SEVERAL INEQUALITIES FOR THE VOLUME OF THE UNIT BALL IN $\mathbb{R}^n$

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Abstract. In the paper, the author establishes several new inequalities involving the volume of the unit ball in $\mathbb{R}^n$.

1. Introduction

In the recent past, inequalities about the Euler gamma function $\Gamma(x)$ have attracted the attention of many authors. In particular, several researchers established interesting properties of the volume of the unit ball in $\mathbb{R}^n$,

$$\Omega_n = \frac{\pi^{n/2}}{\Gamma(n/2 + 1)}, \quad n = 1, 2, \ldots$$  \hspace{1cm} (1.1)

In the paper [5], it was proved that the sequence $\{\Omega_n\}_{n \geq 1}$ attains its maximum at $n = 5$. In the paper [4], the sequence $\{(\Omega_n)^{1/n}\}_{n \geq 1}$ is proved to be monotonically decreased to zero. Other results have been established by Anderson and Qiu[3], and Klain and Rota[9] who proved that the sequence $\{(\Omega_n)^{1/n \ln n}\}_{n \geq 1}$ decreases to $e^{-1/2}$, and the sequence $\left\{\frac{n\Omega_n}{\Omega_{n+1}}\right\}_{n \geq 1}$ is increasing, respectively. Motivated by the following inequalities

$$(\Omega_{n+1})^{n/(n+1)} < \Omega_n, \quad n = 1, 2, \ldots$$  \hspace{1cm} (1.2)

and

$$1 < \frac{\Omega_n^2}{\Omega_{n-1}\Omega_{n+1}} < 1 + \frac{1}{n}$$  \hspace{1cm} (1.3)

stated in [3] and [9], Alzer proved in [1] that for all $n \geq 1$,

$$a(\Omega_{n+1})^{n/(n+1)} \leq \Omega_n \leq b(\Omega_{n+1})^{n/(n+1)}$$  \hspace{1cm} (1.4)

with the best possible constants $a = 2/\sqrt{\pi} = 1.1283\ldots$ and $b = \sqrt{e} = 1.6487\ldots$. An improvement of the double inequality (1.4) was given in [11]: for $n \geq 4$,

$$\frac{k}{2\sqrt{2\pi}} \leq \frac{\Omega_n}{(\Omega_{n+1})^{n/(n+1)}} \leq \frac{\sqrt{e}}{2\sqrt{2\pi}}$$  \hspace{1cm} (1.5)

where $k = \frac{94.720^{1/12}2^{1/22}}{10395^{1/11}\pi^{1/11}} = 1.5714\ldots$. Equality in the left-hand side of (1.5) occurs if and only if $n = 11$.

2000 Mathematics Subject Classification. 33B15, 41A10, 42A16.

Key words and phrases. volume of the unit $n$-dimensional ball; gamma function; monotonicity; inequalities.

The author was supported by Natural Science Foundation of Shandong Province (ZR2012AQ028).
The following class of inequalities
\[ \sqrt{\frac{n + a}{2\pi}} \leq \frac{\Omega_n - 1}{\Omega_n} \leq \sqrt{\frac{n + b}{2\pi}} \]  
(1.6)
was studied by Alzer [1] and Qiu [14] where \( a, b \) are real parameters. Later, the inequality (1.6) was recovered in [6]. Furthermore, Mortici established the following new sharp bounds
\[ \sqrt{\frac{n + \frac{1}{2}}{2\pi}} \leq \frac{\Omega_n - 1}{\Omega_n} \leq \sqrt{\frac{n + \frac{1}{2}}{2\pi}} + \frac{1}{16n\pi} \]  
(1.7)
which improves the previous results of Alzer et in [11]. Therefore, Alzer proved in [1] that for \( n \geq 1 \),
\[ \left(1 + \frac{1}{n}\right)^{\alpha} < \frac{\Omega_n^2}{\Omega_{n-1}\Omega_{n+1}} < \left(1 + \frac{1}{n}\right)^{\beta} \]  
(1.8)
in which the best possible constants \( \alpha = 2 - \log_2 \pi \) and \( \beta = \frac{1}{2} \). Later, in [11], Mortici showed that for every \( n \geq 4 \),
\[ \left(1 + \frac{1}{n}\right)^{\frac{1}{2} - \frac{1}{4n}} < \frac{\Omega_n^2}{\Omega_{n-1}\Omega_{n+1}} < \left(1 + \frac{1}{n}\right)^{\frac{1}{2}} \].  
(1.9)
Related references see [7][8][13][15].

The aim of this paper is to establish some new inequalities involving the volume of the unit ball in \( \mathbb{R}^n \).

2. Lemmas

In order to prove the main results, following lemmas are useful.

Lemma 2.1. [10] p. 390] Let \( x_i \in \mathbb{R}^+, i = 1, 2 \cdots n \) and \( \sum_{i=1}^{n} x_i = nx \), then
\[ \prod_{i=1}^{n} \Gamma(x_i) \geq (\Gamma(x))^n. \]  
(2.1)

Lemma 2.2. [2] Legendre] For every \( z \neq -1, -2 \cdots \), then
\[ 2^{2z-1}\Gamma(z)\Gamma(z + 1/2) = \pi^{1/2}\Gamma(2z). \]  
(2.2)

Lemma 2.3. [4] p. 131] For every integer \( n \geq 1 \), the sequence \( \{\Omega_n^{1/n}\}_{n \geq 1} \) is monotonically decreasing to zero.

Lemma 2.4. [12] p. 612] For every \( x \in [1, \infty) \), we have
\[ \sqrt{\pi} \left(\frac{x}{e}\right)^x \sqrt{2x + \alpha} < \Gamma(x + 1) < \sqrt{\pi} \left(\frac{x}{e}\right)^x \sqrt{2x + \beta} \]  
(2.3)
where \( \alpha = \frac{1}{3} \) and \( \beta = \sqrt{\frac{91}{30}} - 2 = 0.3533 \cdots \).
3. Main Results

In what follows, we always suppose $\beta = \sqrt{\frac{231}{29}} - 2 = 0.3533 \ldots$

**Theorem 3.1.** For all natural number $n$, we have
\[
\Omega_n \leq (\Omega_1 \Omega_2 \cdots \Omega_{n-1})^{1/(n-1)}.
\] (3.1)

If $n$ is odd integer, then
\[
(\Omega_1 \Omega_2 \cdots \Omega_n)^{1/n} \leq \Omega_{(n+1)/2}.
\] (3.2)

**Proof.** Using Lemma 2.3, we easily prove inequality (3.1). Next, we only prove inequality (3.2). By virtue of Lemma 2.1, we get
\[
(\Omega_1 \Omega_2 \cdots \Omega_n)^{1/n} = \left(\frac{\pi^{1/2}}{\Gamma(1/2 + 1)} \frac{\pi^{2/2}}{\Gamma(2/2 + 1)} \cdots \frac{\pi^{n/2}}{\Gamma(n/2 + 1)}\right)^{1/n}
\]
and
\[
= \frac{\pi^{(n+1)/4}}{(\Gamma(1/2 + 1)\Gamma(2/2 + 1) \cdots \Gamma(n/2 + 1))^{1/n}} \leq \frac{\pi^{(n+1)/4}}{\Gamma((n+1)/4 + 1)} = \Omega_{(n+1)/2}.
\]

**Theorem 3.2.** For every integer $n \geq 1$, we have
\[
\frac{(n+1)(n+\frac{1}{2})}{(n+\beta)^2} < \frac{\Omega_n^2}{\Omega_{n-1}\Omega_{n+1}} < \frac{(n+1)(n+\frac{3}{2})}{(n+\frac{1}{2})^2}.
\] (3.3)

**Proof.** Easy computation and simplification yield
\[
\frac{\Omega_n^2}{\Omega_{n-1}\Omega_{n+1}} = \frac{\Gamma((n+1)/2)\Gamma((n+3)/2)}{(\Gamma(n+2)/2)^2}.
\] (3.4)

Setting $z = \frac{n+1}{2}$ and $z = \frac{n+3}{2}$ in (2.2) of Lemma 2.2, we obtain
\[
2^n\Gamma((n+1)/2)\Gamma((n+2)/2) = \pi^{1/2}n!
\] (3.5)
and
\[
2^{n+2}\Gamma((n+3)/2)\Gamma((n+4)/2) = \pi^{1/2}\Gamma(n+3) = \pi^{1/2}(n+2)!.
\] (3.6)

Combining (3.4), (3.5) and (3.6) leads to
\[
\frac{\Omega_n^2}{\Omega_{n-1}\Omega_{n+1}} = \frac{\sqrt{(n+2)!}\sqrt{n!}}{2^n2^{n+2}\Gamma((n+4)/2)\Gamma((n+2)/2)^3} = \frac{\pi(n+1)!n!}{2^{2n+1}(\Gamma(n+2+1))^{4}}
\] (3.7)

where we apply $\Gamma((n+4)/2) = \frac{n+2}{2}\Gamma((n+2)/2)$.

Using Lemma 2.4, we have
\[
\sqrt{\pi} \left(\frac{n}{2e}\right)^{n/2} \sqrt{n + \frac{1}{3}} < \Gamma(n/2 + 1) < \sqrt{\pi} \left(\frac{n}{2e}\right)^{n/2} \sqrt{n + \beta}
\] (3.8)
and
\[
\sqrt{\pi} \left(\frac{n}{2e}\right)^n \sqrt{2n + \frac{1}{3}} < n! < \sqrt{\pi} \left(\frac{n}{e}\right)^n \sqrt{2n + \beta}.
\] (3.9)

Applying (3.8) and (3.9), we have
\[
\frac{\Omega_n^2}{\Omega_{n-1}\Omega_{n+1}} > \frac{\pi \left(\sqrt{\pi} \left(\frac{n}{2e}\right)^n \sqrt{2n + \frac{1}{3}}\right)^2 (n+1)}{2^{2n+1} \left(\sqrt{\pi} \left(\frac{n}{2e}\right)^{n/2} \sqrt{n + \beta}\right)^2} = \frac{(n+1)(n+\frac{1}{2})}{(n+\beta)^2}
\] (3.10)
and
\[
\frac{\Omega_n^2}{\Omega_{n-1}\Omega_{n+1}} < \frac{\pi (\sqrt{\pi} (\frac{n}{2})^n \sqrt{2n+\beta})^2 (n+1)}{2^{2n+1}\left(\sqrt{\pi} \left(\frac{n}{2e}\right)^{n/2} \sqrt{n+\frac{1}{4}}\right)^4} = \frac{(n+1) \left(n+\frac{\beta}{2}\right)}{(n+\frac{1}{4})^2}.
\] (3.11)

The proof of Theorem 3.2 is complete. \hfill \Box

Noting simple inequalities
\[
\frac{(n+1) \left(n+\frac{1}{6}\right)}{(n+\beta)} > \frac{n+\frac{1}{6}}{n+\beta}
\]
and
\[
\frac{(n+1) \left(n+\frac{3}{2}\right)}{(n+\frac{1}{4})^2} < \frac{n+1}{n+\frac{1}{4}},
\]
we get the Corollary 3.1.

**Corollary 3.1.** For every integer \( n \geq 1 \), it holds
\[
\frac{n+\frac{1}{6}}{n+\beta} < \frac{\Omega_n^2}{\Omega_{n-1}\Omega_{n+1}} < \frac{n+\frac{1}{6}}{n+\beta}.
\] (3.12)

**Theorem 3.3.** For every integer \( n \geq 1 \), it holds
\[
\frac{\sqrt{\pi}}{2^{n+1/2}2\pi} \left(\frac{\sqrt{n+\frac{1}{4}}}{n+1}\right)^{(2n+1)/(n+1)} < \frac{\Omega_n}{\Omega_{n-1}\Omega_{n+1}} < \frac{\sqrt{\pi}}{2^{n+1/2}2\pi} \left(\frac{\sqrt{n+1+\beta}}{n+1}\right)^{(2n+1)/(n+1)}.
\] (3.13)

**Proof.** Setting \( z = \frac{n+\beta}{2} \) in (2.2) of Lemma 2.2, we get
\[
2^{n+1}\Gamma((n+2)/2)\Gamma((n+3)/2) = \pi^{1/2}\Gamma(n+2) = \pi^{1/2}(n+1)!. \] (3.14)

Easy computation and simplification yield
\[
\frac{\Omega_n}{(\Omega_n)^{(n+1)/2}} = \frac{\pi^{n/2} (\Gamma((n+1)/2+1))^{n/(n+1)}}{(\pi(n+1/2))^{n/(n+1)}}
\]
\[
\frac{2^{n+1} \Gamma((n+1)/2+1))^{n/(n+1)}}{\sqrt{\pi} (n+1)!}.
\] (3.15)

Similarly to proof of Theorem 3.2, we have
\[
\frac{\Omega_n}{(\Omega_{n+1})^{(n+1)/2}} > \frac{2^{n+1} \left(\sqrt{\pi} \left(\frac{n+1}{2e}\right)^{(n+1)/2} \sqrt{n+\frac{1}{4}}\right)^{(2n+1)/(n+1)}}{\sqrt{\pi} \left(\frac{n+1}{2e}\right)^{(n+1)/2} \sqrt{2n+2+\beta}}
\]
\[
= \frac{\sqrt{\pi}}{2^{n+1/2}2\pi} \left(\frac{\sqrt{n+\frac{1}{4}}}{n+1}\right)^{(2n+1)/(n+1)}.
\]
and

\[
\frac{\Omega_n}{(\Omega_{n+1})^{n/(n+1)}} < \frac{2^{n+1} \left( \sqrt{\pi} \left( \frac{n+1}{e} \right)^{(n+1)/2} \sqrt{n+1+\beta} \right)^{(2n+1)/(n+1)}}{\sqrt{\pi} \left( \frac{n+1}{e} \right)^{n+1} \sqrt{2n+2+\frac{\beta}{2}}}
\]

\[
= \frac{\sqrt{\pi}}{2^{n+1} \sqrt{2\pi}} \frac{(\sqrt{n+1+\beta})^{(2n+1)/(n+1)}}{\sqrt{n+1} \left( n + \frac{\beta}{2} \right)}.
\]

The proof of Theorem 3.3 is complete.

Noting simple inequalities

\[
\frac{(\sqrt{n+1+\beta})^{(2n+1)/(n+1)}}{\sqrt{n+1} \left( n + \frac{\beta}{2} \right)} \leq \frac{(n+1+\beta)^{(2n+2)/(2n+2)}}{(n+1)(n+1)} = \frac{n+1+\beta}{n+1}
\]

and

\[
\frac{(\sqrt{n+1+\beta})^{(2n+1)/(n+1)}}{\sqrt{n+1} \left( n + \frac{\beta}{2} \right)} \geq \frac{(\sqrt{n+1+\frac{\beta}{2}})^{(2n+1)/(n+1)}}{n+1+\frac{\beta}{2}} > \frac{1}{2^{n+1} \sqrt{n+1+\frac{\beta}{2}}},
\]

we easily get the Corollary 3.2.

**Corollary 3.2.** For every integer \( n \geq 1 \), we have

\[
\frac{\Omega_n}{(\Omega_{n+1})^{n/(n+1)}} < \frac{\sqrt{\pi}}{2^{n+1} \sqrt{2\pi}} \frac{(\sqrt{n+1+\beta})^{(2n+1)/(n+1)}}{\sqrt{n+1} \left( n + \frac{\beta}{2} \right)} < \frac{n+1+\beta}{n+1}.
\] (3.16)

Finally, we give a monotone result related to the volume of the unit ball in \( \mathbb{R}^n \).

**Theorem 3.4.** For every integer \( n \geq 3 \), the sequence \( \{(\Omega_n)^{1/H_n}\}_{n \geq 3} \) is monotonically decreasing to zero, where \( H_n \) denotes the \( n \)-th harmonic number. Further, the sequence \( \{(\Omega_n)^{1/H_n}\}_{n \geq 1} \) attains its maximum at \( n = 3 \).

**Proof.** By taking the logarithm, we only prove that

\[
\frac{\ln \Omega_n}{H_n} \geq \frac{\ln \Omega_{n+1}}{H_{n+1}}.
\] (3.17)

For \( n \geq 5 \), using (1.7), we have

\[
\frac{\ln \Omega_n}{H_n} < \frac{\ln \Omega_{n+1}}{H_{n+1}} < \frac{\ln \sqrt{n+2}}{2\pi} < 0.
\]

Direct computation can yield

\[
\frac{\ln \Omega_1}{H_1} < \frac{\ln \Omega_2}{H_2} < \frac{\ln \Omega_3}{H_3} > \frac{\ln \Omega_4}{H_4} > \frac{\ln \Omega_5}{H_5}.
\]

Furthermore, by Stolz’s theorem, we get

\[
\lim_{n \to \infty} (\Omega_n)^{1/H_n} = \exp \left\{ \lim_{n \to \infty} \ln \frac{\Omega_n}{H_n} \right\} = \exp \left\{ \lim_{n \to \infty} \frac{\ln \Omega_n - \ln \Omega_{n-1}}{H_n - H_{n-1}} \right\} = \exp \left\{ \lim_{n \to \infty} n \ln \frac{\Omega_n}{\Omega_{n-1}} \right\} = 0.
\]

The proof of Theorem 2.5 is complete.
Remark 3.1. The sequence \((\Omega_n)^{1/H_n}\) can be rearranged as \(\{(\Omega_n)^{1/n})^{n/H_n}\}\). Since \((\Omega_n)^{1/n}\) is decreasing to 0 and \(n^{n/H_n}\) can be easily proved to be increasing to \(\infty\), so \(\lim_{n \to \infty} (\Omega_n)^{1/H_n} = 0\) can be proved easily.

Remark 3.2. By the well-known software MATHEMATICA Version 7.0.0, we can show that

1. the double inequality (3.3) is better than (1.9),
2. the double inequality (3.13) and (1.5) are not included each other.

Acknowledgments. The author appreciate the referee for his helpful and valuable comments on this manuscript.

References


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