.

SEVERAL INEQUALITIES FOR THE VOLUME OF THE UNIT BALL IN $\mathbb{R}^n$

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ABSTRACT. In the paper, the author establishes several new inequalities involving the volume of the unit ball in $\mathbb{R}^n$.

1. INTRODUCTION

In the recent past, inequalities about the Euler gamma function $\Gamma(x)$ have attracted the attention of many authors. In particular, several researchers established interesting properties of the volume of the unit ball in $\mathbb{R}^n$,

$$\Omega_n = \frac{\pi^{n/2}}{\Gamma(n/2+1)}, n = 1, 2\cdots . \tag{1.1}$$

In the paper [5], it was proved that the sequence $\{\Omega_n\}_{n\geqslant 1}$ attains its maximum at $n = 5$. In the paper [4], the sequence $\left\{(\Omega_n)^{1/n}\right\}_{n\geqslant 1}$ is proved to be monotonically decreased to zero. Other results have been established by Anderson and Qiu[3], and Klain and Rota[9] who proved that the sequence $\left\{(\Omega_n)^{1/n\ln n}\right\}_{n\geqslant 1}$ decreases to $e^{-1/2}$, and the sequence $\left\{\frac{n\Omega_n}{\Omega_{n-1}}\right\}_{n\geqslant 1}$ is increasing, respectively. Motivated by the following inequalities

$$(\Omega_{n+1})^{n/(n+1)} < \Omega_n, n = 1, 2\cdots \tag{1.2}$$

and

$$1 < \frac{\Omega_n^2}{\Omega_{n-1}\Omega_{n+1}} < 1 + \frac{1}{n} \tag{1.3}$$

stated in [4] and [9], Alzer proved in [1] that for all $n \geqslant 1$,

$$a(\Omega_{n+1})^{n/(n+1)} \leqslant \Omega_n \leqslant b(\Omega_{n+1})^{n/(n+1)} \tag{1.4}$$

with the best possible constants $a = 2/\sqrt{\pi} = 1.1283\cdots$ and $b = \sqrt{e} = 1.6487\cdots$. An improvement of the double inequality (1.4) was given in [11]: for $n \geqslant 4$,

$$\frac{k}{\sqrt[2n]{2\pi}} \leqslant \frac{\Omega_n}{(\Omega_{n+1})^{n/(n+1)}} \leqslant \frac{\sqrt{e}}{\sqrt[2n]{2\pi}} \tag{1.5}$$

where $k = \frac{64\cdot 720^{11/12}\cdot 2^{1/22}}{10395\pi^{5/11}} = 1.5714\cdots$. Equality in the left-hand side of (1.5) occurs if and only if $n = 11$.

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The following class of inequalities

$$\sqrt{\frac{n+a}{2\pi}} \leqslant \frac{\Omega_{n-1}}{\Omega_n} \leqslant \sqrt{\frac{n+b}{2\pi}} \tag{1.6}$$

was studied by Alzer [1] and Qiu[14] where a, b are real parameters. Later, the inequality (1.6) was recovered in [6]. Furthermore, Mortici established the following new sharp bounds

$$\sqrt{\frac{n+\frac{1}{2}}{2\pi}} \leqslant \frac{\Omega_{n-1}}{\Omega_n} \leqslant \sqrt{\frac{n+\frac{1}{2}}{2\pi} + \frac{1}{16n\pi}} \tag{1.7}$$

which improves the previous results of Alzer et in [11]. Therefore, Alzer proved in [1] that for $n \geqslant 1$,

$$\left(1+\frac{1}{n}\right)^{\alpha} < \frac{\Omega_n^2}{\Omega_{n-1}\Omega_{n+1}} < \left(1+\frac{1}{n}\right)^{\beta} \tag{1.8}$$

in which the best possible constants $\alpha = 2 - \log_2 \pi$ and $\beta = \frac{1}{2}$. Later, in [11], Mortici showed that for every $n \geqslant 4$,

$$\left(1+\frac{1}{n}\right)^{1/2-1/4n} < \frac{\Omega_n^2}{\Omega_{n-1}\Omega_{n+1}} < \left(1+\frac{1}{n}\right)^{1/2}. \tag{1.9}$$

Related references see [7][8][13][15].

The aim of this paper is to establish some new inequalities involving the volume of the unit ball in $\mathbb{R}^n$.

2. Lemmas

In order to prove the main results, following lemmas are useful.

Lemma 2.1. [10, p. 390] *Let* $x_i \in \mathbb{R}^+, i = 1, 2 \cdots n$ *and* $\sum_{i=1}^{n} x_i = nx$, *then*

$$\prod_{i=1}^{n} \Gamma(x_i) \geqslant (\Gamma(x))^n. \tag{2.1}$$

Lemma 2.2. [2, Legendre] *For every* $z \neq -1, -2 \cdots$, *then*

$$2^{2z-1}\Gamma(z)\Gamma(z+1/2) = \pi^{1/2}\Gamma(2z). \tag{2.2}$$

Lemma 2.3. [4, p. 131] *For every integer* $n \geqslant 1$, *the sequence* $\left\{(\Omega_n)^{1/n}\right\}_{n\geqslant 1}$ *is monotonically decreasing to zero.*

Lemma 2.4. [12, p. 612] *For every* $x \in [1, \infty)$, *we have*

$$\sqrt{\pi}\left(\frac{x}{e}\right)^x \sqrt{2x+\alpha} < \Gamma(x+1) < \sqrt{\pi}\left(\frac{x}{e}\right)^x \sqrt{2x+\beta} \tag{2.3}$$

where $\alpha = \frac{1}{3}$ *and* $\beta = \sqrt[3]{\frac{391}{30}} - 2 = 0.3533\cdots$.

3. Main Results

In what follows, we always suppose $\beta = \sqrt[3]{\frac{391}{30}} - 2 = 0.3533\cdots$.

Theorem 3.1. *For all natural number n, we have*

$$\Omega_n \leqslant (\Omega_1\Omega_2\cdots\Omega_{n-1})^{1/(n-1)}. \tag{3.1}$$

If n is odd integer, then

$$(\Omega_1\Omega_2\cdots\Omega_n)^{1/n} \leqslant \Omega_{(n+1)/2}. \tag{3.2}$$

Proof. Using Lemma 2.3, we easily prove inequality (3.1). Next, we only prove inequality (3.2). By virtue of Lemma 2.1, we get

$$(\Omega_1\Omega_2\cdots\Omega_n)^{1/n} = \left(\frac{\pi^{1/2}}{\Gamma(1/2+1)}\frac{\pi^{2/2}}{\Gamma(2/2+1)}\cdots\frac{\pi^{n/2}}{\Gamma(n/2+1)}\right)^{1/n}$$
$$= \frac{\pi^{(n+1)/4}}{(\Gamma(1/2+1)\Gamma(2/2+1)\cdots\Gamma(n/2+1))^{1/n}} \leqslant \frac{\pi^{(n+1)/4}}{\Gamma((n+1)/4+1)} = \Omega_{(n+1)/2}.$$

□

Theorem 3.2. *For every integer $n \geqslant 1$, we have*

$$\frac{(n+1)\left(n+\frac{1}{6}\right)}{(n+\beta)^2} < \frac{\Omega_n^2}{\Omega_{n-1}\Omega_{n+1}} < \frac{(n+1)\left(n+\frac{\beta}{2}\right)}{\left(n+\frac{1}{3}\right)^2}. \tag{3.3}$$

Proof. Easy computation and simplification yield

$$\frac{\Omega_n^2}{\Omega_{n-1}\Omega_{n+1}} = \frac{\Gamma((n+1)/2)\Gamma((n+3)/2)}{(\Gamma((n+2)/2))^2}. \tag{3.4}$$

Setting $z = \frac{n+1}{2}$ and $z = \frac{n+3}{2}$ in (2.2) of Lemma 2.2, we obtain

$$2^n\Gamma((n+1)/2)\Gamma((n+2)/2) = \pi^{1/2}n! \tag{3.5}$$

and

$$2^{n+2}\Gamma((n+3)/2)\Gamma((n+4)/2) = \pi^{1/2}\Gamma(n+3) = \pi^{1/2}(n+2)!. \tag{3.6}$$

Combining (3.4), (3.5) and (3.6) leads to

$$\frac{\Omega_n^2}{\Omega_{n-1}\Omega_{n+1}} = \frac{\sqrt{\pi}(n+2)!\sqrt{\pi}n!}{2^n2^{n+2}\Gamma((n+4)/2)\left(\Gamma((n+2)/2)\right)^3} = \frac{\pi(n+1)!n!}{2^{2n+1}\left(\Gamma(n/2+1)\right)^4} \tag{3.7}$$

where we apply $\Gamma((n+4)/2) = \frac{n+2}{2}\Gamma((n+2)/2)$.

Using Lemma 2.4, we have

$$\sqrt{\pi}\left(\frac{n}{2e}\right)^{n/2}\sqrt{n+\frac{1}{3}} < \Gamma(n/2+1) < \sqrt{\pi}\left(\frac{n}{2e}\right)^{n/2}\sqrt{n+\beta} \tag{3.8}$$

and

$$\sqrt{\pi}\left(\frac{n}{e}\right)^n\sqrt{2n+\frac{1}{3}} < n! < \sqrt{\pi}\left(\frac{n}{e}\right)^n\sqrt{2n+\beta}. \tag{3.9}$$

Applying (3.8) and (3.9), we have

$$\frac{\Omega_n^2}{\Omega_{n-1}\Omega_{n+1}} > \frac{\pi\left(\sqrt{\pi}\left(\frac{n}{e}\right)^n\sqrt{2n+\frac{1}{3}}\right)^2(n+1)}{2^{2n+1}\left(\sqrt{\pi}\left(\frac{n}{2e}\right)^{n/2}\sqrt{n+\beta}\right)^4} = \frac{(n+1)\left(n+\frac{1}{6}\right)}{(n+\beta)^2} \tag{3.10}$$

and

$$\frac{\Omega_n^2}{\Omega_{n-1}\Omega_{n+1}} < \frac{\pi\left(\sqrt{\pi}\left(\frac{n}{e}\right)^n\sqrt{2n+\beta}\right)^2(n+1)}{2^{2n+1}\left(\sqrt{\pi}\left(\frac{n}{2e}\right)^{n/2}\sqrt{n+\frac{1}{3}}\right)^4} = \frac{(n+1)\left(n+\frac{\beta}{2}\right)}{\left(n+\frac{1}{3}\right)^2}. \tag{3.11}$$

The proof of Theorem 3.2 is complete. □

Noting simple inequalities

$$\frac{(n+1)\left(n+\frac{1}{6}\right)}{(n+\beta)^2} > \frac{n+\frac{1}{6}}{n+\beta}$$

and

$$\frac{(n+1)\left(n+\frac{\beta}{2}\right)}{\left(n+\frac{1}{3}\right)^2} < \frac{n+1}{n+\frac{1}{3}},$$

we get the Corollary 3.1.

Corollary 3.1. *For every integer $n \geqslant 1$, it holds*

$$\frac{n+\frac{1}{6}}{n+\beta} < \frac{\Omega_n^2}{\Omega_{n-1}\Omega_{n+1}} < \frac{n+1}{n+\frac{1}{3}}. \tag{3.12}$$

Theorem 3.3. *For every integer $n \geqslant 1$, it holds*

$$\frac{\sqrt{e}}{\sqrt[2n+2]{2\pi}}\frac{\left(\sqrt{n+\frac{4}{3}}\right)^{(2n+1)/(n+1)}}{\sqrt{(n+1)\left(n+1+\frac{\beta}{2}\right)}} < \frac{\Omega_n}{(\Omega_{n+1})^{n/(n+1)}} < \frac{\sqrt{e}}{\sqrt[2n+2]{2\pi}}\frac{\left(\sqrt{n+1+\beta}\right)^{(2n+1)/(n+1)}}{\sqrt{(n+1)\left(n+\frac{7}{6}\right)}}. \tag{3.13}$$

Proof. Setting $z = \frac{n+2}{2}$ in (2.2) of Lemma 2.2, we get

$$2^{n+1}\Gamma((n+2)/2)\Gamma((n+3)/2) = \pi^{1/2}\Gamma(n+2) = \pi^{1/2}(n+1)!. \tag{3.14}$$

Easy computation and simplification yield

$$\begin{aligned}\frac{\Omega_n}{(\Omega_n)^{n/(n+1)}} &= \frac{\pi^{n/2}}{\Gamma(n/2+1)}\frac{(\Gamma((n+1)/2+1))^{n/(n+1)}}{\left(\pi^{(n+1)/2}\right)^{n/(n+1)}}\\ &= \frac{2^{n+1}\left(\Gamma((n+1)/2+1)\right)^{n/(n+1)}}{\sqrt{\pi}\,(n+1)!}.\end{aligned} \tag{3.15}$$

Similarly to proof of Theorem 3.2, we have

$$\begin{aligned}\frac{\Omega_n}{(\Omega_{n+1})^{n/(n+1)}} &> \frac{2^{n+1}\left(\sqrt{\pi}\left(\frac{n+1}{2e}\right)^{(n+1)/2}\sqrt{n+1+\frac{1}{3}}\right)^{(2n+1)/(n+1)}}{\sqrt{\pi}\left(\frac{n+1}{e}\right)^{n+1}\sqrt{2n+2+\beta}}\\ &= \frac{\sqrt{e}}{\sqrt[2n+2]{2\pi}}\frac{\left(\sqrt{n+\frac{4}{3}}\right)^{(2n+1)/(n+1)}}{\sqrt{(n+1)\left(n+1+\frac{\beta}{2}\right)}}\end{aligned}$$

and

$$\frac{\Omega_n}{(\Omega_{n+1})^{n/(n+1)}} < \frac{2^{n+1}\left(\sqrt{\pi}\left(\frac{n+1}{2e}\right)^{(n+1)/2}\sqrt{n+1+\beta}\right)^{(2n+1)/(n+1)}}{\sqrt{\pi}\left(\frac{n+1}{e}\right)^{n+1}\sqrt{2n+2+\frac{1}{3}}}$$

$$= \frac{\sqrt{e}}{\sqrt[2n+2]{2\pi}}\frac{\left(\sqrt{n+1+\beta}\right)^{(2n+1)/(n+1)}}{\sqrt{(n+1)\left(n+\frac{7}{6}\right)}}.$$

The proof of Theorem 3.3 is complete. □

Noting simple inequalities

$$\frac{\left(\sqrt{n+1+\beta}\right)^{(2n+1)/(n+1)}}{\sqrt{(n+1)\left(n+\frac{7}{6}\right)}} \leqslant \frac{(n+1+\beta)^{(2n+2)/(2n+2)}}{\sqrt{(n+1)(n+1)}} = \frac{n+1+\beta}{n+1}$$

and

$$\frac{\left(\sqrt{n+\frac{4}{3}}\right)^{(2n+1)/(n+1)}}{\sqrt{(n+1)\left(n+1+\frac{\beta}{2}\right)}} \geqslant \frac{\left(\sqrt{n+1+\frac{\beta}{2}}\right)^{(2n+1)/(n+1)}}{n+1+\frac{\beta}{2}} > \frac{1}{\sqrt[2n+2]{n+1+\frac{\beta}{2}}},$$

we easily get the Corollary 3.2.

Corollary 3.2. *For every integer $n \geqslant 1$, we have*

$$\frac{\sqrt{e}}{\sqrt[2n+2]{2\pi}}\frac{1}{\sqrt[2n+2]{n+1+\frac{\beta}{2}}} < \frac{\Omega_n}{(\Omega_{n+1})^{n/(n+1)}} < \frac{\sqrt{e}}{\sqrt[2n+2]{2\pi}}\frac{n+1+\beta}{n+1}. \tag{3.16}$$

Finally, we give a monotone result related to the volume of the unit ball in $\mathbb{R}^n$.

Theorem 3.4. *For every integer $n \geqslant 3$, the sequence $\left\{(\Omega_n)^{1/H_n}\right\}_{n\geqslant 3}$ is monotonically decreasing to zero, where H_n denotes the n-th harmonic number. Further, the sequence $\left\{(\Omega_n)^{1/H_n}\right\}_{n\geqslant 1}$ attains its maximum at $n = 3$.*

Proof. By taking the logarithm, we only prove that

$$\frac{\ln\Omega_n}{H_n} \geqslant \frac{\ln\Omega_{n+1}}{H_{n+1}}. \tag{3.17}$$

For $n \geqslant 5$, using (1.7), we have

$$\frac{\ln\Omega_n}{H_n} - \frac{\ln\Omega_{n+1}}{H_{n+1}} > \frac{\ln\sqrt{\frac{n+\frac{3}{2}}{2\pi}}}{H_{n+1}} > 0.$$

Direct computation can yield

$$\frac{\ln\Omega_1}{H_1} < \frac{\ln\Omega_2}{H_2} < \frac{\ln\Omega_3}{H_3} > \frac{\ln\Omega_4}{H_4} > \frac{\ln\Omega_5}{H_5}.$$

Furthermore, by Stolz's theorem, we get

$$\lim_{n\to\infty}(\Omega_n)^{\frac{1}{H_n}} = \exp\{\lim_{n\to\infty}\frac{\ln\Omega_n}{H_n}\} =$$

$$\exp\{\lim_{n\to\infty}\frac{\ln\Omega_n - \ln\Omega_{n-1}}{H_n - H_{n-1}}\} = \exp\{\lim_{n\to\infty} n\ln\frac{\Omega_n}{\Omega_{n-1}}\} = 0.$$

The proof of Theorem 2.5 is complete. □

Remark 3.1. The sequence $(\Omega_n)^{1/H_n}$ can be rearranged as $\left\{[(\Omega_n)^{1/n}]^{n/H_n}\right\}$. Since $(\Omega_n)^{1/n}$ is decreasing to 0 and $\frac{n}{H_n}$ can be easily proved to be increasing to ∞, so $\lim_{n\to\infty}(\Omega_n)^{1/H_n}=0$ can be proved easily.

Remark 3.2. By the well-known software MATHEMATICA Version 7.0.0, we can show that
(1) the double inequality (3.3) is better than (1.9),
(2) the double inequality (3.13) and (1.5) are not included each other.

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