)$ is a solution of fractional boundary value problem (4.1)-(4.2). Since the function $f(t, u)$ is nondecreasing in u, we know that

$$f(t, \theta(t)) \le g(t, u^*(t)) \le f(t, \gamma(t)) \quad \text{for } t \in [0,1].$$

Thus

$$-(D_q^\alpha z)(t) \ge f(t, \gamma(t)) - g(t, u^*(t)) \ge 0,$$
$$z(0) = (D_q z)(0) = 0, \quad (D_q z)(1) = \beta (D_q z)(\eta),$$

where $z(t) = \gamma(t) - u^*(t)$. By virtue of Remark 3.1, $z(t) \ge 0$, i.e., $u^*(t) \le \theta(t)$ for $t \in [0,1]$. Similarly, $\beta(t) \le u^*(t)$ for $t \in [0,1]$. Therefore, $u^*(t)$ is a positive solution of fractional boundary value problem (1.1) and (1.2).

In the following, we will prove that this positive solution $u(t)$ is strictly increasing function. As $u(0) = \int_0^1 G(0, qs) f(s, u(s)) d_q s$ and $G(0, qs) = 0$ we have $u(0) = 0$.

Moreover, if we take $t_1, t_2 \in [0,1]$ with $t_1 < t_2$, we can consider the following cases.
Case 1: $t_1 = 0$, in this case, $u(t_1) = 0$ and, as $u(t) \ge 0$, suppose that $u(t_2) = 0$. Then

$$0 = u(t_2) = \int_0^1 G(t_2, qs) f(s, u(s)) d_q s + \frac{\beta t_2^{\alpha-1}}{[\alpha-1]_q (1 - \beta \eta^{\alpha-2})} \int_0^1 H(\eta, qs) f(s, u(s)) d_q s.$$

This implies that

$$G(t_2, qs) \cdot f(s, u(s)) = 0, \quad a.e.(s)$$

and as $G(t_2, s) \neq 0$ a.e.(s) we get $f(s, u(s)) = 0$ a.e.(s).

On the other hand, f is nondecreasing with respect to the second variable and therefore we get

$$f(s, 0) \leq f(s, u(s)) = 0, \quad a.e.(s)$$

which contradicts the condition (i) $f(t, 0) \neq 0$ for $t \in Z \subset [0, 1](\mu(Z) \neq 0)$. Thus $u(t_1) = 0 < u(t_2)$.

Case 2: $0 < t_1$. In this case, let us take $t_2, t_1 \in [0, 1]$ with $t_1 < t_2$, then

$$\begin{aligned} u(t_2) - u(t_1) &= \int_0^1 (G(t_2, qs) - G(t_1, qs)) f(s, u(s)) d_q s \\ &\quad + \frac{\beta(t_2^{\alpha-1} - t_1^{\alpha-1})}{[\alpha - 1]_q (1 - \beta\eta^{\alpha-2})} \int_0^1 H(\eta, qs) f(s, u(s)) d_q s. \end{aligned}$$

Taking into account Lemma 3.2 (2) and the fact that $f \geq 0$, we get $u(t_2) - u(t_1) \geq 0$.

Suppose that $u(t_2) = u(t_1)$ then

$$\int_0^1 (G(t_2, qs) - G(t_1, qs)) f(s, u(s)) d_q s = 0$$

and this implies

$$(G(t_2, qs) - G(t_1, qs)) f(s, u(s)) = 0 \quad a.e.(s).$$

Again, Lemma 3.2 (2) gives us

$$f(s, u(s)) = 0 \quad a.e.(s)$$

and using the same reasoning as above we have that this contradicts condition (i) $f(t, 0) \neq 0$ for $t \in Z \subset [0, 1](\mu(Z) \neq 0)$. Thus $u(t_1) = 0 < u(t_2)$. We have finished the proof of Theorem 4.1. □

5. Example

Example 5.1. *As an example we mention the following fractional boundary value problem*

$$\begin{cases} (D_q^{\frac{5}{2}} u)(t) + f(t, u(t)) = 0, \quad 0 < t < 1, \\ u(0) = (D_q u)(0) = 0, \quad (D_q u)(1) = \beta (D_q u)(\eta), \end{cases} \tag{5.1}$$

where $0 < q < 1$ *and* $0 < \beta\eta^{\alpha-2} < 1$ *and* $f(t, u) = t + u^\mu$, $0 < \mu < 1$.

Proof. Since $k^\mu \leq 1$ for $0 < \mu < 1$ and $0 \leq k \leq 1$. It is easy to check that

$$k^\mu f(t, u) = k^\mu t + k^\mu u^\mu \leq t + (ku)^\mu = f(t, ku).$$

Thus, by Theorem 4.1 we know that the boundary value problem (5.1) has a positive and strictly increasing solution $u(t)$. □

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