CONGRUENCE LATTICES OF SYMMETRIC EXTENDED DE MORGAN ALGEBRAS

JIE FANG AND LEI-BO WANG

Abstract. In this paper, we characterize the congruence lattice of a symmetric extended de Morgan algebra L. We show that the congruence lattice of the algebra L is a pseudocomplemented lattice, and that such a congruence lattice is a Stone lattice if and only if the lattice of the compact congruences on L forms a complete boolean lattice. In particular, we prove that the congruence lattice of L is a boolean lattice if and only if, it is a relative Stone lattice, which is the case, if and only if L is finite.

1. Introduction

The investigation in the structures of congruence lattices has been done in several special classes of lattice-ordered algebras. For example, Janowitz [12] showed that the congruence lattice of a boolean algebra B is a Stone lattice if and only if B is complete. Beazer [1, 2] studied the congruence lattice in the context of pseudocomplemented algebras and regular double pseudocomplemented algebras, and in another publication [3], he proved that the congruence lattice $\operatorname{Con} B$ of a boolean algebra B is a relative Stone lattice if and only if every homomorphic image of B is complete. Years latter, in 1987, Haviar and Katriňák [11] showed that the congruence lattice $\operatorname{Con} L$ of a distributive lattice L is a relative Stone lattice if and only if $\operatorname{Con} L$ is a boolean lattice. In [13], Sankappanvar characterized this notion in the context of de Morgan algebras, and he particularly showed that the congruence lattice of a de Morgan algebra L is boolean if and only if L is finite. Subsequently, Blyth and Varlet extended this result to the class of Ockham algebras with de Morgan skeleton (see [4, Theorem 8.15]). In this connection, we mean the wide class of Ockham algebras as introduced by Urquhart [14]. An Ockham algebra $(L; f)$ is a bounded distributive lattice L together with a dual endomorphism f (see also [14]). The special case where $f^2 = id_L$ gives a de Morgan algebra.

In this paper we shall consider the congruence lattices in a particular subclass of the class of extended Ockham algebras; namely the class of symmetric extended de Morgan algebras. We shall show that the congruence lattice $\operatorname{Con} L$ of a symmetric extended de Morgan algebra L is a pseudocomplemented lattice, and that it is a Stone lattice if and only if the lattice $K(L)$ of the compact congruences on L forms a complete boolean lattice with $\operatorname{Con} K(L) \simeq \operatorname{Con} L$. In particular, we shall prove

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that the congruence lattice of the algebra L is a boolean lattice if and only if, it is a relative Stone lattice, which is the case, if and only if L is finite.

2. Preliminary

Definition 2.1. [5] By an extended Ockham algebra $L \equiv (L; \wedge, \vee, f, k, 0, 1)$ we shall mean a bounded distributive lattice L with two unary operations f and k such that

(1) f is a dual lattice endomorphism with $f(0) = 1$ and $f(1) = 0$;

(2) k is a lattice endomorphism with $k(0) = 0$ and $k(1) = 1$;

(3) f and k commute.

An extended de Morgan algebra $L \equiv (L; \wedge, \vee, f, k, 0, 1)$ in which $k^2 = \mathrm{id}_L$ is said to be *symmetric*.

The notion of class of extended Ockham algebras is first introduced by Blyth and Fang in [5]. For more details of extended Ockham algebras and those of symmetric extended de Morgan algebras, we refer the reader to [5, 6].

Throughout what follows we shall denote by $\mathbf{e_2M}$ the class of symmetric extended de Morgan algebras, and for convenience, we shall write x° for $f(x)$ and x^+ for $k(x)$. We note that when $^+ = id_L$ we regard an $\mathbf{e_2M}$-algebra $L \equiv (L; \wedge, \vee, ^\circ, ^+, 0, 1)$ simply as a de Morgan algebra. Clearly, the smallest non-trivial subalgebra of L that is the case where $^+ = \mathrm{id}_L$ is $\{0, 1\}$, and the biggest is $\mathrm{Fix}_+ L = \{x \in L \mid x^+ = x\}$.

By a *congruence* on a symmetric extended de Morgan algebra $(L; ^\circ, ^*)$ we shall mean a lattice congruence ϑ such that

$$(x, y) \in \vartheta \Longrightarrow (x^\circ, y^\circ) \in \vartheta \text{ and } (x^+, y^+) \in \vartheta.$$

We shall denote by $\mathrm{Con}\, L$ the lattice of congruences on a symmetric extended de Morgan algebra L, and use throughout the standard notation $\theta(a, b)$ for the principal congruence on L that identifies a and b with $a \leqslant b$, and $\theta_{\mathrm{lat}}(a, b)$ for the corresponding principal lattice congruence.

As a particular case of [5, Theorem 2.1] (or see [9, Theorem 1.11]), a description of principal congruence on a symmetric extended de Morgan algebra L can be given as follows

Theorem 2.1. *Let $L \in \mathbf{e_2M}$. If $a, b \in L$ are such that $a \leqslant b$ then*

$$\theta(a, b) = \theta_{\mathrm{lat}}(a, b) \vee \theta_{\mathrm{lat}}(b^\circ, a^\circ) \vee \theta_{\mathrm{lat}}(a^+, b^+) \vee \theta_{\mathrm{lat}}(b^{\circ +}, a^{\circ +}).$$

3. Congruence lattices

Throughout what follows, we shall use the symbols ω and ι to stand for the equality and universal relation, respectively. We begin with the following result that shall be proved to be very useful.

Theorem 3.1. *Let $L \in \mathbf{e_2M}$. If $a, b \in L$ are such that $a \leqslant b$. Then $\theta(a, b)$ is complemented with the complement $\theta^c(a, b)$ that is described as follows*

$$\begin{aligned}\theta^c(a, b) = {} & [\theta_{\mathrm{lat}}(0, a) \vee \theta_{\mathrm{lat}}(b, 1)] \wedge [\theta_{\mathrm{lat}}(0, b^\circ) \vee \theta_{\mathrm{lat}}(a^\circ, 1)] \\ & \wedge [\theta_{\mathrm{lat}}(0, a^+) \vee \theta_{lat}(b^+, 1)] \wedge [\theta_{\mathrm{lat}}(0, b^{\circ +}) \vee \theta_{\mathrm{lat}}(a^{\circ +}, 1)].\end{aligned}$$

Proof. If $a, b \in L$ are such that $a \leqslant b$. Then by Theorem 2.1, we have

$$\theta(a,b) = \theta_{\rm lat}(a,b) \vee \theta_{\rm lat}(b^\circ, a^\circ) \vee \theta_{\rm lat}(a^+, b^+) \vee \theta_{\rm lat}(b^{\circ +}, a^{\circ +}).$$

Let φ be the right side of the stated equality. Then we have $\varphi \vee \theta(a,b) = \iota$ and $\varphi \wedge \theta(a,b) = \omega$. To see that $\theta(a,b)$ is complemented, it suffices to verify that φ is a congruence on L. Now, by the well-known fact that for $u, v \in L$ with $u \leqslant v$,

$$(x,y) \in \theta_{\rm lat}(0,u) \vee \theta_{\rm lat}(v,1) \iff (x \vee u) \wedge v = (y \vee u) \wedge v$$

we can obtain the following:

$$(x,y) \in \theta_{\rm lat}(0,a) \vee \theta_{\rm lat}(b,1) \Longrightarrow (x^\circ, y^\circ) \in \theta_{\rm lat}(0,b^\circ) \vee \theta_{lat}(a^\circ,1);$$
$$(x,y) \in \theta_{\rm lat}(0,b^\circ) \vee \theta_{\rm lat}(a^\circ,1) \Longrightarrow (x^\circ, y^\circ) \in \theta_{lat}(0,a) \vee \theta_{\rm lat}(b,1);$$
$$(x,y) \in \theta_{\rm lat}(0,a^+) \vee \theta_{\rm lat}(b^+,1) \Longrightarrow (x^\circ, y^\circ) \in \theta_{\rm lat}(0,b^{\circ +}) \vee \theta_{\rm lat}(a^{\circ +},1);$$
$$(x,y) \in \theta_{\rm lat}(0,b^{\circ +}) \vee \theta_{\rm lat}(a^{\circ +},1) \Longrightarrow (x^\circ, y^\circ) \in \theta_{\rm lat}(0,a^+) \vee \theta_{\rm lat}(b^+,1).$$

Thus it follows from the above observations that $(x,y) \in \varphi$ implies $(x^\circ, y^\circ) \in \varphi$. In a similar way we have also that if $(x,y) \in \varphi$ then $(x^+, y^+) \in \varphi$. Hence φ is a congruence on L, and consequently, $\theta(a,b)$ is complemented with the complement $\theta^c(a,b) = \varphi$. □

Theorem 3.2. *If $L \in \mathbf{e_2M}$ then the lattice* Con L *of the congruences on L is pseudocomplemented.*

Proof. It is enough to show that every $\varphi \in {\rm Con}\, L$ is pseudocomplemented. In order to do so, we observe first that, for $\alpha_i \in {\rm Con}\, L$ $(i \in I)$

$$(\dagger) \qquad \varphi \wedge \bigvee_{i\in I} \alpha_i = \bigvee_{i \in I} (\varphi \wedge \alpha_i).$$

In fact, we have clearly that $\varphi \wedge \bigvee_{i\in I} \alpha_i \geqslant \bigvee_{i\in I} (\varphi \wedge \alpha_i)$. For the reverse inequality, let $x, y \in L$ with $x \leqslant y$ be such that $(x,y) \in \varphi \wedge \bigvee_{i\in I} \alpha_i$. Then there exist $x_j \in L$ with $x \leqslant x_j \leqslant y$ $(j = 0, 1, 2, \cdots, n)$ and α_{i_j} such that

$$x = x_0 \overset{\alpha_{i_1}}{\equiv} x_1 \overset{\alpha_{i_2}}{\equiv} x_2 \overset{\alpha_{i_3}}{\equiv} \cdots \overset{\alpha_{i_n}}{\equiv} x_n = y.$$

Since $x \leqslant x_j \leqslant y$ and $(x,y) \in \varphi$, we have $(x_{j-1}, x_j) \in \varphi$, and so, for each j, $(x_{j-1}, x_j) \in \varphi \wedge \alpha_{i_j}$, from which it follows that

$$(x,y) \in \bigvee_{j=1}^{n} (\varphi \wedge \alpha_{i_j}) \leqslant \bigvee_{i\in I} (\varphi \wedge \alpha_i).$$

Thus we have $\varphi \wedge \bigvee_{i\in I} \alpha_i \leqslant \bigvee_{i\in I} (\varphi \wedge \alpha_i)$, whence the equality $(\dagger)$ holds.

Now, we have by Theorem 3.1 that each $\theta(a,b)$ is complemented with the complement $\theta^c(a,b)$. Let $\varphi = \bigvee\{\theta(a,b) \mid (a,b) \in \varphi \text{ with } a \leqslant b\}$, and let

$$\varphi^* = \bigwedge\{\theta^c(a,b) \mid (a,b) \in \varphi \text{ with } a \leqslant b\}.$$

Then clearly, for every pair of $(a,b) \in \varphi$ with $a \leqslant b$, we have $\theta(a,b) \wedge \varphi^* = \omega$. Thus it follows by $(\dagger)$ that

$$\begin{aligned} \varphi^* \wedge \varphi &= \varphi^* \wedge \bigvee\{\theta(a,b) \mid (a,b) \in \varphi \text{ with } a \leqslant b\} \\ &= \bigvee\{\varphi^* \wedge \theta(a,b) \mid (a,b) \in \varphi \text{ with } a \leqslant b\} \\ &= \bigvee \omega = \omega. \end{aligned}$$

Suppose now $\alpha \in \operatorname{Con} L$ is such that $\alpha \wedge \varphi = \omega$. Then for $(a, b) \in \varphi$ with $a \leqslant b$, we have $\alpha \wedge \theta(a, b) = \omega$, from which it follows that $\alpha \leqslant \theta^c(a, b)$, and consequently, we have

$$\alpha \leqslant \bigwedge \{\theta^c(a, b) \mid (a, b) \in \varphi \text{ with } a \leqslant b\} = \varphi^*.$$

Hence φ^* is the pseudocomplement of φ. □

It is shown by Sankappanavar in [13, Theorem 3.8] that the congruence lattice $\operatorname{Con} L$ of a de Morgan algebra L is boolean if and only if L is finite. This result can be extended to symmetric extended de Morgan algebras. In order to do so, we require first the following technical result.

Theorem 3.3. *Let $L \in \mathbf{e_2M}$. If $\operatorname{Con} L$ is boolean then $\operatorname{Fix}_+ L$ is finite.*

Proof. Since $x^+ = x$ for $x \in \operatorname{Fix}_+ L$, we can regard $\operatorname{Fix}_+ L$ simply as a de Morgan algebra. Thus by Sankappanavar's result [13, Theorem 3.8], we need only show that $\operatorname{Con} \operatorname{Fix}_+ L$ is boolean. Let $\varphi \in \operatorname{Con} \operatorname{Fix}_+ L$. Then, since L has the congruence extension property by [9, Corollary 2 to Theorem 1.11], there exists an extension $\overline{\varphi}$ of φ to L such that $\overline{\varphi}|_{\operatorname{Fix}_+ L} = \varphi$. Since $\operatorname{Con} L$ is boolean, there exists the complement $\overline{\varphi}^c$ of $\overline{\varphi}$ in $\operatorname{Con} L$, namely $\overline{\varphi}^c \vee \overline{\varphi} = \iota$ and $\overline{\varphi}^c \wedge \overline{\varphi} = \omega$. Let $\psi = \overline{\varphi}^c|_{\operatorname{Fix}_+ L}$. Then we have clearly that $\psi \wedge \varphi = \omega$. Now, since $\overline{\varphi}^c \vee \overline{\varphi} = \iota$, we have $(0, 1) \in \overline{\varphi}^c \vee \overline{\varphi}$, and so, there are $x_i \in L$ $(i = 0, 1, 2, \cdots, n)$ such that

$$0 = x_0 \equiv x_1 \equiv x_2 \equiv \cdots \equiv x_n = 1$$

where $(x_i, x_{i+1}) \in \overline{\varphi}^c$ or $(x_i, x_{i+1}) \in \overline{\varphi}$. Note that for $\alpha = \overline{\varphi}^c$ or $\alpha = \overline{\varphi}$, $(x_i, x_{i+1}) \in \alpha$ gives $(x_i \wedge x_i^+, x_{i+1} \wedge x_{i+1}^+) \in \alpha$. Thus we have

$$0 = x_0 \wedge x_0^+ \equiv x_1 \wedge x_1^+ \equiv x_2 \wedge x_2^+ \equiv \cdots \equiv x_n \wedge x_n^+ = 1$$

where $(x_i \wedge x_i^+, x_{i+1} \wedge x_{i+1}^+) \in \overline{\varphi}^c$ or $(x_i \wedge x_i^+, x_{i+1} \wedge x_{i+1}^+) \in \overline{\varphi}$. Since each $x_i \wedge x_i^+ \in \operatorname{Fix}_+ L$, we have

$$(x_i \wedge x_i^+, x_{i+1} \wedge x_{i+1}^+) \in \overline{\varphi}^c|_{\operatorname{Fix}_+ L} = \psi \text{ or } (x_i \wedge x_i^+, x_{i+1} \wedge x_{i+1}^+) \in \overline{\varphi}|_{\operatorname{Fix}_+ L} = \varphi$$

from which it follows that $(0, 1) \in \psi \vee \varphi$ whence $\psi \vee \varphi = \iota$. Consequently, ψ is the complement of φ in $\operatorname{Fix}_+ L$. Hence $\operatorname{Con} \operatorname{Fix}_+ L$ is boolean. It then follows by [13, Theorem 3.8] that $\operatorname{Fix}_+ L$ is finite. □

Theorem 3.4. *Let $L \in \mathbf{e_2M}$. Then $\operatorname{Con} L$ is boolean if and only if L is finite.*

Proof. ($\Leftarrow$:) The argument is same as that of [4, Theorem 2.17] for a $\mathbf{K}_{n,0}$-algebra. We omit it.

($\Rightarrow$:) Let $\operatorname{Con} L$ be boolean. Then by Theorem 3.3, $\operatorname{Fix}_+ L$ is finite. If, on the contrary, L is infinite. Then there exists an infinite chain of L being of one of the following forms:

(1) $a_1 < a_2 < \cdots < a_n < \cdots$;

(2) $\cdots < a_n < \cdots < a_2 < a_1$.

We may assume that L contains a chain of the form (1). Then for each n, we have $a_n < a_{n+1}$ and $a_n^+ < a_{n+1}^+$. Let $x_n = a_n \wedge a_n^+$ and $y_n = a_n \vee a_n^+$. Then $x_n, y_n \in \operatorname{Fix}_+ L$ and

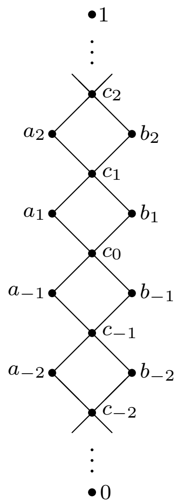
(3) $x_1 \leqslant x_2 \leqslant \cdots \leqslant x_n \leqslant \cdots$;

(4) $y_1 \leqslant y_2 \leqslant \cdots \leqslant y_n \leqslant \cdots$.

Since $\operatorname{Fix}_+ L$ is finite, the chains (3) and (4) must be finite, and so, there is some n such that $x_n = x_{n+1}$ and $y_n = y_{n+1}$; namely $a_n \wedge a_n^+ = a_{n+1} \wedge a_{n+1}^+$ and

$a_n \vee a_n^+ = a_{n+1} \vee a_{n+1}^+$. The former equality gives $a_n \wedge a_n^+ = a_{n+1} \wedge a_n^+$; and since $a_n \vee a_n^+ \leqslant a_{n+1} \vee a_n^+ \leqslant a_{n+1} \vee a_{n+1}^+$, the latter equality gives $a_n \vee a_n^+ = a_{n+1} \vee a_n^+$, there follows by the distributivity the contradiction that $a_n = a_{n+1}$. Hence we must have that L is finite. □

Example 3.1. [7, Example 2] Consider an infinite symmetric extended de Morgan algebra $(L; {}^\circ, {}^+)$ depicted as follows:



where ${}^\circ : L \to L$ is described by
$0^\circ = 1,\ 1^\circ = 0,$
$z_i^\circ = z_{-i}$ for $z_i \in \{a_i, b_i, c_i\}$;
and ${}^+ : L \to L$ is described by
$0^+ = 0,\ 1^+ = 1,$
$a_i^+ = b_i,\ b_i^+ = a_i,\ c_i^+ = c_i.$

By a simple observation we can see that for each $i \geqslant 0$, $\alpha_i = \theta(c_i, c_{i+1})$ is an atom of $\operatorname{Con} L$, where $\theta(c_i, c_{i+1}) = \theta(c_{-(i+1)}, c_{-i})$ for $i \geqslant 1$, and $\theta(c_0, c_1) = \theta(c_{-1}, c_1)$. Let $\Psi = \bigvee_{i \geqslant 0} \alpha_i$. Then Ψ is the comonolith of $\operatorname{Con} L$ (in the sense that Ψ is the maximum of $\operatorname{Con} L \setminus \{\iota\}$). Clearly, all Ψ-classes are $\{0\}, \{1\}$ and $L \setminus \{0, 1\}$. Thus Ψ is not complemented, and consequently, $\operatorname{Con} L$ is not boolean.

By a congruence α on an algebra L being *compact* we shall mean that there exist $a_i, b_i \in L$ with $a_i \leqslant b_i$ $(i = 1, 2, \cdots, n)$ such that $\alpha = \bigvee_{i=1}^{n} \theta(a_i, b_i)$. Throughout what follows we shall denote by $K(L)$ the set of compact congruences on L. Clearly, $K(L)$ forms a distributive lattice.

Theorem 3.5. *Let $L \in \mathbf{e_2M}$ and $\varphi \in \operatorname{Con} L$. Then φ is compact if and only if it is complemented.*

Proof. $(\Rightarrow:)$ If φ is a compact congruence on L. Then there exist $a_i, b_i \in L$ with $a_i \leqslant b_i$ $(i = 1, 2, \cdots, n)$ such that $\varphi = \bigvee_{i=1}^{n} \theta(a_i, b_i)$. By Theorem 3.1, each $\theta(a_i, b_i)$ is complemented with the complement $\theta^c(a_i, b_i)$. Let

$$\psi = \bigwedge_{i=1}^{n} \theta^c(a_i, b_i).$$

Then clearly, $\varphi \wedge \psi = \omega$ and

$$\begin{aligned}\varphi \vee \psi &= \bigvee_{i=1}^{n} \theta(a_i, b_i) \vee \bigwedge_{i=1}^{n} \theta^c(a_i, b_i) \\ &= \bigwedge_{i=1}^{n} \bigvee_{j=1}^{n} [\theta(a_j, b_j) \vee \theta^c(a_i, b_i)] \\ &= \bigwedge_{i=1}^{n} \iota = \iota.\end{aligned}$$

Thus φ is complemented with the complement $\varphi^c = \psi$.

($\Leftarrow$:) Suppose that φ is complemented with the complement φ^c in $\operatorname{Con} L$. Then $\varphi \vee \varphi^c = \iota$, and so $(0,1) \in \varphi \vee \varphi^c$. Thus there exist $x_i \in L$ with $x_i \leqslant x_{i+1}$ $(i = 0, 1, 2, \cdots, n-1)$ such that

$$0 = x_0 \equiv x_1 \equiv x_2 \equiv \cdots \equiv x_n = 1$$

where $(x_i, x_{i+1}) \in \varphi$ or $(x_i, x_{i+1}) \in \varphi^c$. Then $\theta(x_i, x_{i+1}) \leqslant \varphi$ or $\theta(x_i, x_{i+1}) \wedge \varphi = \omega$. Since $\bigvee_{i=0}^{n-1} \theta(x_i, x_{i+1}) = \iota$, we have

$$\begin{aligned}\varphi &= \varphi \wedge \bigvee_{i=0}^{n-1} \theta(x_i, x_{i+1}) \\ &= \bigvee_{i=0}^{n-1} [\varphi \wedge \theta(x_i, x_{i+1})] \\ &= \bigvee_{j} \{\theta(x_{i_j}, x_{i_j+1}) \mid (x_{i_j}, x_{i_j+1}) \in \varphi\}\end{aligned}$$

from which it follows that φ is compact. Hence (1) holds. □

By Theorem 3.5 the following corollary is immediate.

Corollary 3.1. *If $L \in \mathbf{e_2M}$ then the lattice $K(L)$ of the compact congruence on L is a boolean sublattice of* $\operatorname{Con} L$.

In what follows for a lattice L, we shall denote by $\mathcal{I}(L)$ the lattice of ideals of L where the lattice operations $\wedge$ and $\vee$ are given as follows

$$I \wedge J = I \cap J \ \text{ and } \ I \vee J = \{x \in L \mid (\exists i \in I)(\exists j \in J) \ \ x \leqslant i \vee j\}.$$

Theorem 3.6. *If $L \in \mathbf{e_2M}$ then* $\operatorname{Con} L \simeq \operatorname{Con} K(L)$.

Proof. By Corollary 3.1, the lattice $K(L)$ of the compact congruences on L is boolean, and so by [10, Theorem 9.7], we have $\mathcal{I}(K(L)) \simeq \operatorname{Con} K(L)$. Thus, in order to obtain the stated isomorphism, it is enough to show that $\operatorname{Con} L \simeq \mathcal{I}(K(L))$. Let $\varphi \in \operatorname{Con} L$ and define

$$E(\varphi) = \{\alpha \in K(L) \mid \alpha \leqslant \varphi\}.$$

Then clearly, $E(\varphi)$ is an ideal of $K(L)$. Consider now the mapping $E : \operatorname{Con} L \to \mathcal{I}(K(L))$ by the prescription $\varphi \mapsto E(\varphi)$. It is readily seen that $E(\varphi \wedge \psi) = E(\varphi) \wedge E(\psi)$ and $E(\varphi) \vee E(\psi) \subseteq E(\varphi \vee \psi)$. To see that E is a lattice homomorphism, we need only show that $E(\varphi \vee \psi) \subseteq E(\varphi) \vee E(\psi)$. Let $\vartheta \in E(\varphi \vee \psi)$. Then $\vartheta \in K(L)$ with $\vartheta \leqslant \varphi \vee \psi$. If $(x,y) \in \vartheta$ with $x \leqslant y$, then $(x,y) \in \varphi \vee \psi$, and so there exist $a_i \in L$ with $a_i \leqslant a_{i+1}$ $(i = 0, 1, \cdots, n-1)$ such that

$$x = a_0 \equiv a_1 \equiv a_2 \equiv \cdots \equiv a_n = y$$

where $(a_i, a_{i+1}) \in \varphi$ or $(a_i, a_{i+1}) \in \psi$. Without loss of generality we may assume that

$$(a_{2i}, a_{2i+1}) \in \varphi \text{ and } (a_{2i+1}, a_{2i+2}) \in \psi \ (i = 0, 1, \cdots).$$

Let $\alpha = \bigvee_i \theta(a_{2i}, a_{2i+1})$ and $\beta = \bigvee_i \theta(a_{2i+1}, a_{2i+2})$. Then we have that

$$\alpha \vee \beta \leqslant E(\varphi) \vee E(\psi) \text{ and } (x, y) \in \alpha \vee \beta$$

from which it follows that $\theta(x, y) \leqslant \alpha \vee \beta \in E(\varphi) \vee E(\psi)$, and whence $\theta(x, y) \in E(\varphi) \vee E(\psi)$. Note that the compact congruence ϑ is a joint of finitely many principal congruences $\theta(x, y)$ for which $(x, y) \in \vartheta$ with $x \leqslant y$, thus we can see that $\vartheta \in E(\varphi) \vee E(\psi)$. Therefore it follows that $E(\varphi \vee \psi) \subseteq E(\varphi) \vee E(\psi)$ whence $E(\varphi \vee \psi) = E(\varphi) \vee E(\psi)$, and consequently, E is a lattice homomorphism.

To see that E is injective, we let $\varphi, \psi \in \operatorname{Con} L$ be such that $E(\varphi) = E(\psi)$. Observe that

$$\begin{aligned} (x, y) \in \varphi &\iff \theta(x, y) \leqslant \varphi \\ &\iff \theta(x, y) \in E(\varphi) = E(\psi) \\ &\iff \theta(x, y) \leqslant \psi \\ &\iff (x, y) \in \psi \end{aligned}$$

we have $\varphi = \psi$. Hence E is injective.

Finally, we shall show that E is surjective. Let $I \in \mathcal{I}(K(L))$ and $\varphi = \bigvee I = \bigvee_{\alpha_i \in I} \alpha_i$. Then we have clearly $I \subseteq E(\varphi)$. If now $\alpha \in E(\varphi)$ then $\alpha \in K(L)$ with $\alpha \leqslant \varphi = \bigvee_{\alpha_i \in I} \alpha_i$. By the compactness of α, there are finitely many of $\alpha_{i_j} \in I$ $(j = 1, 2, \cdots, m)$ such that $\alpha \leqslant \bigvee_{j=1}^{m} \alpha_{i_j}$. Since $\bigvee_{j=1}^{m} \alpha_{i_j} \in I$, it follows that $\alpha \in I$ whence $E(\varphi) \subseteq I$, and consequently, $E(\varphi) = I$. Hence E is surjective.

Therefore, we obtain from the observations above that E is a lattice isomorphism. Hence we obtain that $\operatorname{Con} L \simeq \mathcal{I}(K(L))$, and consequently, $\operatorname{Con} L \simeq \operatorname{Con} K(L)$. □

Corollary 3.2. *Let $L \in \mathbf{e_2M}$. Then $K(L)$ is finite if and only if L is finite.*

Proof. (⇐:) It is clear.

(⇒:) Suppose that $K(L)$ is finite. Then by [13, Theorem 3.8], $\operatorname{Con} K(L)$ is boolean, and by Theorem 3.6, so is $\operatorname{Con} L$. Thus it follows by Theorem 3.4 that L is finite. □

Using the fact established by Janowitz [12, Corollary to Theorem 4] that *the congruence lattice* $\operatorname{Con} B$ *of a boolean algebra* B *is a Stone lattice if and only if* B *is complete*, the following theorem can be obtained immediately by Theorem 3.6 and Corollary 3.1.

Theorem 3.7. *Let $L \in \mathbf{e_2M}$. Then* $\operatorname{Con} L$ *is Stone if and only if* $K(L)$ *is complete.*

A lattice L is said to be *relative Stone* if every interval of L is a Stone lattice. In [11, Theorem 7], Haviar and Katriňák showed that if L is a distributive lattice then the congruence lattice $\operatorname{Con} L$ is a relative Stone lattice if and only if $\operatorname{Con} L$ is boolean. Here we shall extend this to a symmetric extended de Morgan algebra. For this purpose, we require the following lemmas.

Lemma 3.1. [3, Corollary 4] *If B is a boolean algebra, then* $\operatorname{Con} B$ *is a relative Stone lattice if and only if every homomorphic image of B is complete.*

Lemma 3.2. [8, Theorem 4.3] *Every infinite complete boolean algebra has an incomplete homomorphic image.*

Theorem 3.8. *Let* $L \in \mathbf{e_2M}$. *Then* $\operatorname{Con} L$ *is a relative Stone lattice if and only if* L *is finite.*

Proof. ($\Leftarrow$:) It is clear.

($\Rightarrow$:) If $\operatorname{Con} L$ is a relative Stone lattice, then by Theorem 3.6, so is $\operatorname{Con} K(L)$. Since, by Corollary 3.1, $K(L)$ is boolean. It follows by Lemma 3.1 that every homomorphic image of $K(L)$ is complete. If now, on the contrary, L is infinite, then by Corollary 3.2, $K(L)$ is an infinite complete boolean algebra, from which it follows by Lemma 3.2 the contradiction that $K(L)$ has an incomplete homomorphic image. Therefore, we must have that L is finite. □

By Theorems 3.4 and 3.8, the following corollary is immediate.

Corollary 3.3. *If* $L \in \mathbf{e_2M}$ *then the following statements are equivalent:*

(1) $\operatorname{Con} L$ *is boolean;*

(2) $\operatorname{Con} L$ *is relative Stone;*

(3) L *is finite.*

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