Solutions and Periodicity of a Rational Recursive Sequences of Order Five

E. M. Elsayed[1,3] and T. F. Ibrahim[2,3]
[1]King Abdulaziz University, Faculty of Science,
Mathematics Department, P. O. Box 80203,
Jeddah 21589, Saudi Arabia.
E-mail: emelsayed@mans.edu.eg, emmelsayed@yahoo.com.
[2]Department of Mathematics, Faculty of Sciences and arts (S. A.)
King Khalid University, Abha , Saudi Arabia
E-mail: tfibrahem@mans.edu.eg, t_fawzyi@yahoo.com.
[3]Mathematics Department, Faculty of Science,
Mansoura University, Mansoura 35516, Egypt.

Abstract

We get in this paper the form of the solutions of the following recursive sequences

$$x_{n+1} = \frac{x_n x_{n-2} x_{n-4}}{x_{n-1} x_{n-3}(\pm 1 \pm x_n x_{n-2} x_{n-4})}, \quad n = 0, 1, ...,$$

where the initial conditions $x_{-4},\ x_{-3},\ x_{-2},\ x_{-1}$ and x_0 are arbitrary non zero real numbers.

Keywords: difference equations, recursive sequences, stability, periodic solution.
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1 Introduction

In this paper we obtain the solutions of the following difference equations of order five

$$x_{n+1} = \frac{x_n x_{n-2} x_{n-4}}{x_{n-1} x_{n-3}(\pm 1 \pm x_n x_{n-2} x_{n-4})}, \quad n = 0, 1, ..., \tag{1}$$

where the initial conditions $x_{-4},\ x_{-3},\ x_{-2},\ x_{-1}$ and x_0 are arbitrary real numbers.

Recently there has been a great interest in studying the qualitative properties of rational difference equations. For the systematical studies of rational and nonrational difference equations, one can refer to the papers [1–40] and references therein.

The study of rational difference equations of order greater than one is quite challenging and rewarding because some prototypes for the development of the basic theory of the global behavior of nonlinear difference equations of order greater than one come from the results for rational difference equations. However, there have not been any effective general methods to deal with the global behavior of rational difference equations of order greater than one so far. Therefore, the study of rational difference equations of order greater than one is worth further consideration.
Recently, Agarwal et al. [2] investigated the global stability, periodicity character and gave the solution of some special cases of the difference equation

$$x_{n+1} = a + \frac{dx_{n-l}x_{n-k}}{b - cx_{n-s}}.$$

Aloqeili [4] has obtained the solutions of the difference equation

$$x_{n+1} = \frac{x_{n-1}}{a - x_n x_{n-1}}.$$

Cinar [7]-[9] investigated the solutions of the following difference equations

$$x_{n+1} = \frac{x_{n-1}}{1 + ax_n x_{n-1}}, \quad x_{n+1} = \frac{x_{n-1}}{-1 + ax_n x_{n-1}}, \quad x_{n+1} = \frac{ax_{n-1}}{1 + bx_n x_{n-1}}.$$

Elabbasy et al. [10], [12] investigated the global stability, periodicity character and gave the solution of special case of the following recursive sequences

$$x_{n+1} = ax_n - \frac{bx_n}{cx_n - dx_{n-1}}, \quad x_{n+1} = \frac{\alpha x_{n-k}}{\beta + \gamma \prod_{i=0}^{k} x_{n-i}}.$$

Elsayed in [18] studied the behavior of the solutions of the third order rational difference equation

$$x_{n+1} = ax_{n-1} + \frac{bx_n x_{n-1}}{cx_n + dx_{n-2}}.$$

Also, he obtained the expressions of the solutions of four special cases of this equation.
Ibrahim [24] get the solutions of the rational difference equation

$$x_{n+1} = \frac{x_n x_{n-2}}{x_{n-1}(a + bx_n x_{n-2})}.$$

Karatas et al. [25] get the form of the solution of the difference equation

$$x_{n+1} = \frac{x_{n-5}}{1 + x_{n-2}x_{n-5}}.$$

Simsek et al. [31]-[32] obtained the solutions of the following difference equations

$$x_{n+1}=\frac{x_{n-3}}{1+x_{n-1}}, \qquad x_{n+1}=\frac{x_{n-5}}{1+x_{n-1}x_{n-3}}.$$

In [39-40] Zayed and A. El-Moneam dealt with the dynamics of the following rational recursive sequences

$$\begin{aligned} x_{n+1} &= ax_n-\frac{bx_n}{cx_n-dx_{n-k}}, \\ x_{n+1} &= \frac{\alpha x_n+\beta x_{n-1}+\gamma x_{n-2}+\delta x_{n-3}}{Ax_n+Bx_{n-1}+Cx_{n-2}+Dx_{n-3}}. \end{aligned}$$

Here, we recall some notations and results which will be useful in our investigation.

Let I be some interval of real numbers and let

$$f: I^{k+1}\to I,$$

be a continuously differentiable function. Then for every set of initial conditions $x_{-k}, x_{-k+1}, ..., x_0 \in I$, the difference equation

$$x_{n+1}=f(x_n,x_{n-1},...,x_{n-k}), \quad n=0,1,..., \tag{2}$$

has a unique solution $\{x_n\}_{n=-k}^{\infty}$ [27].

Definition 1. (Equilibrium Point)

A point $\overline{x}\in I$ is called an equilibrium point of Eq.(2) if

$$\overline{x}=f(\overline{x},\overline{x},...,\overline{x}).$$

That is, $x_n=\overline{x}$ for $n\geq 0$, is a solution of Eq.(2), or equivalently, $\overline{x}$ is a fixed point of f.

Definition 2. (Stability)

(i) The equilibrium point $\overline{x}$ of Eq.(2) is locally stable if for every $\epsilon>0$, there exists $\delta>0$ such that for all $x_{-k}, x_{-k+1}, ..., x_{-1}, x_0\in I$ with

$$|x_{-k}-\overline{x}|+|x_{-k+1}-\overline{x}|+...+|x_0-\overline{x}|<\delta,$$

we have

$$|x_n-\overline{x}|<\epsilon \qquad \text{for all} \quad n\geq -k.$$

(ii) The equilibrium point $\overline{x}$ of Eq.(2) is locally asymptotically stable if $\overline{x}$ is locally stable solution of Eq.(2) and there exists $\gamma>0$, such that for all $x_{-k}, x_{-k+1}, ..., x_{-1}, x_0 \in I$ with

$$|x_{-k}-\overline{x}|+|x_{-k+1}-\overline{x}|+...+|x_0-\overline{x}|<\gamma,$$

we have

$$\lim_{n\to\infty} x_n=\overline{x}.$$

(iii) The equilibrium point $\overline{x}$ of Eq.(2) is global attractor if for all $x_{-k}, x_{-k+1}, ..., x_{-1}, x_0 \in I$, we have

$$\lim_{n\to\infty} x_n = \overline{x}.$$

(iv) The equilibrium point $\overline{x}$ of Eq.(2) is globally asymptotically stable if $\overline{x}$ is locally stable, and $\overline{x}$ is also a global attractor of Eq.(2).
(v) The equilibrium point $\overline{x}$ of Eq.(2) is unstable if $\overline{x}$ is not locally stable.
Definition 3. (Periodicity)
A solution $\{x_n\}_{n=-k}^{\infty}$ of Eq.(2) is called periodic with period p if there exists an integer $p \geq 1$ such that

$$x_{n+p} = x_n, \quad \text{for all} \quad n \geq -k.$$

A solution is called periodic with prime period p if p is the smallest positive integer for which the previous equation holds.

The linearized equation of Eq.(2) about the equilibrium point $\overline{x}$ is the linear difference equation

$$y_{n+1} = \sum_{i=0}^{k} \frac{\partial f(\overline{x}, \overline{x}, ..., \overline{x})}{\partial x_{n-i}} y_{n-i}, \tag{3}$$

and the equation

$$\lambda^{k+1} - q_0\lambda^k - q_1\lambda^{k-1} - ... - q_{k-1}\lambda - q_k = 0, \tag{4}$$

where $q_i = \dfrac{\partial f(\overline{x}, \overline{x}, ..., \overline{x})}{\partial x_{n-i}}$, for $i = 0, 1, ..., k$, is called the characteristic equation of Eq.(3) about $\overline{x}$.

The following result, known as the Linearized Stability Theorem, is very useful in determining the local stability character of the equilibrium point $\overline{x}$ of Eq.(2).

Theorem A [6] (The Linearized Stability Theorem)

Assume that the function f is a continuously differentiable function defined on some open neighborhood of an equilibrium point $\overline{x}$. Then the following statements are true:

1. When all the roots of Eq.(4) have absolute value less than one, then the equilibrium point $\overline{x}$ of Eq.(2) is locally asymptotically stable.

2. If at least one root of Eq.(4) has absolute value greater than one, then the equilibrium point $\overline{x}$ of Eq.(2) is unstable.

Definition 4. (Hyperbolic)

The equilibrium point $\overline{x}$ of Eq.(2) is called hyperbolic if no root of Eq.(4) has absolute value equal to one. If there exists a root of Eq.(4) with absolute value equal to one, then the equilibrium $\overline{x}$ is called nonhyperbolic.

2 The First Equation $x_{n+1} = \dfrac{x_n x_{n-2} x_{n-4}}{x_{n-1}x_{n-3}(1+x_n x_{n-2} x_{n-4})}$

In this section we give a specific form of the solution of the first equation in the form

$$x_{n+1} = \frac{x_n x_{n-2} x_{n-4}}{x_{n-1}x_{n-3}(1+x_n x_{n-2} x_{n-4})}, \quad n = 0, 1, ..., \tag{5}$$

where the initial values are arbitrary non zero real numbers.

Theorem 1 *Let $\{x_n\}_{n=-4}^{\infty}$ be a solution of Eq.(5). Then for $n = 0, 1, ...$*

$$x_{6n-4} = e\prod_{i=0}^{n-1}\left(\frac{1+6iace}{1+(6i+2)\,ace}\right), \qquad x_{6n-3} = d\prod_{i=0}^{n-1}\left(\frac{1+(6i+1)ace}{1+(6i+3)\,ace}\right),$$

$$x_{6n-2} = c\prod_{i=0}^{n-1}\left(\frac{1+(6i+2)ace}{1+(6i+4)\,ace}\right), \qquad x_{6n-1} = b\prod_{i=0}^{n-1}\left(\frac{1+(6i+3)ace}{1+(6i+5)\,ace}\right),$$

$$x_{6n} = a\prod_{i=0}^{n-1}\left(\frac{1+(6i+4)ace}{1+(6i+6)\,ace}\right), \qquad x_{6n+1} = \frac{ace}{bd(1+ace)}\prod_{i=0}^{n-1}\left(\frac{1+(6i+5)ace}{1+(6i+7)\,ace}\right),$$

where $x_{-4} = e,\ x_{-3} = d,\ x_{-2} = c,\ x_{-1} = b,\ x_0 = a.$

Proof: For $n = 0$ the result holds. Now suppose that $n > 0$ and that our assumption holds for $n - 1$. That is;

$$x_{6n-10} = e\prod_{i=0}^{n-2}\left(\frac{1+6iace}{1+(6i+2)\,ace}\right), \qquad x_{6n-9} = d\prod_{i=0}^{n-2}\left(\frac{1+(6i+1)ace}{1+(6i+3)\,ace}\right),$$

$$x_{6n-8} = c\prod_{i=0}^{n-2}\left(\frac{1+(6i+2)ace}{1+(6i+4)\,ace}\right), \qquad x_{6n-7} = b\prod_{i=0}^{n-2}\left(\frac{1+(6i+3)ace}{1+(6i+5)\,ace}\right),$$

$$x_{6n-6} = a\prod_{i=0}^{n-2}\left(\frac{1+(6i+4)ace}{1+(6i+6)\,ace}\right), \qquad x_{6n-5} = \frac{ace}{bd(1+ace)}\prod_{i=0}^{n-2}\left(\frac{1+(6i+5)ace}{1+(6i+7)\,ace}\right).$$

Now, it follows from Eq.(5) that

$$x_{6n-4} = \frac{x_{6n-5}x_{6n-7}x_{6n-9}}{x_{6n-6}x_{6n-8}(1+x_{6n-5}x_{6n-7}x_{6n-9})}$$

$$= \frac{\frac{ace}{bd(1+ace)}\prod_{i=0}^{n-2}\left(\frac{1+(6i+5)ace}{1+(6i+7)ace}\right) b\prod_{i=0}^{n-2}\left(\frac{1+(6i+3)ace}{1+(6i+5)ace}\right) d\prod_{i=0}^{n-2}\left(\frac{1+(6i+1)ace}{1+(6i+3)ace}\right)}{\left(a\prod_{i=0}^{n-2}\left(\frac{1+(6i+4)ace}{1+(6i+6)ace}\right) c\prod_{i=0}^{n-2}\left(\frac{1+(6i+2)ace}{1+(6i+4)ace}\right)\right)\left(1+\frac{ace}{bd(1+ace)}\prod_{i=0}^{n-2}\left(\frac{1+(6i+5)ace}{1+(6i+7)ace}\right) b\prod_{i=0}^{n-2}\left(\frac{1+(6i+3)ace}{1+(6i+5)ace}\right) d\prod_{i=0}^{n-2}\left(\frac{1+(6i+1)ace}{1+(6i+3)ace}\right)\right)}$$

$$= \frac{\frac{ace}{(1+ace)}\prod_{i=0}^{n-2}\left(\frac{1+(6i+1)ace}{1+(6i+7)\,ace}\right)}{\left(ac\prod_{i=0}^{n-2}\left(\frac{1+(6i+2)ace}{1+(6i+6)\,ace}\right)\right)\left(1+\frac{ace}{(1+ace)}\prod_{i=0}^{n-2}\left(\frac{1+(6i+1)ace}{1+(6i+7)\,ace}\right)\right)}$$

$$\begin{aligned}
&= \frac{\left(\frac{e}{1+(6n-5)ace}\right)}{\left(\prod_{i=0}^{n-2}\left(\frac{1+(6i+2)ace}{1+(6i+6)\,ace}\right)\right)\left(1+\frac{ace}{1+(6n-5)ace}\right)} \\
&= \frac{e}{\left(\prod_{i=0}^{n-2}\left(\frac{1+(6i+2)ace}{1+(6i+6)\,ace}\right)\right)(1+(6n-5)ace+ace)} \\
&= e\prod_{i=0}^{n-2}\left(\frac{1+(6i+6)\,ace}{1+(6i+2)ace}\right)\left(\frac{1}{1+(6n-4)ace}\right).
\end{aligned}$$

Hence, we have

$$x_{6n-4} = e\prod_{i=0}^{n-1}\left(\frac{1+6iace}{1+(6i+2)ace}\right).$$

Similarly

$$x_{6n-3} = \frac{x_{6n-4}x_{6n-6}x_{6n-8}}{x_{6n-5}x_{6n-7}(1+x_{6n-4}x_{6n-6}x_{6n-8})}$$

$$= \frac{e\prod_{i=0}^{n-1}\left(\frac{1+6iace}{1+(6i+2)ace}\right)a\prod_{i=0}^{n-2}\left(\frac{1+(6i+4)ace}{1+(6i+6)ace}\right)c\prod_{i=0}^{n-2}\left(\frac{1+(6i+2)ace}{1+(6i+4)ace}\right)}{\left(\frac{ace}{bd(1+ace)}\prod_{i=0}^{n-2}\left(\frac{1+(6i+5)ace}{1+(6i+7)ace}\right)b\prod_{i=0}^{n-2}\left(\frac{1+(6i+3)ace}{1+(6i+5)ace}\right)\right)\left(1+e\prod_{i=0}^{n-1}\left(\frac{1+6iace}{1+(6i+2)ace}\right)a\prod_{i=0}^{n-2}\left(\frac{1+(6i+4)ace}{1+(6i+6)ace}\right)c\prod_{i=0}^{n-2}\left(\frac{1+(6i+2)ace}{1+(6i+4)ace}\right)\right)}$$

$$= \frac{ace\left(\frac{1}{1+(6n-4)\,ace}\right)}{\left(\frac{ace}{d(1+ace)}\prod_{i=0}^{n-2}\left(\frac{1+(6i+3)ace}{1+(6i+7)\,ace}\right)\right)\left(1+ace\left(\frac{1}{1+(6n-4)\,ace}\right)\right)}$$

$$= d\prod_{i=0}^{n-2}\left(\frac{1+(6i+7)\,ace}{1+(6i+3)ace}\right)\frac{(1+ace)}{(1+(6n-3)\,ace)}.$$

Hence, we have

$$x_{6n-3} = d\prod_{i=0}^{n-1}\left(\frac{1+(6i+1)\,ace}{1+(6i+3)ace}\right).$$

Similarly, one can easily obtain the other relations. Thus, the proof is completed.

Theorem 2 *Eq.(5) has a unique equilibrium point which is the number zero and this equilibrium point is nonhyperbolic.*

Proof: For the equilibrium points of Eq.(5), we can write

$$\overline{x} = \frac{\overline{x}^3}{\overline{x}^2\left(1+\overline{x}^3\right)}.$$

Then we have

$$\begin{aligned}\overline{x}^3\left(1+\overline{x}^3\right) &= \overline{x}^3,\\ \overline{x}^3\left(1+\overline{x}^3-1\right) &= 0,\end{aligned}$$

or,

$$\overline{x}^6 = 0.$$

Thus the equilibrium point of Eq.(5) is $\overline{x} = 0$.
Let $f : (0,\infty)^5 \longrightarrow (0,\infty)$ be a function defined by

$$f(u,v,w,t,p) = \frac{uwp}{vt\left(1+uwp\right)}.$$

Therefore it follows that

$$\begin{aligned}f_u(u,v,w,t,p) &= \frac{wp}{vt\left(1+uwp\right)^2},\quad f_v(u,v,w,t,p) = -\frac{uwp}{v^2t\left(1+uwp\right)},\\ f_w(u,v,w,t,p) &= \frac{up}{vt\left(1+uwp\right)^2},\quad f_t(u,v,w,t,p) = -\frac{uwp}{vt^2\left(1+uwp\right)},\\ f_p(u,v,w,t,p) &= \frac{uw}{vt\left(1+uwp\right)^2},\end{aligned}$$

we see that

$$\begin{aligned}f_u(\overline{x},\overline{x},\overline{x},\overline{x},\overline{x}) &= 1,\quad f_v(\overline{x},\overline{x},\overline{x},\overline{x},\overline{x}) = -1,\quad f_w(\overline{x},\overline{x},\overline{x},\overline{x},\overline{x}) = 1,\\ f_t(\overline{x},\overline{x},\overline{x},\overline{x},\overline{x}) &= -1,\quad f_p(\overline{x},\overline{x},\overline{x},\overline{x},\overline{x}) = 1,\end{aligned}$$

and the characteristic equation about the equilibrium point $\overline{x} = 0$ is given by

$$\lambda^5 - \lambda^4 + \lambda^3 - \lambda^2 + \lambda - 1 = 0,$$

then we obtain that $\lambda = 1$, is one of the roots of the previous equation, then the equilibrium point $\overline{x} = 0$ is nonhyperbolic.

Numerical examples

For confirming the results of this section, we consider numerical examples which represent different types of solutions to Eq. (5).

Example 1. We assume the initial condition as follows: $x_{-4} = 5,\ x_{-3} = 13,\ x_{-2} = 7,\ x_{-1} = 3,\ x_0 = 9$. See Fig. 1.

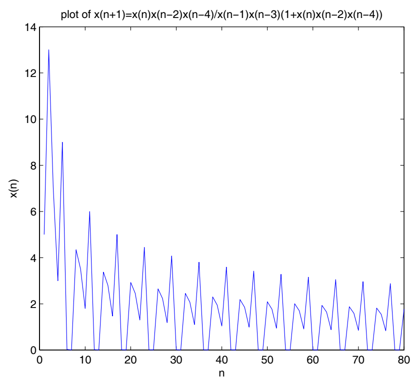

Figure 1.

Example 2. See Fig. 2, since $x_{-4} = 11,\ x_{-3} = 3,\ x_{-2} = 9,\ x_{-1} = 3,\ x_0 = 2$.

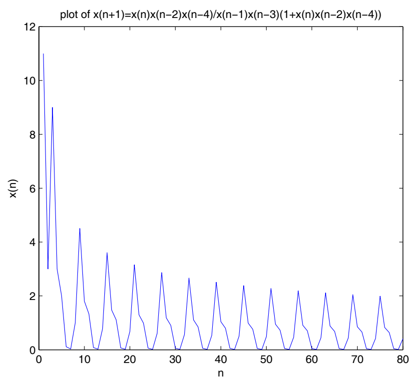

Figure 2.

3 The Second Equation $x_{n+1} = \dfrac{x_n x_{n-2} x_{n-4}}{x_{n-1}x_{n-3}(-1 + x_n x_{n-2} x_{n-4})}$

In this section we obtain the solution of the second equation in the form

$$x_{n+1} = \frac{x_n x_{n-2} x_{n-4}}{x_{n-1}x_{n-3}(-1 + x_n x_{n-2} x_{n-4})}, \quad n = 0, 1, ..., \tag{6}$$

where the initial values are arbitrary non zero real numbers with $x_0x_{-2}x_{-4} \neq 1$.

Theorem 3 *Let $\{x_n\}_{n=-4}^{\infty}$ be a solution of Eq.(6). Then every solution of Eq.(6) is periodic with period 6 and for $n = 0, 1, ...$*

$$\begin{aligned} x_{6n-4} &= e, & x_{6n-3} &= d, & x_{6n-2} &= c, \\ x_{6n-1} &= b, & x_{6n} &= a, & x_{6n+1} &= \frac{ace}{bd(-1+ace)}, \end{aligned}$$

where $x_{-4} = e,\ x_{-3} = d,\ x_{-2} = c,\ x_{-1} = b,\ x_0 = a.$

Proof: For $n = 0$ the result holds. Now suppose that $n > 0$ and that our assumption holds for $n - 1$. That is;

$$\begin{aligned} x_{6n-10} &= e, & x_{6n-9} &= d, & x_{6n-8} &= c, \\ x_{6n-7} &= b, & x_{6n-6} &= a, & x_{6n-5} &= \frac{ace}{bd(-1+ace)}. \end{aligned}$$

Now, it follows from Eq.(6) that

$$\begin{aligned} x_{6n-4} &= \frac{x_{6n-5}x_{6n-7}x_{6n-9}}{x_{6n-6}x_{6n-8}(-1+x_{6n-5}x_{6n-7}x_{6n-9})} = \frac{acebd}{bd(-1+ace)ac\left(-1+\frac{acebd}{bd(-1+ace)}\right)} \\ &= \frac{e}{(-1+ace)\left(-1+\frac{ace}{(-1+ace)}\right)} = \frac{e}{(1-ace+ace)} = e, \end{aligned}$$

$$x_{6n-3} = \frac{x_{6n-4}x_{6n-6}x_{6n-8}}{x_{6n-5}x_{6n-7}(-1+x_{6n-4}x_{6n-6}x_{6n-8})} = \frac{eac}{\left(\frac{ace}{bd(-1+ace)}\right)b(-1+ace)} = d,$$

$$x_{6n-2} = \frac{x_{6n-3}x_{6n-5}x_{6n-7}}{x_{6n-4}x_{6n-6}(-1+x_{6n-3}x_{6n-5}x_{6n-7})} = \frac{d\left(\frac{ace}{bd(-1+ace)}\right)b}{ea\left(-1+d\left(\frac{ace}{bd(-1+ace)}\right)b\right)} = c,$$

$$x_{6n-1} = \frac{x_{6n-2}x_{6n-4}x_{6n-6}}{x_{6n-3}x_{6n-5}(-1+x_{6n-2}x_{6n-4}x_{6n-6})} = \frac{cea}{d\left(\frac{ace}{bd(-1+ace)}\right)(-1+cea)} = b,$$

$$x_{6n} = \frac{x_{6n-1}x_{6n-3}x_{6n-5}}{x_{6n-2}x_{6n-4}(-1+x_{6n-1}x_{6n-3}x_{6n-5})} = \frac{bd\left(\frac{ace}{bd(-1+ace)}\right)}{ce\left(-1+bd\left(\frac{ace}{bd(-1+ace)}\right)\right)} = a.$$

Finally,

$$x_{6n+1} = \frac{x_{6n}x_{6n-2}x_{6n-4}}{x_{6n-1}x_{6n-3}(-1+x_{6n}x_{6n-2}x_{6n-4})} = \frac{ace}{db\,(-1+ace)}.$$

Thus, the proof is completed.

Theorem 4 *Eq. (6) has a periodic solution of period three iff* $e = b,\ d = a,\ ace = 2$ *and it will be taken the following form* $\{x_n\} = \{b, a, c, b, a, ...\}$.

Proof: First suppose that there exists a prime period three solution $\{x_n\} = \{b, a, c, b, a, ...\}$ of Eq. (6), we see from the form of the solution of Eq. (6) that

$$\begin{aligned} x_{6n-4} &= e = b, \qquad x_{6n-3} = d = a, \qquad x_{6n-2} = c, \\ x_{6n-1} &= b, \quad x_{6n} = a, \quad x_{6n+1} = \frac{ace}{bd(-1+ace)} = c, \end{aligned}$$

Then we get

$$e = b, \quad d = a, \quad ace = 2.$$

Second assume that $e = b, \quad d = a, \quad ace = 2$. Then we see that

$$\begin{aligned} x_{6n-4} &= b, \qquad x_{6n-3} = a, \qquad x_{6n-2} = c, \\ x_{6n-1} &= b, \qquad x_{6n} = a, \qquad x_{6n+1} = c, \end{aligned}$$

Thus we have a periodic solution of period three and the proof is complete.

Theorem 5 *Eq.(6) has two equilibrium points which are* $0, \sqrt[3]{2}$ *and the equilibrium point* $\overline{x} = \sqrt[3]{2}$ *is nonhyperbolic.*

Proof: For the equilibrium points of Eq.(6), we can write

$$\overline{x} = \frac{\overline{x}^3}{\overline{x}^2\left(-1+\overline{x}^3\right)}.$$

Then we have

$$\overline{x}^3\left(-1+\overline{x}^3\right) = \overline{x}^3,$$

or

$$\overline{x}^3\left(\overline{x}^3 - 2\right) = 0,$$

Thus the equilibrium points of Eq.(6) are $0, \sqrt[3]{2}$.
Let $f : (0,\infty)^5 \longrightarrow (0,\infty)$ be a function defined by

$$f(u,v,w,t,p) = \frac{uwp}{vt\left(-1+uwp\right)}.$$

Therefore it follows that

$$\begin{aligned} f_u(u,v,w,t,p) &= -\frac{wp}{vt\left(-1+uwp\right)^2}, \quad f_v(u,v,w,t,p) = -\frac{uwp}{v^2t\left(-1+uwp\right)}, \\ f_w(u,v,w,t,p) &= -\frac{up}{vt\left(-1+uwp\right)^2}, \quad f_t(u,v,w,t,p) = -\frac{uwp}{vt^2\left(-1+uwp\right)}, \\ f_p(u,v,w,t,p) &= -\frac{uw}{vt\left(-1+uwp\right)^2}, \end{aligned}$$

we see that (at $\overline{x} = \sqrt[3]{2}$)

$$\begin{aligned} f_u(\overline{x},\overline{x},\overline{x},\overline{x},\overline{x}) &= -1, \quad f_v(\overline{x},\overline{x},\overline{x},\overline{x},\overline{x}) = -1, \quad f_w(\overline{x},\overline{x},\overline{x},\overline{x},\overline{x}) = -1, \\ f_t(\overline{x},\overline{x},\overline{x},\overline{x},\overline{x}) &= -1, \quad f_p(\overline{x},\overline{x},\overline{x},\overline{x},\overline{x}) = -1. \end{aligned}$$

Thus the characteristic equation about the equilibrium point $\overline{x} = \sqrt[3]{2}$ is given by

$$\lambda^5 + \lambda^4 + \lambda^3 + \lambda^2 + \lambda + 1 = 0.$$

Also, we see that $\lambda = -1$, one of the roots of this equation, then the equilibrium point $\overline{x} = \sqrt[3]{2}$ is nonhyperbolic.

Numerical examples

Here we will represent different types of solutions of Eq. (6).

Example 3. We consider Eq.(6) with $x_{-4} = 11,\ x_{-3} = 3,\ x_{-2} = 9,\ x_{-1} = 3,\ x_0 = 2$ See Fig. 3.

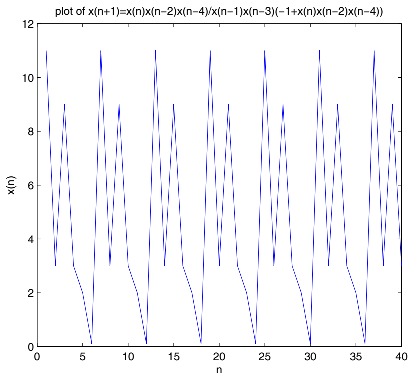

Figure 3.

Example 4. Figure 4 shows the behavior of the solutions of Eq.(6) with the initial conditions: $x_{-4} = 5,\ x_{-3} = -3,\ x_{-2} = -2/15,\ x_{-1} = 5,\ x_0 = -3$.

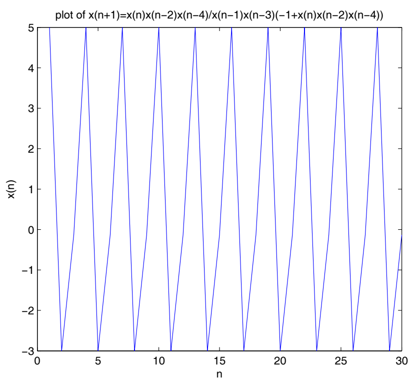

Figure 4.

The following cases can be proved similarly.

4 The Third Equation $x_{n+1}=\dfrac{x_nx_{n-2}x_{n-4}}{x_{n-1}x_{n-3}(1-x_nx_{n-2}x_{n-4})}$

In this section, we get the expressions of the solution of the third equation which in the following form

$$x_{n+1}=\frac{x_nx_{n-2}x_{n-4}}{x_{n-1}x_{n-3}(1-x_nx_{n-2}x_{n-4})},\quad n=0,1,..., \tag{7}$$

where the initial values are arbitrary non zero real numbers.

Theorem 6 *Let $\{x_n\}_{n=-4}^{\infty}$ be a solution of Eq.(7). Then the solutions of Eq.(7) takes the following form for $n=0,1,...$*

$$x_{6n-4} = e\prod_{i=0}^{n-1}\left(\frac{1-6iace}{1-(6i+2)\,ace}\right),\qquad x_{6n-3}=d\prod_{i=0}^{n-1}\left(\frac{1-(6i+1)ace}{1-(6i+3)\,ace}\right),$$

$$x_{6n-2} = c\prod_{i=0}^{n-1}\left(\frac{1-(6i+2)ace}{1-(6i+4)\,ace}\right),\qquad x_{6n-1}=b\prod_{i=0}^{n-1}\left(\frac{1-(6i+3)ace}{1-(6i+5)\,ace}\right),$$

$$x_{6n} = a\prod_{i=0}^{n-1}\left(\frac{1-(6i+4)ace}{1-(6i+6)\,ace}\right),\qquad x_{6n+1}=\frac{ace}{bd(1-ace)}\prod_{i=0}^{n-1}\left(\frac{1-(6i+5)ace}{1-(6i+7)\,ace}\right),$$

where $x_{-4}=e,\ x_{-3}=d,\ x_{-2}=c,\ x_{-1}=b,\ x_0=a.$

Theorem 7 *Eq.(7) has a unique equilibrium point which is the number zero and this equilibrium point is nonhyperbolic.*

Example 5. Assume that the initial values for Eq.(7) $x_{-4}=10,\ x_{-3}=4,\ x_{-2}=9,\ x_{-1}=6,\ x_0=2$ see Fig. 5

Example 6. See Fig. 6 since $x_{-4}=2,\ x_{-3}=7,\ x_{-2}=5,\ x_{-1}=8,\ x_0=12.$

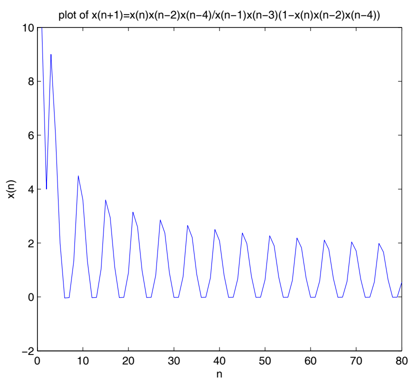

Figure 5.

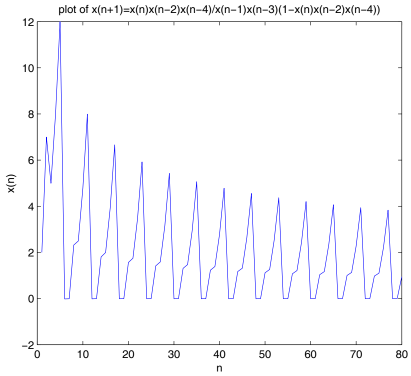

Figure 6.

5 The Fourth Equation $x_{n+1} = \frac{x_n x_{n-2} x_{n-4}}{x_{n-1} x_{n-3}(-1 - x_n x_{n-2} x_{n-4})}$

Here we obtain a form of the solutions of the equation

$$x_{n+1} = \frac{x_n x_{n-2} x_{n-4}}{x_{n-1} x_{n-3}(-1 - x_n x_{n-2} x_{n-4})}, \quad n = 0, 1, ..., \tag{8}$$

where the initial values are arbitrary non zero real numbers with $x_{-4}x_{-2}x_0 \neq -1$.

Theorem 8 *Let $\{x_n\}_{n=-4}^{\infty}$ be a solution of Eq.(8). Then every solution of Eq.(8) is periodic with period 6 and for $n = 0, 1, ...$*

$$\begin{aligned} x_{6n-4} &= e, \qquad x_{6n-3} = d, \qquad x_{6n-2} = c, \\ x_{6n-1} &= b, \qquad x_{6n} = a, \qquad x_{6n+1} = \frac{ace}{bd(-1 - ace)}, \end{aligned}$$

where $x_{-4} = e,\ x_{-3} = d,\ x_{-2} = c,\ x_{-1} = b,\ x_0 = a.$

Theorem 9 *Eq. (8) has a periodic solution of period three iff $e = b,\ d = a,\ ace = -2$ and it will be taken the following form $\{x_n\} = \{b, a, c, b, a, ...\}$.*

Theorem 10 *Eq.(8) has two equilibrium points which are $0, \sqrt[3]{-2}$ and the equilibrium point $\overline{x} = \sqrt[3]{-2}$ is nonhyperbolic.*

Example 7. Consider $x_{-4} = -2,\ x_{-3} = 7,\ x_{-2} = 1/7,\ x_{-1} = -2,\ x_0 = 7$ see Fig. 7.

Example 8. Fig. 8 shows the solution of Eq.(8) with the initial conditions $x_{-4} = 11,\ x_{-3} = -7,\ x_{-2} = 13,\ x_{-1} = 8,\ x_0 = -3.$

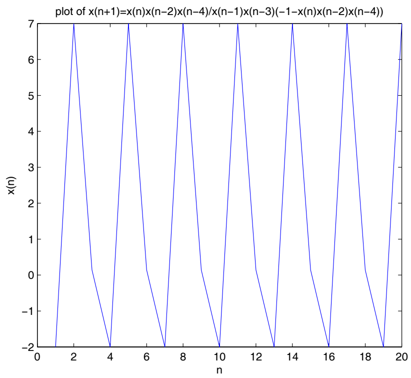

Figure 7.

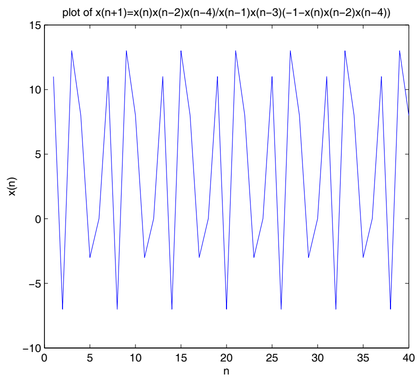

Figure 8.

References

[1] R. P. Agarwal, Difference Equations and Inequalities, 1^{st} edition, Marcel Dekker, New York, 1992, 2^{nd} edition, 2000.

[2] R. P. Agarwal and E. M. Elsayed, Periodicity and stability of solutions of higher order rational difference equation, Advanced Studies in Contemporary Mathematics, 17 (2) (2008), 181-201.

[3] R. P. Agarwal and E. M. Elsayed, On the solution of fourth-order rational recursive sequence, Advanced Studies in Contemporary Mathematics, 20 (4) (2010), 525–545.

[4] M. Aloqeili, Dynamics of a rational difference equation, Appl. Math. Comp., 176(2), (2006), 768-774.

[5] M. Atalay, C. Cinar and I. Yalcinkaya, On the positive solutions of systems of difference equations, International Journal of Pure and Applied Mathematics, 24(4) (2005), 443-447.

[6] E. Camouzis and G. Ladas, Dynamics of Third-Order Rational Difference Equations with Open Problems and Conjectures, Chapman & Hall/CRC Press, 2008.

[7] C. Cinar, On the positive solutions of the difference equation $x_{n+1} = \frac{x_{n-1}}{1+ax_nx_{n-1}}$, Appl. Math. Comp., 158(3) (2004), 809-812.

[8] C. Cinar, On the positive solutions of the difference equation $x_{n+1} = \frac{x_{n-1}}{-1+ax_nx_{n-1}}$, Appl. Math. Comp., 158(3) (2004), 793-797.

[9] C. Cinar, On the positive solutions of the difference equation $x_{n+1} = \frac{ax_{n-1}}{1+bx_nx_{n-1}}$, Appl. Math. Comp., 156 (2004) 587-590.

[10] E. M. Elabbasy, H. El-Metwally and E. M. Elsayed, On the difference equation $x_{n+1} = ax_n - \frac{bx_n}{cx_n - dx_{n-1}}$, Advances in Difference Equations, Volume 2006, Article ID 82579,1–10.

[11] E. M. Elabbasy, H. El-Metwally and E. M. Elsayed, Qualitative behavior of higher order difference equation, Soochow Journal of Mathematics, 33 (4) (2007), 861-873.

[12] E. M. Elabbasy, H. El-Metwally and E. M. Elsayed, On the difference equations $x_{n+1} = \frac{\alpha x_{n-k}}{\beta + \gamma \prod_{i=0}^{k} x_{n-i}}$, J. Conc. Appl. Math., 5(2) (2007), 101-113.

[13] E. M. Elabbasy, H. El-Metwally and E. M. Elsayed, Global behavior of the solutions of difference equation, Advances in Difference Equations 2011, 2011:28.

[14] E. M. Elabbasy, H. El-Metwally and E. M. Elsayed, Some properties and expressions of solutions for a class of nonlinear difference equation, Utilitas Mathematica, 87 (2012), 93-110.

[15] E. M. Elabbasy and E. M. Elsayed, Global attractivity and periodic nature of a difference equation, World Applied Sciences Journal, 12 (1) (2011), 39–47.

[16] H. El-Metwally, Global behavior of an economic model, Chaos, Solitons and Fractals, 33 (2007), 994–1005.

[17] E. M. Elsayed, Behavior of a rational recursive sequences, Studia Univ. " Babes — Bolyai ", Mathematica, LVI (1) (2011), 27–42.

[18] E. M. Elsayed, Solution and attractivity for a rational recursive sequence, Discrete Dynamics in Nature and Society, Volume 2011, Article ID 982309, 17 pages.

[19] E. M. Elsayed, On the solution of some difference equations, European Journal of Pure and Applied Mathematics, 4 (3) (2011), 287-303.

[20] E. M. Elsayed, On the dynamics of a higher order rational recursive sequence, Communications in Mathematical Analysis, 12 (1) (2012), 117–133.

[21] E. M. Elsayed, Solutions of rational difference system of order two, Mathematical and Computer Modelling, 55 (2012), 378–384.

[22] E. M. Elsayed and M. M. El-Dessoky, Dynamics and behavior of a higher order rational recursive sequence, Advances in Difference Equations 2012, 2012:69.

[23] C. Hengkrawit, V. Laohakosol and P. Udomkavanich, Rational recursive equations characterizing cotangent-tangent and hyperbolic cotangent-tangent functions, Bull. Malays. Math. Sci. Soc. (2) 33(3) (2010), 421–428.

[24] T. F. Ibrahim, On the third order rational difference equation $x_{n+1} = \frac{x_n x_{n-2}}{x_{n-1}(a + b x_n x_{n-2})}$, Int. J. Contemp. Math. Sciences, 4 (27) (2009), 1321-1334.

[25] R. Karatas, C. Cinar and D. Simsek, On positive solutions of the difference equation $x_{n+1} = \frac{x_{n-5}}{1 + x_{n-2} x_{n-5}}$, Int. J. Contemp. Math. Sci., 1(10) (2006), 495-500.

[26] V. L. Kocic and G. Ladas, Global Behavior of Nonlinear Difference Equations of Higher Order with Applications, Kluwer Academic Publishers, Dordrecht, 1993.

[27] M. R. S. Kulenovic and G. Ladas, Dynamics of Second Order Rational Difference Equations with Open Problems and Conjectures, Chapman & Hall / CRC Press, 2001.

[28] C. Peng and Z. Chen, On a conjecture concerning some nonlinear difference equations, Bull. Malays. Math. Sci. Soc. (2) 36(1) (2013), 221–227.

[29] M. Saleh and S. Abu-Baha, Dynamics of a higher order rational difference equation, Appl. Math. Comp., 181 (2006), 84–102.

[30] S. H. Saker, Oscillation of a certain class of third order nonlinear difference equations, Bull. Malays. Math. Sci. Soc. (2) 35(3) (2012), 651–669.

[31] D. Simsek, C. Cinar and I. Yalcinkaya, On the recursive sequence $x_{n+1} = \frac{x_{n-3}}{1 + x_{n-1}}$, Int. J. Contemp. Math. Sci., 1 (10) (2006), 475-480.

[32] D. Simsek, C. Cınar, R. Karatas and I. Yalcinkaya, On the recursive sequence $x_{n+1} = \frac{x_{n-5}}{1 + x_{n-1}x_{n-3}}$, Int. J. of Pure and Appl. Math., 28 (2006), 117-124.

[33] N. Touafek and E. M. Elsayed, On the solutions of systems of rational difference equations, Mathematical and Computer Modelling, 55 (2012), 1987–1997.

[34] N. Touafek and E. M. Elsayed, On the periodicity of some systems of nonlinear difference equations, Bull. Math. Soc. Sci. Math. Roumanie, Tome 55 (103) (2), (2012), 217–224.

[35] C. Wang, F. Gong, S. Wang, L. LI and Q. Shi, Asymptotic behavior of equilibrium point for a class of nonlinear difference equation, Advances in Difference Equations, Volume 2009, Article ID 214309, 8 pages.

[36] I. Yalçınkaya, C. Cinar and M. Atalay, On the solutions of systems of difference equations, Advances in Difference Equations, Vol. 2008, Article ID 143943, 9 pages, doi: 10.1155/2008/ 143943.

[37] I. Yalçınkaya, On the global asymptotic stability of a second-order system of difference equations, Discrete Dynamics in Nature and Society, Vol. 2008, Article ID 860152, 12 pages, doi: 10.1155/2008/ 860152.

[38] I. Yalçınkaya, On the difference equation $x_{n+1} = \alpha + \frac{x_{n-m}}{x_n^k}$, Discrete Dynamics in Nature and Society, Vol. 2008, Article ID 805460, 8 pages, doi: 10.1155/2008/ 805460.

[39] E. M. E. Zayed and M. A. El-Moneam, On the rational recursive sequence $x_{n+1} = ax_n - \frac{bx_n}{cx_n - dx_{n-k}}$, Comm. Appl. Nonlinear Analysis, 15 (2008), 47-57.

[40] E. M. E. Zayed and M. A. El-Moneam, On the rational recursive sequence $x_{n+1} = \frac{\alpha x_n + \beta x_{n-1} + \gamma x_{n-2} + \delta x_{n-3}}{Ax_n + Bx_{n-1} + Cx_{n-2} + Dx_{n-3}}$; Comm. Appl. Nonlinear Analysis, 12 (2005), 15-28.