A new operational matrix of fractional integration for shifted Jacobi polynomials

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Abstract. A new shifted Jacobi operational matrix (SJOM) of fractional integration of arbitrary order is introduced and applied together with spectral tau method for solving linear fractional differential equations (FDEs). The fractional integration is described in the Riemann-Liouville sense. The numerical approach is based on the shifted Jacobi tau method. The main characteristic behind the approach using this technique is that only a small number of shifted Jacobi polynomials is needed to obtain a satisfactory result. Illustrative examples reveal that the present method is very effective and convenient for linear muti-term FDEs.

Keywords: Operational matrix; Shifted Jacobi polynomials; Tau method; Multi-term FDEs; Riemann-Liouville derivative

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1 Introduction

In recent years, the study of fractional ODEs and PDEs has attracted much attention due to an exact description of nonlinear phenomena in fluid mechanics, viscoelasticity, biology, physics, engineering and other areas of science [2, 20, 36, 37]. On this kind of equations the derivatives of fractional order are involved. The interest of the study of fractional-order differential equations lies in the fact that fractional-order models are more accurate than integer-order models, that is, there are more degrees of freedom in the fractional-order models. Furthermore, fractional derivatives provide an excellent instrument for the description of memory and hereditary properties of various materials and processes due to the existence of a memory term in a model. This memory term insures the history and its impact to the present and future, see [23]. In consequence, the subject of fractional differential equations is gaining much importance and attention. For details, see [2, 21, 40] and the references therein. Recent results on fractional differential equations can be seen in [17, 18, 25, 30, 28].

It is well known that the spectral methods have gained increasing popularity for several decades, especially in solving differential equations and in the field of computational fluid dynamics (see, e.g., [7, 34, 39, 9, 13] and the references therein). The main advantage of these methods lies in their accuracy for a given number of unknowns. For smooth problems in simple geometries, they offer exponential rates of convergence/spectral accuracy. In contrast, finite difference and finite-element methods yield only algebraic convergence rates. The three most widely used spectral versions are the Galerkin, collocation, and tau methods. In the Lanczos tau-method [22], the auxiliary conditions imposed on the problem, such as initial, boundary or more general conditions may be imposed as constraints on the expansions coefficients.

A number of algorithms have been proposed to solve the multi-term fractional differential equations. Some recent techniques are spectral methods [11, 14, 16], Haar wavelet [24, 6], Legendre wavelet method [29, 19] and Piecewise polynomial collocation [33]. Moreover, the authors in [11, 12, 15] constructed an efficient spectral methods for the numerical approximation of the FDEs and fractional integro-differential equations based on tau and pseudo-spectral methods. Bhrawy et al. [1] introduced a quadrature shifted Legendre tau method based on Gauss-Lobatto interpolation for solving the multi-order FDEs with variable coefficients and in [4], the shifted Legendre spectral methods have been developed for solving the fractional-order multi-point boundary value problems.

For spectral and pseudospectral methods; explicit formulae for operational matrices of fractional derivatives for classical orthogonal polynomials are needed. The operational matrices of fractional derivatives have been determined for Chebyshev polynomials [12] and Legendre polynomials [1], and are applied together with tau and pseudospectral methods to solve some types of FDEs.

The operational matrix of integration has been determined for several types of orthogonal polynomials, such as Chebyshev polynomials of the first kind [31], Chebyshev polynomials of third and fourth kinds [10] and Legendre polynomials [32]. Recently, Singh et al. [38] derived the Bernstein operational matrix of integration. The Bernstein operational matrix approach is developed for solving a system of high order linear Volterra-Fredholm integro-differential equations in [26]. The Haar wavelet operational matrix of fractional order integration has been developed for solving FDEs [24]. In [5], the authors derived a new explicit formula for the integrals of shifted Chebyshev polynomials of any degree for any fractional-order in terms of shifted Chebyshev polynomials themselves. In their article, and as an important application, they described how to use these formulae to solve multi-term FDEs. However in [3], the authors introduced a shifted Chebyshev operational matrix of fractional integration and applied it together with spectral tau method for the same FDEs.

The Jacobi polynomials have become increasingly important in numerical analysis, from both theoretical and practical points of view. Recently, Doha et al. [14] derived the shifted Jacobi operational matrix of fractional derivatives which is applied together with spectral tau method for numerical solution of general linear multi-term fractional differential equations. In this paper, we derive an operational matrix of fractional integration of the shifted Jacobi polynomials; in the Riemann-Liouville sense. Subsequently, we use this operational matrix for Jacobi polynomials to introduce a direct solution technique for solving the FDEs. We note that the shifted Chebyshev operational matrix of fractional integration has been introduced by Bhrawy and Alofi [3], and some other very interesting cases, can be obtained directly as special cases from the shifted Jacobi operational matrix of fractional integration. Finally, the accuracy of the proposed algorithm is demonstrated by test problems.

The paper is organized as follows. In Section 2 we introduce some necessary definitions and give some relevant properties of Jacobi polynomials. In Section 3 the SJOM of fractional integration is introduced. In Section 4 we apply SJOM of fractional integration for solving linear multi-order FDEs. In Section 5 the proposed method is applied to several examples. Also a conclusion is given in Section 6.

2 Preliminaries and notation

2.1 The fractional integration in the Riemann-Liouville sense

There are several definitions of a fractional integration of order $\nu > 0$, and not necessarily equivalent to each other, see [27]. The most used definition is due to Riemann-Liouville, which is defined as

$$\begin{aligned} I^{\nu} f(x) &= \frac{1}{\Gamma(\nu)} \int_0^x (x-t)^{\nu-1} f(t) dt, \quad \nu > 0, \quad x > 0, \\ I^0 f(x) &= f(x). \end{aligned} \tag{2.1}$$

One of the basic property of the operator I^{ν} is

$$I^{\nu} x^{\beta} = \frac{\Gamma(\beta+1)}{\Gamma(\beta+1+\nu)} \, x^{\beta+\nu}. \tag{2.2}$$

The Riemann-Liouville fractional derivative of order ν will be denoted by D^{ν}. The next equation define Riemann-Liouville fractional derivative of order ν

$$D^{\nu} f(x) = \frac{d^m}{dx^m} (I^{m-\nu} f(x)), \tag{2.3}$$

where $m-1 < \nu \leq m, \; m \in N$ and m is the smallest integer greater than ν.

Lemma 2.1. *If $m-1 < \nu \leq m, \; m \in N$, then*

$$D^{\nu} I^{\nu} f(x) = f(x), \qquad I^{\nu} D^{\nu} f(x) = f(x) - \sum_{i=0}^{m-1} f^{(i)}(0^+) \frac{x^i}{i!}, \quad x > 0. \tag{2.4}$$

2.2 Properties of shifted Jacobi polynomials

The well-known Jacobi polynomials $P_i^{(\alpha,\beta)}(x)$ are defined on the interval $(-1,1)$. In order to use these polynomials on the interval $x \in (0,t)$ we defined the so-called shifted Jacobi polynomials by introducing the change of variable $x = \frac{2x}{t} - 1$. Let the shifted Jacobi polynomials $P_i^{(\alpha,\beta)}(\frac{2x}{t} - 1)$ be denoted by $P_{t,i}^{(\alpha,\beta)}(x)$. The analytic form of the shifted Jacobi polynomials $P_{t,i}^{(\alpha,\beta)}(x)$ of degree i is given by

$$P_{t,i}^{(\alpha,\beta)}(x) = \sum_{k=0}^{i} \frac{(-1)^{i-k} \Gamma(i+\beta+1) \Gamma(i+k+\alpha+\beta+1)}{\Gamma(k+\beta+1) \Gamma(i+\alpha+\beta+1) \; (i-k)! \; k! \; t^k} \, x^k, \tag{2.5}$$

where $P_{t,i}^{(\alpha,\beta)}(0) = (-1)^i \frac{\Gamma(i+\beta+1)}{\Gamma(\beta+1) \; i!}$. The orthogonality condition is

$$\int_0^t P_{t,j}^{(\alpha,\beta)}(x) P_{t,k}^{(\alpha,\beta)}(x) w_t^{(\alpha,\beta)}(x) dx = h_{t,k}^{(\alpha,\beta)} \, \delta_{jk}, \tag{2.6}$$

where $w_t^{(\alpha,\beta)}(x) = (t-x)^{\alpha} x^{\beta}$ and $h_{t,k}^{(\alpha,\beta)} = \frac{t^{\alpha+\beta+1} \Gamma(k+\alpha+1) \Gamma(k+\beta+1)}{(2k+\alpha+\beta+1) \Gamma(k+1) \Gamma(k+\alpha+\beta+1)}$. The special values

$$D^q P_{t,i}^{(\alpha,\beta)}(0) = \frac{(-1)^{i-q} \Gamma(i+\beta+1) (i+\alpha+\beta+1)_q}{t^q \Gamma(i-q+1) \Gamma(q+\beta+1)}, \tag{2.7}$$

will be of important use later. A function $u(x)$, square integrable in $(0,t)$, may be expressed in terms of shifted Jacobi polynomials as

$$u(x) = \sum_{j=0}^{\infty} c_j P_{t,j}^{(\alpha,\beta)}(x),$$

where the coefficients c_j are given by

$$c_j = \frac{1}{h_{t,j}^{(\alpha,\beta)}} \int_0^t u(x) P_{t,j}^{(\alpha,\beta)}(x) w_t^{(\alpha,\beta)}(x) dx, \quad j = 0, 1, 2, \cdots . \tag{2.8}$$

In practice, only the first $(N+1)$-terms shifted Jacobi polynomials are considered. Hence $u(x)$ can be expanded in the form

$$u_N(x) \simeq \sum_{j=0}^{N} c_j P_{t,j}^{(\alpha,\beta)}(x) = C^T \phi(x), \tag{2.9}$$

where the shifted Jacobi coefficient vector C and the shifted Jacobi vector $\phi(x)$ are given by

$$\begin{aligned} C^T &= [c_0, c_1, \cdots, c_N], \\ \phi(x) &= [P_{t,0}^{(\alpha,\beta)}(x), P_{t,1}^{(\alpha,\beta)}(x), \cdots, P_{t,N}^{(\alpha,\beta)}(x)]^T. \end{aligned} \tag{2.10}$$

If we define the q times repeated integration of Jacobi vector $\phi(x)$ by $J^q\phi(x)$.

$$J^q \phi(x) \simeq \mathbf{P}^{(q)} \phi(x), \tag{2.11}$$

where q is an integer value and $\mathbf{P}^{(q)}$ is the operational matrix of integration of $\phi(x)$.

3 Operational matrix of fractional integration

The main objective of this section is to generalize the SJOM of integration (2.11) for fractional calculus.

Theorem 3.1. *Let $\phi(x)$ be the shifted Jacobi vector and $\nu > 0$ then*

$$J^\nu \phi(x) \simeq \boldsymbol{P}^{(\nu)} \phi(x), \tag{3.1}$$

where $\boldsymbol{P}^{(\nu)}$ is the $(N+1) \times (N+1)$ operational matrix of fractional integration of order ν in the Riemann-Liouville sense and is defined as follows:

$$\boldsymbol{P}^{(\nu)} = \begin{pmatrix} \Omega_\nu(0,0,\alpha,\beta) & \Omega_\nu(0,1,\alpha,\beta) & \cdots & \Omega_\nu(0,N,\alpha,\beta) \\ \Omega_\nu(1,0,\alpha,\beta) & \Omega_\nu(1,1,\alpha,\beta) & \cdots & \Omega_\nu(1,N,\alpha,\beta) \\ \vdots & \vdots & \cdots & \vdots \\ \Omega_\nu(i,0,\alpha,\beta) & \Omega_\nu(i,1,\alpha,\beta) & \cdots & \Omega_\nu(i,N,\alpha,\beta) \\ \vdots & \vdots & \cdots & \vdots \\ \Omega_\nu(N,0,\alpha,\beta) & \Omega_\nu(N,1,\alpha,\beta) & \cdots & \Omega_\nu(N,N,\alpha,\beta) \end{pmatrix} \tag{3.2}$$

where

$$\begin{aligned} \Omega_\nu(i,j,\alpha,\beta) = &\sum_{k=0}^{i} \frac{(-1)^{i-k}\ \Gamma(i+\beta+1)\ \Gamma(i+k+\alpha+\beta+1)}{\Gamma(k+\beta+1)\ \Gamma(i+\alpha+\beta+1)(i-k)!\ \Gamma(k+\nu+1)} \\ &\times \sum_{f=0}^{j} \frac{(-1)^{j-f}\ \Gamma(j+f+\alpha+\beta+1)\ \Gamma(\alpha+1)\Gamma(f+k+\nu+\beta+1)\ (2j+\alpha+\beta+1)\ j!\ t^\nu}{\Gamma(j+\alpha+1)\ \Gamma(f+\beta+1)(j-f)!\ f!\Gamma(f+k+\alpha+\beta+\nu+2)}. \end{aligned} \tag{3.3}$$

Proof. The analytic form of the shifted Jacobi polynomials $P_{t,i}^{(\alpha,\beta)}(x)$ of degree i is given by (2.5). Using (2.1) and (2.2), and since the Riemann-Liouville's fractional integration is a linear operation, we get

$$\begin{aligned} J^{\nu}P_{t,i}^{(\alpha,\beta)}(x) &= \sum_{k=0}^{i} \frac{(-1)^{i-k}\Gamma(i+\beta+1)\Gamma(i+k+\alpha+\beta+1)}{\Gamma(k+\beta+1)\Gamma(i+\alpha+\beta+1)\ (i-k)!\ k!\ t^{k}}\ J^{\nu}x^{k} \\ &= \sum_{k=0}^{i} \frac{(-1)^{i-k}\Gamma(i+\beta+1)\Gamma(i+k+\alpha+\beta+1)}{\Gamma(k+\beta+1)\Gamma(i+\alpha+\beta+1)\ (i-k)!\Gamma(k+\nu+1)\ t^{k}}\ x^{k+\nu}, \quad i=0,1,\cdots,N. \end{aligned} \tag{3.4}$$

Now, approximate $x^{k+\nu}$ by $N+1$ terms of shifted Jacobi series, yields

$$x^{k+\nu} = \sum_{j=0}^{N} c_{kj} P_{t,j}^{(\alpha,\beta)}(x), \tag{3.5}$$

where c_{kj} is given from (2.8) with $u(x) = x^{k+\nu}$, that is

$$\begin{aligned} c_{kj} =& \frac{(2j+\alpha+\beta+1)\Gamma(j+1)}{\Gamma(j+\alpha+1)} t^{k+\nu} \\ &\times \sum_{f=0}^{j} \frac{(-1)^{j-f}\ \Gamma(j+f+\alpha+\beta+1)\ \Gamma(\alpha+1)\ \Gamma(f+k+\beta+\nu+1)}{\Gamma(f+\beta+1)\ (j-f)!\ f!\ \Gamma(f+k+\alpha+\beta+\nu+2)}, \quad j=1,2,\cdots,N. \end{aligned} \tag{3.6}$$

In virtue of (3.4) and (3.5), we get

$$J^{\nu}P_{t,i}^{(\alpha,\beta)}(x) = \sum_{j=0}^{N} \Omega_{\nu}(i,j) P_{t,j}^{(\alpha,\beta)}(x), \quad i=0,1,\cdots,N, \tag{3.7}$$

where $\Omega_{\nu}(i,j) = \sum_{k=0}^{i} \zeta_{ijk}$, and

$$\begin{aligned} \zeta_{ijk} =& \frac{(-1)^{i-k}\ \Gamma(i+\beta+1)\ \Gamma(i+k+\alpha+\beta+1)}{\Gamma(k+\beta+1)\ \Gamma(i+\alpha+\beta+1)(i-k)!\ \Gamma(k+\nu+1)} \\ &\times \sum_{f=0}^{j} \frac{(-1)^{j-f}\ \Gamma(j+f+\alpha+\beta+1)\ \Gamma(\alpha+1)\Gamma(f+k+\nu+\beta+1)\ (2j+\alpha+\beta+1)\ j!\ t^{\nu}}{\Gamma(j+\alpha+1)\ \Gamma(f+\beta+1)(j-f)!\ f!\Gamma(f+k+\alpha+\beta+\nu+2)}, \\ & j=1,2,\cdots N. \end{aligned}$$

Accordingly, Eq. (3.7) can be written in a vector form as follows:

$$J^{\nu}P_{t,i}^{(\alpha,\beta)}(x) \simeq \Big[\Omega_{\nu}(i,0,\alpha,\beta), \Omega_{\nu}(i,1,\alpha,\beta), \Omega_{\nu}(i,2,\alpha,\beta), \cdots, \Omega_{\nu}(i,N,\alpha,\beta)\Big]\phi(x), \qquad i=0,1,\cdots,N. \tag{3.8}$$

Eq. (3.8) leads to the desired result.

4 Fractional SJOM for solving Linear multi-order FDEs

In this section, the proposed multi-order FDE is integrated ν times, in the Riemann-Liouville sense, where ν is the highest fractional-order and making use of the formula relating the expansion coefficients

of fractional integration appearing in this integrated form of the proposed multi-order FDE to shifted Jacobi polynomials themselves, and then we apply tau approximations based on operational matrix.
In order to show the fundamental importance of SJOM of fractional integration, we apply it to solve the following multi-order FDE:

$$D^{\nu}u(x) = \sum_{i=1}^{k} \gamma_j D^{\beta_i} u(x) + \gamma_{k+1} u(x) + f(x), \quad \text{in } I = (0,t), \tag{4.1}$$

with initial conditions

$$u^{(i)}(0) = d_i, \qquad i = 0, \cdots, m-1, \tag{4.2}$$

where γ_i $(i = 1, 2, \cdots, k+1)$ are real constant coefficients and also $m-1 < \nu \leq m,\ 0 < \beta_1 < \beta_2 < \cdots < \beta_k < \nu$. Moreover $D^{\nu}u(x) \equiv u^{(\nu)}(x)$ denotes the Riemann-Liouville fractional derivative of order ν for $u(x)$ and the values of d_i $(i = 0, \cdots, m-1)$ describe the initial state of $u(x)$ and $g(x)$ is a given source function. For the existence and uniqueness and continuous dependence of the solution to the problem, see [8].

If we apply the Riemann-Liouville integral of order ν on (4.1) and after making use of (2.4), we get the integrated form of (4.1), namely

$$\begin{aligned} &u(x) - \sum_{j=0}^{m-1} u^{(j)}(0^+)\frac{x^j}{j!} = \sum_{i=1}^{k} \gamma_i I^{\nu-\beta_i}\Big[u(x) - \sum_{j=0}^{m_i-1} u^{(j)}(0^+)\frac{x^j}{j!}\Big] + \gamma_{k+1} I^{\nu} u(x) + I^{\nu} f(x), \\ &u^{(i)}(0) = d_i, \qquad i = 0, \cdots, m-1, \end{aligned} \tag{4.3}$$

where $m_i - 1 < \beta_i \leq m_i,\ m_i \in N$, this implies that

$$\begin{aligned} &u(x) = \sum_{i=1}^{k} \gamma_i I^{\nu-\beta_i} u(x) + \gamma_{k+1} I^{\nu} u(x) + g(x), \\ &u^{(i)}(0) = d_i, \qquad i = 0, \cdots, m-1, \end{aligned} \tag{4.4}$$

where

$$g(x) = I^{\nu} f(x) + \sum_{j=0}^{m-1} d_j \frac{x^j}{j!} + \sum_{i=1}^{k} \gamma_i I^{\nu-\beta_i}\Big(\sum_{j=0}^{m_i-1} d_j \frac{x^j}{j!}\Big).$$

In order to use the tau method with SJOM for solving the fully integrated problem (4.4) with initial conditions (4.2). We approximate $u(x)$ and $g(x)$ by the shifted Jacobi polynomials as

$$u_N(x) \simeq \sum_{i=0}^{N} c_i P_{t,i}^{(\alpha,\beta)}(x) = C^T \phi(x), \tag{4.5}$$

$$g(x) \simeq \sum_{i=0}^{N} g_i P_{t,i}^{(\alpha,\beta)}(x) = G^T \phi(x), \tag{4.6}$$

where the vector $G = [g_0, g_1, \cdots, g_N]^T$ is given but $C = [c_0, c_1, \cdots, c_N]^T$ is an unknown vector.
Now, the Riemann-Liouville integral of orders ν- and $(\nu - \beta_j)$ of the approximate solution (4.5), after making use of Theorem 3.1 (relation (3.1)), can be written as

$$I^{\nu} u_N(x) \simeq C^T I^{\nu} \phi(x) \simeq C^T \mathbf{P}^{(\nu)} \phi(x), \tag{4.7}$$

and

$$I^{\nu-\beta_j}u_N(x) \simeq C^T I^{\nu-\beta_j}\phi(x) \simeq C^T \mathbf{P}^{(\nu-\beta_j)}\phi(x), \quad j=1,\cdots,k, \tag{4.8}$$

respectively, where $\mathbf{P}^{(\nu)}$ is the $(N+1)\times(N+1)$ operational matrix of fractional integration of order ν.

Employing Eqs. (4.5)-(4.8) the residual $R_N(x)$ for Eq. (4.4) can be written as

$$R_N(x) = (C^T - C^T\sum_{j=1}^{k}\gamma_j \mathbf{P}^{(\nu-\beta_j)} - \gamma_{k+1}C^T\mathbf{P}^{(\nu)} - G^T)\phi(x). \tag{4.9}$$

As in a typical tau method, see [7, 12], we generate $N-m+1$ linear algebraic equations by applying

$$\langle R_N(x), P_{t,j}^{(\alpha,\beta)}(x)\rangle = \int_0^t R_N(x)P_{t,j}^{(\alpha,\beta)}(x)dx = 0, \qquad j=0,1,\cdots,N-m. \tag{4.10}$$

Also by substituting Eqs. (2.7) and (4.5) in Eq (4.2), we get

$$u^{(i)}(0) = \sum_{i=0}^{N} c_i D^{(i)}P_{t,i}^{(\alpha,\beta)}(0) = d_i, \qquad i=0,1,\cdots,m-1. \tag{4.11}$$

Eqs. (4.10) and (4.11) generate $N-m+1$ and m set of linear equations, respectively. These linear equations can be solved for unknown coefficients of the vector C. Consequently, $u_N(x)$ given in Eq. (4.5) can be calculated, which give a solution of Eq. (4.1) with the initial conditions (4.2).

5 Illustrative examples

To illustrate the effectiveness of the proposed method in the present paper, some test examples are carried out in this section. The results obtained by the present methods reveal that the present method is very effective and convenient for linear FDEs.

Example 1. *As the first example, we consider the following initial value problem,*

$$D^{\frac{3}{2}}u(x) + 3u(x) = 3x^3 + \frac{8}{\Gamma(0.5)}x^{1.5}, \quad u(0)=0,\ u'(0)=0, \quad x\in[0,t], \tag{5.1}$$

whose exact solution is given by $u(x) = x^3$.

By applying the technique described in Section 4 with $N=3$, we may write the approximate solution and the right hand side in the forms

$$u(x) = \sum_{i=0}^{3} c_i P_{t,i}^{(\alpha,\beta)}(x) = C^T\phi(x), \quad \text{and} \quad g(x) \simeq \sum_{i=0}^{3} g_i P_{t,i}^{(\alpha,\beta)}(x) = G^T\phi(x).$$

From Eq. (3.2) one can write

$$\mathbf{P}^{(\frac{3}{2})} = \begin{pmatrix} \Omega_{\frac{3}{2}}(0,0,\alpha,\beta) & \Omega_{\frac{3}{2}}(0,1,\alpha,\beta) & \Omega_{\frac{3}{2}}(0,2,\alpha,\beta) & \Omega_{\frac{3}{2}}(0,3,\alpha,\beta) \\ \Omega_{\frac{3}{2}}(1,0,\alpha,\beta) & \Omega_{\frac{3}{2}}(1,1,\alpha,\beta) & \Omega_{\frac{3}{2}}(1,2,\alpha,\beta) & \Omega_{\frac{3}{2}}(1,3,\alpha,\beta) \\ \Omega_{\frac{3}{2}}(2,0,\alpha,\beta) & \Omega_{\frac{3}{2}}(2,1,\alpha,\beta) & \Omega_{\frac{3}{2}}(2,2,\alpha,\beta) & \Omega_{\frac{3}{2}}(2,3,\alpha,\beta) \\ \Omega_{\frac{3}{2}}(3,0,\alpha,\beta) & \Omega_{\frac{3}{2}}(3,1,\alpha,\beta) & \Omega_{\frac{3}{2}}(3,2,\alpha,\beta) & \Omega_{\frac{3}{2}}(3,3,\alpha,\beta) \end{pmatrix}, \quad G = \begin{pmatrix} g_0 \\ g_1 \\ g_2 \\ g_3 \end{pmatrix},$$

where $\Omega_{\frac{3}{2}}(i,j,\alpha,\beta)$ is given in Eq. (3.3) and

$$g_j = \frac{(2j+\alpha+\beta+1)j!}{t^{\alpha+\beta+1}\Gamma(j+\alpha+1)}\sum_{f=0}^{j}\frac{(-1)^{j-f}\Gamma(f+j+\alpha+\beta+1)}{t^f f!(j-f)!\Gamma(f+\beta+1)}$$
$$\times\int_0^t\left(\frac{64x^{9/2}}{105\sqrt{\pi}}+x^3\right)x^{\beta+f}(t-x)^{\alpha}dx.$$

Making use of (4.8) and (4.10) yields

$$3\Omega_{\frac{3}{2}}(0,2,\alpha,\beta)c_0+3\Omega_{\frac{3}{2}}(1,2,\alpha,\beta)c_1+3\Omega_{\frac{3}{2}}(2,2,\alpha,\beta)c_2+3\Omega_{\frac{3}{2}}(3,2,\alpha,\beta)c_3+c_2-g_2=0, \tag{5.2}$$

$$3\Omega_{\frac{3}{2}}(0,3,\alpha,\beta)c_0+3\Omega_{\frac{3}{2}}(1,3,\alpha,\beta)c_1+3\Omega_{\frac{3}{2}}(2,3,\alpha,\beta)c_2+3\Omega_{\frac{3}{2}}(3,3,\alpha,\beta)c_3+c_3-g_3=0. \tag{5.3}$$

Applying Eq. (4.11) for the initial conditions gives

$$\begin{aligned}C^T\phi(0)&=c_0-(\beta+1)c_1+\frac{(\beta+1)(\beta+2)}{2}c_2-\frac{(\beta+1)(\beta+2)(\beta+3)}{6}c_3=0,\\ C^TD^{(1)}\phi(0)&=\frac{(\alpha+\beta+2)}{t}c_1-\frac{(\beta+2)(\alpha+\beta+3)}{t}c_2+\frac{(\beta+2)(\beta+3)(\alpha+\beta+4)}{2t}c_3=0.\end{aligned} \tag{5.4}$$

Finally by solving Eqs. ((5.2)-(5.6)) we get the approximate solution.
In particular, the special cases for ultraspherical basis ($\alpha=\beta$ and each is replaced by $(\alpha-\frac{1}{2})$) and for Chebyshev basis of the first, second, third and fourth kinds may be obtained directly by taking $\alpha=\beta=\mp\frac{1}{2}$, $\alpha=-\beta=\pm\frac{1}{2}$, respectively, and for the Legendre basis by taking $\alpha=\beta=0$.

If $\alpha=\beta=0$, then

$$c_0=\frac{t^3}{4},\qquad c_1=\frac{9t^3}{20},\qquad c_2=\frac{t^3}{4},\qquad c_3=\frac{t^3}{20}.$$

and the approximate solution is given by

$$u_N(x)=\sum_{i=0}^{3}c_iP_{t,i}^{(0,0)}(x)=x^3,$$

which is the exact solution.

Also if we choose $\alpha=-\frac{1}{2},\beta=\frac{1}{2}$, then

$$c_0=\frac{35t^3}{64},\qquad c_1=\frac{21t^3}{32},\qquad c_2=\frac{7t^3}{24},\qquad c_3=\frac{t^3}{20},$$

and

$$u_N(x)=\sum_{i=0}^{3}c_iP_{t,i}^{(-\frac{1}{2},\frac{1}{2})}(x)=x^3,$$

which is the exact solution.

In the case of $\alpha=\frac{1}{2},\beta=-\frac{1}{2}$, we have

$$c_0=\frac{5t^3}{64},\qquad c_1=\frac{9t^3}{32},\qquad c_2=\frac{5t^3}{24},\qquad c_3=\frac{t^3}{20},$$

and

$$u_N(x)=\sum_{i=0}^{3}c_iP_{t,i}^{(\frac{1}{2},-\frac{1}{2})}(x)=x^3,$$

which is the exact solution.

Example 2. *Consider the equation*

$$D^2u(x)-2Du(x)+D^{\frac{1}{2}}u(x)+u(x)=x^3-6x^2+6x+\frac{16}{5\sqrt{\pi}}x^{2.5},\quad u(0)=0,\ u'(0)=0,\quad x\in[0,t],\ (5.5)$$

whose exact solution is given by $u(x)=x^3$.

Now, we can apply the technique described in Example 1 with $N=3$
The approximate solution obtained by using the proposed method for some special cases of α *and* β are listed in the following cases

Case 1. If $\alpha=\beta=0$, then

$$c_0=\frac{t^3}{4},\qquad c_1=\frac{9t^3}{20},\qquad c_2=\frac{t^3}{4},\qquad c_3=\frac{t^3}{20},$$

and

$$u_N(x)=\sum_{i=0}^{3}c_iP_{t,i}^{(0,0)}(x)=x^3,$$

which is the exact solution.

Case 2. If $\alpha=-\frac{1}{2},\beta=\frac{1}{2}$, then

$$c_0=\frac{35t^3}{64},\qquad c_1=\frac{21t^3}{32},\qquad c_2=\frac{7t^3}{24},\qquad c_3=\frac{t^3}{20},$$

and

$$u_N(x)=\sum_{i=0}^{3}c_iP_{t,i}^{(-\frac{1}{2},\frac{1}{2})}(x)=x^3,$$

which is the exact solution.

Case 3. If $\alpha=\frac{1}{2},\beta=-\frac{1}{2}$, then

$$c_0=\frac{5t^3}{64},\qquad c_1=\frac{9t^3}{32},\qquad c_2=\frac{5t^3}{24},\qquad c_3=\frac{t^3}{20},$$

and

$$u_N(x)=\sum_{i=0}^{3}c_iP_{t,i}^{(\frac{1}{2},-\frac{1}{2})}(x)=x^3,$$

which is the exact solution.

Case 4. If $\alpha=\beta=-\frac{1}{2}$, then

$$c_0=\frac{5t^3}{16},\qquad c_1=\frac{15t^3}{32},\qquad c_2=\frac{3t^3}{16},\qquad c_3=\frac{t^3}{32},$$

and

$$u_N(x)=\sum_{i=0}^{3}c_iP_{t,i}^{(-\frac{1}{2},-\frac{1}{2})}(x)=x^3,$$

which is the exact solution.

Example 3. *Consider the equation*

$$D^2u(x) - 2Du(x) + D^{\frac{1}{2}}u(x) + u(x) = x^7 + \frac{2048}{429\sqrt{\pi}}x^{6.5} - 14x^6 + 42x^5 - x^2 - \frac{8}{3\sqrt{\pi}}x^{1.5} + 4x - 2, \qquad (5.6)$$
$$u(0) = 0, \quad u'(0) = 0, \quad x \in [0, t],$$

whose exact solution is given by $u(x) = x^7 - x^2$.

Now, applying the technique described in Example 1 with $N = 9$ for some special choices of α *and* β, gives the following cases

Case 1. If $\alpha = \beta = 0$, then

$$c_0 = \frac{t^2}{24}(3t^5 - 8), \quad c_1 = \frac{t^2}{24}(7t^5 - 12), \quad c_2 = \frac{t^2}{24}(7t^5 - 4), \quad c_3 = \frac{49t^7}{264},$$
$$c_4 = \frac{7t^7}{88}, \quad c_5 = \frac{7t^7}{312}, \quad c_6 = \frac{t^7}{264}, \quad c_7 = \frac{t^7}{3432}, \quad c_8 = 0, \quad c_9 = 0,$$

and

$$u_N(x) = \sum_{i=0}^{9} c_i P_{t,i}^{(0,0)}(x) = x^7 - x^2,$$

which is the exact solution.

Case 2. If $\alpha = \beta = \frac{1}{2}$, then

$$c_0 = \frac{5t^2}{8192}(143t^5 - 512), \quad c_1 = \frac{t^2}{6144}(1001t^5 - 2048), \quad c_2 = \frac{t^2}{5120}(819t^5 - 512), \quad c_3 = \frac{13t^7}{128},$$
$$c_4 = \frac{25t^7}{576}, \quad c_5 = \frac{15t^7}{1232}, \quad c_6 = \frac{7t^7}{3432}, \quad c_7 = \frac{t^7}{6435}, \quad c_8 = 0, \quad c_9 = 0,$$

and

$$u_N(x) = \sum_{i=0}^{9} c_i P_{t,i}^{(\frac{1}{2},\frac{1}{2})}(x) = x^7 - x^2,$$

which is the exact solution.

Case 3. If $\alpha = -\frac{1}{2}, \beta = \frac{1}{2}$,

$$c_0 = \frac{t^2(6435t^5 - 10240)}{16384}, \quad c_1 = \frac{t^2(5005t^5 - 5120)}{8192}, \quad c_2 = \frac{t^2(3003t^5 - 1024)}{6144}, \quad c_3 = \frac{273t^7}{1024},$$
$$c_4 = \frac{13t^7}{128}, \quad c_5 = \frac{5t^7}{192}, \quad c_6 = \frac{5t^7}{1232}, \quad c_7 = \frac{t^7}{3432}, \quad c_8 = 0, \quad c_9 = 0,$$

and

$$u_N(x) = \sum_{i=0}^{9} c_i P_{t,i}^{(-\frac{1}{2},\frac{1}{2})}(x) = x^7 - x^2,$$

which is the exact solution.

Case 4. If $\alpha = \frac{1}{2}, \beta = -\frac{1}{2}$, then

$$c_0 = \frac{t^2(429t^5 - 2048)}{16384}, \quad c_1 = \frac{t^2(1001t^5 - 3072)}{8192}, \quad c_2 = \frac{t^2(1001t^5 - 1024)}{6144}, \quad c_3 = \frac{637t^7}{5120},$$
$$c_4 = \frac{39t^7}{640}, \quad c_5 = \frac{11t^7}{576}, \quad c_6 = \frac{13t^7}{3696}, \quad c_7 = \frac{t^7}{3432}, \quad c_8 = 0, \quad c_9 = 0,$$

and

$$u_N(x) = \sum_{i=0}^{9} c_i P_{t,i}^{(\frac{1}{2},-\frac{1}{2})}(x) = x^7 - x^2,$$

which is the exact solution.

6 Concluding remarks

In this article, we have presented the operational matrix of fractional integration of the shifted Jacobi polynomials, and as an important application, we describe how to use the operational tau technique to numerically solve the FDEs. The basic idea of this technique is as follows:

(i) The FDE is converted to an fully integrated form via multiple integration in the Riemann-Liouville sense.

(ii) Subsequently, the various signals involved in the integrated form equation are approximated by representing them as linear combinations of shifted Jacobi polynomials.

(iii) Finally, the integrated form equation is converted to an algebraic equation by introducing the operational matrix of fractional integration of the shifted Jacobi polynomials.

To the best of our knowledge, the presented theoretical formula for SJOM is completely new and we do believe that this formula may be used to solve some other kinds of fractional-order initial value problems

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