GENERALIZED SPACES OF SOBOLEV TYPES AND APPLICATIONS

HATEM MEJJAOLI

Dedicated to Khalifa Trimèche for his 66 birthday

Abstract. In this paper we introduce and we study Sobolev type spaces associated to Jacobi-Cherednik operator on $\mathbb{R}$. Next we define the generalized Besov and Triebel spaces and study some of properties. As applications on these spaces we establish the Sobolev embedding, the hypoellipticity for the Jacobi-Cherednik operator. We give some properties including some estimates for the solution of the generalized wave equation and the generalized Schrödinger equation. Also, some applications are given for these spaces.

1. Introduction

In this paper, we are interested in generalized spaces of Sobolev types. In the classical case the theory of function spaces appears at first to be a disconnected subject, because of the variety of spaces and the different considerations involved in their definitions. There are the Lebesgue spaces $L^p(\mathbb{R}^d)$, the Sobolev spaces $H^s(\mathbb{R}^d)$, the Besov spaces $B^s_{p,q}(\mathbb{R}^d)$, the Triebel $F^s_{p,q}(\mathbb{R}^d)$ spaces and others.

In this paper, we consider the differential-difference operator $T_{k,k'}$, called the Jacobi-Cherednik operator (cf. [4]), defined for a function f of class C^1 on $\mathbb{R}$ by

(1.1)
$$T_{k,k'}f(x) = \begin{cases} f'(x) + \Big(k\coth(x) + k'\tanh(x)\big)(f(x) - f(-x)\Big) - (k+k')f(-x), & \text{for } x \neq 0 \\ (2k+1)f'(0) - (k+k')f(0) & \end{cases}$$

and where $k > 0$ and $k' \geq 0$ are two parameters satisfying the following condition (C):

$$\begin{array}{c} \text{either } \ k' = 0 \ \text{ and } 0 < k, \\ \text{either } \ 0 < k' \leq k. \end{array} \qquad (C)$$

The one dimensional Cherednik operator (cf. [3]) is a particular case of $T_{k,k'}$. Such operators have been used by Heckman and Opdam to develop a theory generalizing harmonic analysis on symmetric spaces (cf. [6, 11, 12]). For recent important results in this direction we refer to [13]. We note also that the operator $T_{k,k'}$ is a particular case of the operator Λ (cf. [10]) given by

$$\Lambda f = \frac{df}{dx} + \frac{A'(x)}{A(x)}\left(\frac{f(x) - f(-x)}{2}\right) - \varrho f(-x),$$

where

$$A(x) = |x|^{2k}B(x), \quad k > 0,$$

B being a positive C^∞ even function on $\mathbb{R}$, and $\varrho > 0$. The operator $T_{k,k'}$ corresponds to the function $A(x) = A_{k,k'}(x)$, where

$$A_{k,k'}(x) = \sinh^{2k}(|x|)\cosh^{2k'}(x), \quad \varrho = k + k'.$$

For $k \geq k' \geq 0$ and $k \neq 0$, a complete spectral analysis of the Jacobi-Cherednik operator has been performed in [1, 4].

1991 *Mathematics Subject Classification.* Primary 35L05. Secondary 22E30.

Key words and phrases. Jacobi-Cherednik operator, generalized Sobolev spaces, generalized Besov spaces, generalized wave equation, generalized wavelet transform.

The author is deeply indebted to the referees for providing constructive comments and helps in improving the contents of this article.

In the context of differential-differences operators, the generalized Sobolev has already been studied in various settings. The rational Dunkl case was treated by Mejjaoli-Trimèche [8], while Ben Salem-Dachraoui [2] studied the generalized Soblev spaces in the Jacobi setting theory. The purpose of this paper is to introduce new spaces associated with the Jacobi-Cherednik operator: the Sobolev spaces $W^{s,p}_{k,k'}(\mathbb{R})$, $\mathcal{W}^{s,p}_{k,k'}(\mathbb{R})$, the Potential space $H^{s,p}_{k,k'}(\mathbb{R})$, the Besov space $B^{s,k,k'}_{p,q}(\mathbb{R})$ and the Triebel space $F^{s,k,k'}_{p,q}(\mathbb{R})$ that generalizes the corresponding classical spaces.

The paper is organized as follows. In §2 we recall the main results about the harmonic analysis associated with the Jacobi-Cherednik operator. In §3 generalized Sobolev spaces associated with the Jacobi-Cherednik operator are studied. Some properties including completeness and Sobolev embedding theorems are established. Next, we define the generalized potential transform and we study the generalized potential space. The §4 is devoted to define the Besov and the Triebel spaces associated with the Jacobi-Cherednik operator and to give some of their properties. In §5 we give some applications. Firstly we study the hypoellipticity for the Jacobi-Cherednik operator. Some estimates of the solution for the generalized wave equation is given. We introduce also the generalized Schrödinger equation, and we study the solution if the initial data belongs to the generalized Sobolev spaces. Finally, we give practical real inversion formulas using the theory of reproducing kernels for the generalized wavelet transform.

2. Preliminaries

This section gives an introduction to the harmonic analysis associated with the Jacobi-Cherednik operator. Main references are [1, 4, 13].

2.1. **Jacobi-Cherednik kernel.** In this subsection we collect some notations and results on Jacobi-Cherednik operator and the Jacobi-Cherednik kernel.

In the following we denote by

$C(\mathbb{R})$	the space of continuous functions on $\mathbb{R}$.
$C_c(\mathbb{R})$	the space of continuous functions on $\mathbb{R}$ with compact support.
$C^p(\mathbb{R})$	the space of functions of class C^p on $\mathbb{R}$.
$C^p_b(\mathbb{R})$	the space of bounded functions of class C^p.
$\mathcal{E}(\mathbb{R})$	the space of C^∞-functions on $\mathbb{R}$.
$\mathcal{S}(\mathbb{R})$	the Schwartz space of rapidly decreasing functions on $\mathbb{R}$.
$D(\mathbb{R})$	the space of C^∞-functions on $\mathbb{R}$ which are of compact support.
$\mathcal{S}'(\mathbb{R})$	the space of temperate distributions on $\mathbb{R}$.
$\mathcal{S}_{k,k'}(\mathbb{R})$	$:= (\cosh x)^{-\rho} S(\mathbb{R})$ the generalized Schwartz space.
$\mathcal{S}'_{k,k'}(\mathbb{R})$	the dual topological space of $\mathcal{S}_{k,k'}(\mathbb{R})$.

We define the generalized Laplace operator on $\mathbb{R}$ by

$$(2.2)\qquad \begin{aligned}\triangle_{k,k'} f(x) := T^2_{k,k'} f(x) &= f''(x) + (k+k')\coth x f'(x) + (k+k')^2 f(x)\\ &- \Big(\frac{k-k'}{\sinh^2 x} + \frac{4k'}{\sinh^2(2x)}\Big)\Big(f(-x) - f(x)\Big).\end{aligned}$$

For every $\lambda \in \mathbb{C}$, let us denote by $G^{(k,k')}_\lambda$ the unique solution of the eigenvalue problem

$$(2.3)\qquad \left\{\begin{array}{l} T_{k,k'} f(x) = i\lambda f(x),\\ f(0) = 1.\end{array}\right.$$

Proposition 1. *([4]). For every $\lambda \in \mathbb{C}$, the eigenfunction equation (2.3) has a unique solution of the form*

$$(2.4)\qquad \forall x \in \mathbb{R},\ G^{(k,k')}_\lambda(x) = \varphi^{(k-\frac{1}{2},k'-\frac{1}{2})}_\lambda(x) - \frac{\rho + i\lambda}{\rho^2+\lambda^2}\frac{d}{dx}\varphi^{(k-\frac{1}{2},k'-\frac{1}{2})}_\lambda(x)$$

$$(2.5)\qquad = \varphi^{(k-\frac{1}{2},k'-\frac{1}{2})}_\lambda(x) + \frac{\rho + i\lambda}{4k+2}\sinh(2x)\varphi^{(k+\frac{1}{2},k'+\frac{1}{2})}_\lambda(x)$$

where $\rho = k + k'$ *and* $\varphi_\lambda^{(\alpha,\beta)}$ *is the Jacobi function of index* (α, β) *given by*

$$\varphi_\lambda^{(\alpha,\beta)}(x) = F(\frac{1}{2}(\rho + i\lambda), \frac{1}{2}(\rho - i\lambda);\ \alpha + 1;\ -(\sinh(x))^2), \tag{2.6}$$

where F *is the hypergeometric function* ${}_2F_1$ *of Gauss.*

Proposition 2. *([13]). Let* p *and* q *be polynomials of degree* m *and* n*. Then there exists a positive constant* C *such that for all* $\lambda \in \mathbb{C}$ *and for all* $x \in \mathbb{R}$*, we have*

$$|p(\frac{\partial}{\partial\lambda})q(\frac{\partial}{\partial x})G_\lambda^{(k,k')}(x)| \leq C(1 + |x|)^n(1 + |\lambda|)^m e^{-\varrho|x|}e^{|Im\lambda||x|}. \tag{2.7}$$

2.2. **Opdam-Cherednik transform.** For a Borel positive measure μ on $\mathbb{R}$, and $1 \leq p \leq \infty$, we write $L^p_\mu(\mathbb{R})$ for the Lebesgue space equipped with the norm $\| \cdot \|_{p,\mu}$ defined by

$$\|f\|_{L^p_\mu(\mathbb{R})} = \left(\int_{\mathbb{R}} |f(x)|^p\, d\mu(x)\right)^{1/p}, \quad \text{if } p < \infty,$$

and $\|f\|_{L^\infty_\mu(\mathbb{R})} =$ ess $\sup_{x\in\mathbb{R}}|f(x)|$. When $\mu(x) = w(x)dx$, with w a nonnegative function on $\mathbb{R}$, we replace the μ in the norms by w.

For $k \geq k' \geq 0$ with $k \neq 0$, and $f \in C_c(\mathbb{R})$, the Opdam-Cherednik transform is defined by

$$\mathcal{F}(f)(\lambda) = \int_{\mathbb{R}} f(x)G_\lambda^{(k,k')}(-x)A_{k,k'}(x)dx, \quad \forall \lambda \in \mathbb{C}. \tag{2.8}$$

Remark 1. *For* $\lambda \in \mathbb{C}$ *and* $g \in C_c(\mathbb{R})$*, we have*

$$\mathcal{F}(g)(\lambda) = 2\mathcal{F}_{k,k'}(g_e)(\lambda) + 2(\varrho + i\lambda)\mathcal{F}_{k,k'}(\mathcal{I}g_o)(\lambda), \tag{2.9}$$

where $\mathcal{F}_{k,k'}$ *denotes the Jacobi transform,* g_e *(resp.* g_o*) denotes the even (resp. odd) part of* g*, and*

$$\mathcal{I}g_o(x) = \int_{-\infty}^x g_o(t)dt.$$

The inverse Opdam-Cherednik transform of a suitable function g on $\mathbb{R}$ is given by:

$$\mathcal{J}g(x) = \mathcal{F}^{-1}g(x) = \int_{\mathbb{R}} g(\lambda)G_\lambda^{(k,k')}(x)(1 - \frac{\rho}{i\lambda})\frac{d\lambda}{8\pi|c_{k,k'}(\lambda)|^2} \tag{2.10}$$

where

$$c_{k,k'}(\lambda) := \frac{2^{\rho - i\lambda}\Gamma(k + \frac{1}{2})\Gamma(i\lambda)}{\Gamma(\frac{1}{2}(\rho + i\lambda))\Gamma(\frac{1}{2}(k - k' + 1 + i\lambda))}, \quad \lambda \in \mathbb{C}\backslash i\mathbb{N}.$$

Remark 2. *i) The function* $\lambda \mapsto \frac{1}{|c_{k,k'}(\lambda)|^2}$ *is continuous on* $\mathbb{R}$*.*

ii) There exists $R > 0$ *such that* C_1*,* $C_2 > 0$ *such that*

$$C_1|\lambda|^{2k} \leq |c_{k,k'}(\lambda)|^{-2} \leq C_2|\lambda|^{2k},$$

when $|\lambda| \geq R$*.*

Proposition 3. *([13]). The Opdam-Cherednik transform* $\mathcal{F}$ *and its inverse* $\mathcal{J}$ *are topological isomorphisms between the generalized Schwartz space* $\mathcal{S}_{k,k'}(\mathbb{R})$ *and the Schwartz space* $S(\mathbb{R})$*.*

Next, we give some properties of this transform.

i) For f in $L^1_{A_{k,k'}}(\mathbb{R})$ we have

$$||\mathcal{F}(f)||_{L^\infty_{\nu_{k,k'}}(\mathbb{R})} \leq ||f||_{L^1_{A_{k,k'}}(\mathbb{R})}, \tag{2.11}$$

where

$$d\nu_{k,k'}(\lambda) = \frac{d\lambda}{16\pi|c_{k,k'}(\lambda)|^2}. \tag{2.12}$$

ii) For f in $\mathcal{S}_{k,k'}(\mathbb{R})$ we have

$$\mathcal{F}(\triangle_{k,k'}f)(y) = -y^2\mathcal{F}(f)(y), \quad \text{for all } y \in \mathbb{R}. \tag{2.13}$$

Proposition 4. *([13]). i) **Plancherel formula for** $\mathcal{F}$. For all f in $\mathcal{S}_{k,k'}(\mathbb{R})$ we have*

$$\int_{\mathbb{R}} |f(x)|^2 A_{k,k'}(x)\, dx = \int_{\mathbb{R}_+} \Big[|\mathcal{F}(f)(\xi)|^2 + \mathcal{F}(\check{f})(\xi)|^2\Big] d\nu_{k,k'}(\xi), \tag{2.14}$$

where $\check{f}(x) := f(-x)$.

ii*) For all f, g in $\mathcal{S}_{k,k'}(\mathbb{R})$ we have*

$$\int_{\mathbb{R}} f(x)g(-x)A_{k,k'}(x)\, dx = 2\int_{\mathbb{R}} \mathcal{F}(f)(\xi)\overline{\mathcal{F}(g)(-\xi)}(1-\frac{\varrho}{i\xi})d\nu_{k,k'}(\xi). \tag{2.15}$$

We denote by

$$D_e(\mathbb{R}) := \Big\{f \in D(\mathbb{R}): \quad f \text{ is even}\Big\}$$

and

$$\mathcal{S}_{k,k',e}(\mathbb{R}) := \Big\{f \in \mathcal{S}_{k,k'}(\mathbb{R}): \quad f \text{ is even}\Big\}.$$

Corollary 1. *i) Plancherel formula.*

For all f, g in $D_e(\mathbb{R})$ (resp . $\mathcal{S}_{k,k',e}(\mathbb{R})$) we have

$$\int_{\mathbb{R}} f(x)\overline{g(x)}A_{k,k'}(x)dx = \int_{\mathbb{R}} \mathcal{F}(f)(\lambda)\overline{\mathcal{F}(g)(\lambda)}d\nu_{k,k'}(\lambda). \tag{2.16}$$

ii) Plancherel theorem.

The transform $\mathcal{F}$ extends uniquely to an isomorphism from $L^2_{A_{k,k'},e}(\mathbb{R})$ onto $L^2_{\nu_{k,k'}}(\mathbb{R})$.

2.3. Jacobi-Cherednik convolution.

Definition 1. *([1]). Let $x \in \mathbb{R}$ and let $f \in C_b(\mathbb{R})$. For $k \geq k' \geq 0$, with $k \neq 0$, we define the generalized translation operator $\tau_x^{(k,k')}$ by*

$$\tau_x^{(k,k')} f(y) = \int_{\mathbb{R}} f(z) d\mu_{x,y}^{(k,k')}(z) \tag{2.17}$$

here

$$d\mu_{x,y}^{(k,k')}(z) = \begin{cases} \mathcal{K}_{k,k'}(x,y,z)A_{k,k'}(z)dz & \text{if} \quad xy \neq 0 \\ d\delta_x(z) & \text{if} \quad y = 0 \\ d\delta_y(z) & \text{if} \quad x = 0 \end{cases}$$

where $\mathcal{K}_{k,k'}(x,y,z)$ is given explicitly in [1]. Moreover

$$supp(d\mu_{x,y}^{(k,k')}) \subset \Big[-|x|-|y|, -\Big||x|-|y|\Big|\Big] \bigcup \Big[\Big||x|-|y|\Big|, |x|+|y|\Big].$$

Proposition 5. *([1]). For a suitable function f on $\mathbb{R}$, we have*

i) $\tau_x^{(k,k')} f(y) = \tau_y^{(k,k')} f(x)$.

ii) $\tau_0^{k,k'} f(y) = f(y)$.

iii) $\tau_x^{(k,k')}\tau_y^{(k,k')} = \tau_y^{(k,k')}\tau_x^{(k,k')}$.

iv) $\tau_x^{(k,k')} G_\lambda^{(k,k')}(y) = G_\lambda^{(k,k')}(x)G_\lambda^{(k,k')}(y)$.

v) $\mathcal{F}(\tau_x^{(k,k')} f)(\lambda) = G_\lambda^{(k,k')}(x)\mathcal{F}(f)(\lambda)$.

vi) $T_{k,k'}(\tau_x^{(k,k')})f = \tau_x^{(k,k')}(T_{k,k'}f)$ *where $T_{k,k'}$ is the Jacobi-Cherednik operator (1.1).*

Lemma 1. *([1]). For $1 \leq p \leq \infty$, $f \in L^p_{A_{k,k'}}(\mathbb{R})$ and $x \in \mathbb{R}$, we have*

$$\|\tau_x^{(k,k')} f\|_{L^p_{A_{k,k'}}(\mathbb{R})} \leq C_{k,k'}\|f\|_{L^p_{A_{k,k'}}(\mathbb{R})}, \tag{2.18}$$

where

$$C_{k,k'} = \begin{cases} 4 + \frac{\Gamma(k+\frac{1}{2})\Gamma(k')}{\Gamma(k)\Gamma(k'+\frac{1}{2})} & \text{if} \quad k > k' > 0, \\ \frac{5}{2}, & \text{if} \quad k = k' > 0. \end{cases} \tag{2.19}$$

Definition 2. *([1]). For suitable functions f and g, we define the convolution product $f *_{k,k'} g$ by*

$$f *_{k,k'} g(x) = \int_{\mathbb{R}} \tau_x^{(k,k')} f(-y) g(y) A_{k,k'}(y) dy. \tag{2.20}$$

Remark 3. *It is clear that this convolution product is both commutative and associative:*
i) $f *_{k,k'} g = g *_{k,k'} f$.
ii) $(f *_{k,k'} g) *_{k,k'} h = f *_{k,k'} (g *_{k,k'} h)$.

Proposition 6. *([1]).*
i) *Assume that $1 \leq p, q, r \leq \infty$ satisfy $\frac{1}{p} + \frac{1}{q} - 1 = \frac{1}{r}$. Then, for every $f \in L^p_{A_{k,k'}}(\mathbb{R})$ and $g \in L^q_{A_{k,k'}}(\mathbb{R})$, we have $f *_{k,k'} g \in L^r_{A_{k,k'}}(\mathbb{R})$, and*

$$\|f *_{k,k'} g\|_{L^r_{A_{k,k'}}(\mathbb{R})} \leq C_{k,k'} \|f\|_{L^p_{A_{k,k'}}(\mathbb{R})} \|g\|_{L^q_{A_{k,k'}}(\mathbb{R})}, \tag{2.21}$$

where $C_{k,k'}$ is given by the relation (2.19).
ii) *Let $1 \leq p < q \leq 2$. Then*

$$L^p_{A_{k,k'}}(\mathbb{R}) *_{k,k'} L^q_{A_{k,k'}}(\mathbb{R}) \hookrightarrow L^q_{A_{k,k'}}(\mathbb{R}). \tag{2.22}$$

iii) *Let $2 < p, q < \infty$ such that $\frac{q}{2} \leq p < q$. Then*

$$L^p_{A_{k,k'}}(\mathbb{R}) *_{k,k'} L^{q'}_{A_{k,k'}}(\mathbb{R}) \hookrightarrow L^q_{A_{k,k'}}(\mathbb{R}) \tag{2.23}$$

where q' is the conjugate exponent of q.
iv) *Let $1 < p < 2$ and $p < q \leq \frac{p}{2-p}$. Then*

$$L^p_{A_{k,k'}}(\mathbb{R}) *_{k,k'} L^p_{A_{k,k'}}(\mathbb{R}) \hookrightarrow L^q_{A_{k,k'}}(\mathbb{R}). \tag{2.24}$$

Proposition 7. *([1]).*
i) *Let $D_a(\mathbb{R})$ be the space of smooth functions on $\mathbb{R}$ supported in $[-a, a]$. For $f \in D_a(\mathbb{R})$ and $g \in D_b(\mathbb{R})$, we have $f *_{k,k'} g \in D_{a+b}(\mathbb{R})$ and*

$$\mathcal{F}(f *_{k,k'} g) = \mathcal{F}(f)(\lambda)\mathcal{F}(f)(\lambda). \tag{2.25}$$

ii) *For $f \in L^2_{A_{k,k'}}(\mathbb{R})$ and $g \in L^p_{A_{k,k'}}(\mathbb{R})$, with $1 \leq p < 2$ we have*

$$\mathcal{F}(f *_{k,k'} g) = \mathcal{F}(f)(\lambda)\mathcal{F}(g)(\lambda). \tag{2.26}$$

Proposition 8. *Let $f, g \in L^2_{A_{k,k'}}(\mathbb{R})$. Then $f *_{k,k'} g \in L^2_{A_{k,k'}}(\mathbb{R})$ if and only if $\mathcal{F}(f)\mathcal{F}(g)$ belongs to $L^2_{\nu_{k,k'}}(\mathbb{R})$, and in this case we have*

$$\mathcal{F}(f *_{k,k'} g) = \mathcal{F}(f)\mathcal{F}(g).$$

The proof of this proposition is a consequence of the two following lemmas.

Lemma 2. *Let $f \in L^\infty_{A_{k,k'}}(\mathbb{R}), g \in L^1_{\nu_{k,k'}}(\mathbb{R})$ and assume that for all $\chi \in L^1_{A_{k,k'}}(\mathbb{R}) \cap L^2_{A_{k,k'}}(\mathbb{R})$ we have*

$$\int_{\mathbb{R}} f(y)\overline{\chi(y)} A_{k,k'}(y) dy = 2 \int_{\mathbb{R}} g(\xi) \mathcal{F}(\overline{\chi})(\xi)(1 - \frac{\varrho}{i\xi}) d\nu_{k,k'}(\xi),$$

where $\overline{\chi}(\xi) = \overline{\chi}(-\xi)$. Then $f \in L^2_{A_{k,k'}}(\mathbb{R})$ if and only if $g \in L^2_{\nu_{k,k'}}(\mathbb{R})$, and in this case we have

$$\mathcal{F}(f) = g \quad a.e.$$

Proof. This follows from an easy application of the Plancherel formula. □

Lemma 3. *Let $f, g \in L^2_{A_{k,k'}}(\mathbb{R}), \chi \in L^1_{A_{k,k'}}(\mathbb{R}) \cap L^2_{A_{k,k'}}(\mathbb{R})$ we have*

$$\int_{\mathbb{R}} f *_{k,k'} g(y)\overline{\chi(y)} A_{k,k'}(y) dy = 2 \int_{\mathbb{R}} \mathcal{F}(f)(\xi)\mathcal{F}(g)(\xi)\mathcal{F}(\overline{\chi})(1 - \frac{\varrho}{i\xi}) d\nu_{k,k'}(\xi).$$

Proof. First note the following general fact: if $f \in L^1_{A_{k,k'}}(\mathbb{R}) \cap L^2_{A_{k,k'}}(\mathbb{R})$ and $g \in L^2_{A_{k,k'}}(\mathbb{R})$ then

$$\mathcal{F}(f *_{k,k'} g) = \mathcal{F}(f)\mathcal{F}(g) \quad a.e.$$

This follows from the analogous fact for $L^1_{A_{k,k'}}(\mathbb{R})$ functions and the possibility to approximate g in $L^2_{A_{k,k'}}(\mathbb{R})$ with functions in $L^1_{A_{k,k'}}(\mathbb{R}) \cap L^2_{A_{k,k'}}(\mathbb{R})$.

Next fix $g \in L^2_{A_{k,k'}}(\mathbb{R})$ and define on $L^1_{A_{k,k'}}(\mathbb{R}) \cap L^2_{A_{k,k'}}(\mathbb{R})$ the two functionals

$$S_1(f) := \int_{\mathbb{R}} f *_{k,k'} g(y)\overline{\chi(y)}A_{k,k'}(y)dy, \quad S_2(f) := 2\int_{\mathbb{R}} \mathcal{F}(f)(\xi)\mathcal{F}(g)(\xi)\mathcal{F}(\overline{\check{\chi}})(1-\frac{\varrho}{i\xi})d\nu_{k,k'}(\xi).$$

By the previous fact and Plancherels identity, S_1 and S_2 coincide on $L^1_{A_{k,k'}}(\mathbb{R}) \cap L^2_{A_{k,k'}}(\mathbb{R})$. It is easy to show that both functionals are bounded with respect to the $L^2_{A_{k,k'}}$ norm, and therefore can be extended to the whole space $L^2_{A_{k,k'}}(\mathbb{R})$, where they still coincide. □

An immediate consequence of Proposition 8 and the Plancherel formula we deduce the following.

Proposition 9. *Let f and g be in $L^2_{A_{k,k'}}(\mathbb{R})$. Then, we have*

$$\int_{\mathbb{R}} |f *_{k,k'} g(x)|^2 A_{k,k'}(x)dx = \int_{\mathbb{R}_+} \Big[|\mathcal{F}(f)(\xi)|^2|\mathcal{F}(g)(\xi)|^2 + \mathcal{F}(\check{f})(\xi)|^2|\mathcal{F}(\check{g})(\xi)|^2\Big]d\nu_{k,k'}(\xi) \tag{2.27}$$

where both sides are finite or infinite.

Definition 3. *The Opdam-Cherednik transform of a distribution τ in $\mathcal{S'}_{k,k'}(\mathbb{R})$ is defined by*

$$\langle \mathcal{F}(\tau), \phi\rangle = \langle \tau, \mathcal{F}^{-1}(\phi)\rangle, \quad \text{for all } \phi \in \mathcal{S}(\mathbb{R}). \tag{2.28}$$

Proposition 10. *The Opdam-Cherednik transform $\mathcal{F}$ is a topological isomorphism from $\mathcal{S'}_{k,k'}(\mathbb{R})$ onto $\mathcal{S}'(\mathbb{R})$.*

Let τ be in $\mathcal{S'}_{k,k'}(\mathbb{R})$. We define the distribution $\triangle_{k,k'}\tau$, by

$$\langle \triangle_{k,k'}\tau, \psi\rangle = \langle \tau, \triangle_{k,k'}\psi\rangle, \quad \text{for all } \psi \in \mathcal{S}_{k,k'}(\mathbb{R}). \tag{2.29}$$

This distribution satisfy the following property

$$\mathcal{F}(\triangle_{k,k'}\tau) = -y^2\mathcal{F}(\tau). \tag{2.30}$$

3. Sobolev and potentials spaces

In this Section we establish the main properties of the Sobolev spaces associated with the Jacobi-Cherednik operator.

Definition 4. *Let $s \in \mathbb{R}$ and $1 \le p < \infty$, we define the space $W^{s,p}_{k,k'}(\mathbb{R})$ as*

$$\Big\{u \in \mathcal{S'}_{k,k'}(\mathbb{R}) : (1+|\xi|^2)^s\mathcal{F}(u) \in L^p_{\nu_{k,k'}}(\mathbb{R})\Big\}.$$

We provide this space with the norm

$$||u||_{W^{s,p}_{k,k'}(\mathbb{R})} = \Big(\int_{\mathbb{R}} (1+|\xi|^2)^{sp}|\mathcal{F}(u)(\xi)|^p d\nu_{k,k'}(\xi)\Big)^{\frac{1}{p}}. \tag{3.31}$$

Proposition 11. i*) Let $1 \le p < \infty$. The space $W^{s,p}_{k,k'}(\mathbb{R})$ provided with the norm $||.||_{W^{s,p}_{k,k'}(\mathbb{R})}$ is a Banach space.*

ii*) Let $1 \le p < \infty$ and s_1, s_2 in $\mathbb{R}$ such that $s_1 \ge s_2$ then*

$$W^{s_1,p}_{k,k'}(\mathbb{R}) \hookrightarrow W^{s_2,p}_{k,k'}(\mathbb{R}).$$

iii*) Let $s \in \mathbb{R}$ and $1 \le p < \infty$. Then $D(\mathbb{R})$ is dense in $W^{s,p}_{k,k'}(\mathbb{R})$.*

Proof. i) It is clear that $L^p(\mathbb{R},(1+|\xi|^2)^{sp}d\nu_{k,k'}(\xi))$ is complete and since $\mathcal{F}$ is an isomorphism from $S'_{k,k'}(\mathbb{R})$ onto $\mathcal{S}'(\mathbb{R})$, $W^{s,p}_{k,k'}(\mathbb{R})$ is then a Banach space.

The result ii) is immediately from definition of the generalized Sobolev space. As in [15], §6, we can obtain iii). □

Proposition 12. *Let $1 \leq p < \infty$, and s_1, s, s_2 be three real numbers : $s_1 < s < s_2$. Then, for all $\varepsilon > 0$, there exists a nonnegative constant C_ε such that for all u in $W^{s,p}_{k,k'}(\mathbb{R})$*

$$||u||_{W^{s,p}_{k,k'}(\mathbb{R})} \leq C_\varepsilon ||u||_{W^{s_1,p}_{k,k'}(\mathbb{R})} + \varepsilon ||u||_{W^{s_2,p}_{k,k'}(\mathbb{R})}. \tag{3.32}$$

Proof. We consider $s = (1-t)s_1 + ts_2$, (with $t \in]0,1[$). Moreover it is easy to see

$$||u||_{W^{s,p}_{k,k'}(\mathbb{R})} \leq ||u||^{1-t}_{W^{s_1,p}_{k,k'}(\mathbb{R})} ||u||^{t}_{W^{s_2,p}_{k,k'}(\mathbb{R})}.$$

Thus

$$\begin{aligned} ||u||_{W^{s,p}_{k,k'}(\mathbb{R})} &\leq (\varepsilon^{-\frac{t}{1-t}} ||u||_{W^{s_1,p}_{k,k'}(\mathbb{R})})^{1-t} (\varepsilon ||u||_{W^{s_2,p}_{k,k'}(\mathbb{R})})^t \\ &\leq \varepsilon^{-\frac{t}{1-t}} ||u||_{W^{s_1,p}_{k,k'}(\mathbb{R})} + \varepsilon ||u||_{W^{s_2,p}_{k,k'}(\mathbb{R})}. \end{aligned}$$

Hence the proof is completed for $C_\varepsilon = \varepsilon^{-\frac{t}{1-t}}$. □

A characterization of $W^{s,p}_{k,k'}(\mathbb{R})$, for $s = m$, a positive integer, is given below.

Proposition 13. *Let $m \in \mathbb{N}$, then for $1 \leq p < \infty$*

$$W^{m,p}_{k,k'}(\mathbb{R}) = \Big\{ u \in S'_{k,k'}(\mathbb{R}) : \quad \mathcal{F}(\triangle^j_{k,k'} u) \in L^p_{\nu_{k,k'}}(\mathbb{R}),\ 0 \leq j \leq m \Big\}.$$

Proof. Let $u \in W^{m,p}_{k,k'}(\mathbb{R})$. Using the formula (2.30) we have

$$\begin{aligned} \int_{\mathbb{R}} |\mathcal{F}(\triangle^j_{k,k'} u)(\xi)|^p d\nu_{k,k'}(\xi) &= \int_{\mathbb{R}} |(-\xi^2)^j \mathcal{F}(u)(\xi)|^p d\nu_{k,k'}(\xi) \\ &\leq \int_{\mathbb{R}} (1+|\xi|^2)^{mp} |\mathcal{F}(u)(\xi)|^p d\nu_{k,k'}(\xi) < \infty. \end{aligned}$$

Conversely assume now that $\mathcal{F}(\triangle^j_{k,k'} u) \in L^p_{\nu_{k,k'}}(\mathbb{R})$, $0 \leq j \leq m$. It is easy to see that there exists a positive constant C such that $(1+|\xi|^2)^{mp} \leq C \sum_{j=0}^{m} |\xi|^{2pj}$. Then

$$\begin{aligned} \int_{\mathbb{R}} (1+|\xi|^2)^{mp} |\mathcal{F}(u)(\xi)|^p d\nu_{k,k'}(\xi) &\leq C \sum_{j=0}^{m} \int_{\mathbb{R}} |(-\xi^2)^j \mathcal{F}(u)(\xi)|^p d\nu_{k,k'}(\xi) \\ &= C \sum_{j=0}^{m} \int_{\mathbb{R}} |\mathcal{F}(\triangle^j_{k,k'} u)(\xi)|^p d\nu_{k,k'}(\xi) < \infty. \end{aligned}$$

□

Proposition 14. *Let $p \in \mathbb{N}$ and $s \in \mathbb{R}$ such that $s > \frac{2k+1+2p}{4}$, then*

$$W^{s,2}_{k,k'}(\mathbb{R}) \hookrightarrow C^p(\mathbb{R}).$$

Proof. Let u be in $W^{s,2}_{k,k'}(\mathbb{R})$ with $s \in \mathbb{R}$ such that $s > \frac{2k+1}{4}$.
We have

$$\int_{\mathbb{R}} |\mathcal{F}(u)(\lambda)| d\nu_{k,k'}(\lambda) = \int_{\mathbb{R}} (1+|\lambda|^2)^{-s} (1+|\lambda|^2)^{s} |\mathcal{F}(u)(\lambda)| d\nu_{k,k'}(\lambda).$$

Using Hölder inequality we obtain

$$\int_{\mathbb{R}} |\mathcal{F}(u)(\lambda)| d\nu_{k,k'}(\lambda) \leq \Big(\int_{\mathbb{R}} (1+|\lambda|^2)^{-2s} d\nu_{k,k'}(\lambda) \Big)^{\frac{1}{2}} \Big(\int_{\mathbb{R}} (1+|\lambda|^2)^{2s} |\mathcal{F}(u)(\lambda)|^2 d\nu_{k,k'}(\lambda) \Big)^{\frac{1}{2}}.$$

Thus from Remark 2, we deduce that there exists a positive constant C such that

$$||\mathcal{F}(u)||_{L^1_{\nu_{k,k'}}(\mathbb{R})} \leq C||u||_{W^{s,2}_{k,k'}(\mathbb{R})}. \tag{3.33}$$

Then $\mathcal{F}(u)$ is in $L^1_{\nu_{k,k'}}(\mathbb{R})$. Hence $\mathcal{F}(u)$ belongs to $L^1_{\nu_{k,k'}}(\mathbb{R}) \cap L^2_{\nu_{k,k'}}(\mathbb{R})$.
Thus from (2.10) we have

$$u(x) = \int_{\mathbb{R}} \mathcal{F}(u)(\lambda) G^{(k,k')}_\lambda(x)(1-\frac{\rho}{i\lambda})\frac{d\lambda}{8\pi|c_{k,k'}(\lambda)|^2}, \quad a.e.\, x \in \mathbb{R}.$$

We identify u with the second member, then we deduce that u belongs to $C(\mathbb{R})$ and using (3.33) we show that the injection of $W^{s,2}_{k,k'}(\mathbb{R})$ into $C(\mathbb{R})$ is continuous.

Now let u be in $W^{s,2}_{k,k'}(\mathbb{R})$ with $s \in \mathbb{R}$ such that $s > \frac{2k+1+2p}{4}$ with $p \in \mathbb{N}\backslash\{0\}$. From (2.7), for all $x, \lambda \in \mathbb{R}$, and $n \in \mathbb{N}$ such that $n \leq p$, we have

$$|D^n_x G^{(k,k')}_\lambda(x)| \leq C|\lambda|^n.$$

Using the same method as for $p = 0$, and the derivation theorem under the integral sign we deduce that

$$\forall\, x \in \mathbb{R},\ D^n u(x) = \int_{\mathbb{R}} \mathcal{F}(u)(\lambda) D^n_x G^{(k,k')}_\lambda(x)(1-\frac{\rho}{i\lambda})\frac{d\lambda}{8\pi|c_{k,k'}(\lambda)|^2}.$$

Thus $D^n u$ belongs to $C(\mathbb{R})$, for all $n \in \mathbb{N}$ such that $n \leq p$. Then we show that u is in $C^p(\mathbb{R})$ and the injection of $W^{s,2}_{k,k'}(\mathbb{R})$ into $C^p(\mathbb{R})$ is continuous. □

Definition 5. *Let n be a nonnegative integer and $1 \leq p < \infty$, we define inhomogeneous Sobolev spaces $\mathcal{W}^{n,p}_{k;k'}(\mathbb{R})$ by:*

$$\mathcal{W}^{n,p}_{k,k'}(\mathbb{R}) := \Big\{f \in \mathcal{S'}_{k,k'}(\mathbb{R}) : \triangle^j_{k,k'} f \in L^p_{A_{k,k'}}(\mathbb{R}),\ 0 \leq j \leq n\Big\}$$

endowed with the norm

$$||f||_{\mathcal{W}^{n,p}_{k;k'}(\mathbb{R})} := \sum_{j=0}^n ||\triangle^j_{k,k'} f||_{L^p_{A_{k,k'}}(\mathbb{R})}$$

where $\triangle^j_{k,k'} = \triangle_{k,k'} \circ \cdots \circ \triangle_{k,k'}$ is the iterated of generalized Laplace operator.

Proposition 15. *Let n be a nonnegative integer. $\mathcal{W}^{n,p}_{k;k'}(\mathbb{R})$ is complete when $1 \leq p < \infty$.*

Proof. Let $(f_l)_l$ be a Cauchy sequence in $\mathcal{W}^{n,p}_{k;k'}(\mathbb{R})$. Therefore $((\triangle_{k,k'})^j f_l)_l$ is a Cauchy sequence in $L^p_{A_{k,k'}}(\mathbb{R}), j = 0, ..., n$. If we denote by g_j to the limit in $L^p_{A_{k,k'}}(\mathbb{R})$ of $((\triangle_{k,k'})^j f)_l$, we have, by the uniqueness of the limit

$$\langle(\triangle_{k,k'})^j g_0, \phi\rangle = \langle g_j, \phi\rangle, \quad \text{for all} \quad \phi \in \mathcal{S}_{k,k'}(\mathbb{R}).$$

Then, $f_l \to g_0$ in $\mathcal{W}^{n,p}_{k;k'}(\mathbb{R})$ as $l \to \infty$. □

Now, we establish in a similar way to that in (cf. [2, 16]), the definition of the generalized Jacobi potential and potentials spaces.

Definition 6. *Let $u \in S'_{k,k'}(\mathbb{R})$ and $s \in \mathbb{R}$, we define the generalized Jacobi potential of order s, as follows*

$$\mathcal{J}^s_{k,k'}(u) = \mathcal{F}^{-1}\Big((\lambda^2+1)^{-s/2}\mathcal{F}(u)(\lambda)\Big).$$

Definition 7. *Let $s \in \mathbb{R}$ and $1 \leq p < \infty$. We define the generalized potentials spaces as*

$$H^{s,p}_{k,k'}(\mathbb{R}) := \Big\{\phi \in S'_{k,k'}(\mathbb{R}) : \mathcal{J}^{-s}_{k,k'}(\phi) \in L^p_{A_{k,k'}}(\mathbb{R})\Big\}.$$

The norm in $H^{s,p}_{k,k'}(\mathbb{R})$ is given by

$$\|\phi\|_{H^{s,p}_{k,k'}(\mathbb{R})} = \|\mathcal{J}^{-s}_{k,k'}(\phi)\|_{L^p_{A_{k,k'}}(\mathbb{R})}.$$

Lemma 4. *Let* $f \in S'_{k,k'}(\mathbb{R})$. *Then*

$$\mathcal{J}^s_{k,k'}(\mathcal{J}^t_{k,k'}(f)) = \mathcal{J}^{s+t}_{k,k'}(f)$$

and

$$\mathcal{J}^0_{k,k'}(f) = f.$$

Proof. By definition $\mathcal{J}^t_{k,k'}(f) = \mathcal{F}^{-1}\left((\lambda^2+1)^{-t/2}\mathcal{F}(f)(\lambda)\right)$.
Then,

$$\begin{aligned}\mathcal{J}^s_{k,k'}(\mathcal{J}^t_{k,k'}(f)) &= \mathcal{F}^{-1}\Big((\lambda^2+1)^{-s/2}(\lambda^2+1)^{-t/2}\mathcal{F}(f)(\lambda)\Big)\\ &= \mathcal{F}^{-1}\Big((\lambda^2+1)^{-(t+s)/2}\mathcal{F}(f)(\lambda)\Big)\\ &= \mathcal{J}^{s+t}_{k,k'}(f)\end{aligned}$$

On the other hand, $\mathcal{J}^0_{k,k'}(f) = \mathcal{F}^{-1}(\mathcal{F}(f)(\lambda)) = f$. □

Lemma 5. *The generalized Jacobi potential* $\mathcal{J}^t_{k,k'}$ *is an isometry of* $H^{s,p}_{k,k'}(\mathbb{R})$ *onto* $H^{s+t,p}_{k,k'}(\mathbb{R})$ *satisfying*

$$\|\mathcal{J}^t_{k,k'}(\phi)\|_{H^{s+t,p}_{k,k'}(\mathbb{R})} = \|\phi\|_{H^{s,p}_{k,k'}(\mathbb{R})}.$$

Proof. Let $\phi \in H^{s,p}_{k,k'}(\mathbb{R})$. By Definition 6 and Lemma 4 we obtain

$$\|\mathcal{J}^t_{k,k'}(\phi)\|_{H^{s+t,p}_{k,k'}(\mathbb{R})} = \|\mathcal{J}^{-s-t}_{k,k'}(\mathcal{J}^t_{k,k'}(\phi))\|_{L^p_{A_{k,k'}}(\mathbb{R})} = \|\mathcal{J}^{-s}_{k,k'}(\phi)\|_{L^p_{A_{k,k'}}(\mathbb{R})} = \|\phi\|_{H^{s,p}_{k,k'}(\mathbb{R})}.$$

Now, let $f \in H^{s+t,p}_{k,k'}(\mathbb{R})$. $\mathcal{J}^{-t}_{k,k'}(f) \in H^{s,p}_{k,k'}(\mathbb{R})$ and $\mathcal{J}^t_{k,k'}(\mathcal{J}^{-t}_{k,k'}(f)) = f$. Therefore we obtain the result. □

Proposition 16. $H^{s,p}_{k,k'}(\mathbb{R})$ *is a Banach space with respect to the norm* $\|.\|_{H^{s,p}_{k,k'}(\mathbb{R})}$.

Proof. Let $(\phi_n)_n$ be a Cauchy sequence in $H^{s,p}_{k,k'}(\mathbb{R})$. By the definition of $H^{s,p}_{k,k'}(\mathbb{R})$ the sequence $\{\mathcal{J}^{-s}_{k,k'}(\phi_n)\}$ is a Cauchy sequence in $L^p_{A_{k,k'}}(\mathbb{R})$. As $L^p_{A_{k,k'}}(\mathbb{R})$ is complete, it follows that there exists a function ϕ in $L^p_{A_{k,k'}}(\mathbb{R})$ such that $\mathcal{J}^{-s}_{k,k'}(\phi_n)$ converge to ϕ in $L^p_{A_{k,k'}}(\mathbb{R})$. Thus, it is easy to see that $\phi_n \to g$ in $H^{s,p}_{k,k'}(\mathbb{R})$ as $n \to \infty$, with $g = \mathcal{J}^s_{k,k'}(\phi)$. □

Proposition 17. *For* $s \in \mathbb{R}$ *and* $1 \le p < \infty$, $\mathcal{S}_{k,k'}(\mathbb{R})$ *is dense in* $H^{s,p}_{k,k'}(\mathbb{R})$.

Proof. Let $f \in H^{s,p}_{k,k'}(\mathbb{R})$. Then, $\mathcal{J}^{-s}_{k,k'}(f) \in L^p_{A_{k,k'}}(\mathbb{R})$. Since $D(\mathbb{R})$ is dense in $L^p_{A_{k,k'}}(\mathbb{R})$, there exists a sequence $(\phi_j)_j \in D(\mathbb{R})$ such that

$$\phi_j \to \mathcal{J}^{-s}_{k,k'}(f) \text{ in } L^p_{A_{k,k'}}(\mathbb{R}). \tag{3.34}$$

Next, we define $g_j = \mathcal{J}^s_{k,k'}(\phi_j) = \mathcal{F}^{-1}\left((\lambda^2+1)^{-s/2}\mathcal{F}(\phi_j)(\lambda)\right)$. From Proposition 3 we deduce that

$$\lambda \mapsto (\lambda^2+1)^{-s/2}\mathcal{F}(\phi_j)(\lambda) \in S(\mathbb{R}).$$

Proposition 3 give that $g_j = \mathcal{F}^{-1}\Big((\lambda^2+1)^{-s/2}\mathcal{F}(\phi_j)(\lambda)\Big) \in \mathcal{S}_{k,k'}(\mathbb{R})$. Hence, by (3.34) we obtain

$$\begin{aligned}\|f - g_j\|_{H^{s,p}_{k,k'}(\mathbb{R})} &= \left(\int_{\mathbb{R}} |\mathcal{J}^{-s}_{k,k'}(f)(x) - \mathcal{J}^{-s}_{k,k'}(g_j)(x)|^p A_{k,k'}(x)dx\right)^{1/p}\\ &= \left(\int_{\mathbb{R}} |\mathcal{J}^{-s}_{k,k'}(f)(x) - \phi_j(x)|^p A_{k,k'}(x)dx\right)^{1/p} \to 0, \text{ for } j \to \infty.\end{aligned}$$

□

Proposition 18. *For $s > k + \frac{1}{2}$ and $1 \leq p \leq 2$, $\mathcal{J}_{k,k'}^{-s}$ maps $L^p_{A_{k,k'}}(\mathbb{R})$ into $L^2_{A_{k,k'}}(\mathbb{R})$. More precisely there exists $g \in L^2_{A_{k,k'}}(\mathbb{R})$ such that for all $f \in L^p_{A_{k,k'}}(\mathbb{R})$, we have*

$$\mathcal{J}_{k,k'}^{-s}(f) = f *_{k,k'} g,$$

and there exists a positive constant C such that

$$\|\mathcal{J}_{k,k'}^{-s}(f)\|_{L^2_{A_{k,k'}}(\mathbb{R})} \leq C||f||_{L^p_{A_{k,k'}}(\mathbb{R})}.$$

Proof. From properties of $\nu_{k,k'}$ (cf. Remark 2), we see that the function $\lambda \mapsto (1+\lambda^2)^{-\frac{s}{2}}$ belongs to

$$L^2_{\nu_{k,k'}}(\mathbb{R}) \cap L^\infty_{\nu_{k,k'}}(\mathbb{R}), \quad \text{for } s > k + \frac{1}{2}.$$

Thus we deduce from Proposition 4 that there exists an even function $g \in L^2_{A_{k,k'}}(\mathbb{R})$ such that

$$(1+\lambda^2)^{-\frac{s}{2}} = \mathcal{F}(g)(\lambda).$$

i) For $p = 2$, the function $\lambda \mapsto (1+\lambda^2)^{-\frac{s}{2}}\mathcal{F}(f)(\lambda)$ belongs to $L^2_{\nu_{k,k'}}(\mathbb{R})$. Using Proposition 8, we deduce that $g *_{k,k'} f$ belongs to $L^2_{A_{k,k'}}(\mathbb{R})$ and

$$\mathcal{F}(g *_{k,k'} f)(\lambda) = \mathcal{F}(g)(\lambda)\mathcal{F}(f)(\lambda) = (1+\lambda^2)^{-\frac{s}{2}}\mathcal{F}(f)(\lambda).$$

On the other hand, we have

$$\mathcal{F}(\mathcal{J}_{k,k'}^{-s}(f))(\lambda) = (1+\lambda^2)^{-\frac{s}{2}}\mathcal{F}(f)(\lambda).$$

We conclude by using Proposition 4 and Proposition 8 that $\mathcal{J}_{k,k'}^{-s}(f) = f *_{k,k'} g$. Moreover, we have

$$\begin{aligned}
\|\mathcal{J}_{k,k'}^{-s}(f)\|^2_{L^2_{A_{k,k'}}(\mathbb{R})} &= \|g *_{k,k'} f\|^2_{L^2_{A_{k,k'}}(\mathbb{R})} = \|\mathcal{F}(g *_{k,k'} f)\|^2_{L^2_{\nu_{k,k'}}(\mathbb{R}_+)} + \|\mathcal{F}(g \check{*_{k,k'}} f)\|^2_{L^2_{\nu_{k,k'}}(\mathbb{R}_+)} \\
&= \|\mathcal{F}(g)\mathcal{F}(f)\|^2_{L^2_{\nu_{k,k'}}(\mathbb{R}_+)} + \|\mathcal{F}(g)\mathcal{F}(\check{f})\|^2_{L^2_{\nu_{k,k'}}(\mathbb{R}_+)} \\
&\leq \|\mathcal{F}(g)\|^2_{L^\infty_{\nu_{k,k'}}(\mathbb{R}_+)}\Big[\|\mathcal{F}(f)\|^2_{L^2_{\nu_{k,k'}}(\mathbb{R}_+)} + \|\mathcal{F}(\check{f})\|^2_{L^2_{\nu_{k,k'}}(\mathbb{R}_+)}\Big] \\
&= \|\mathcal{F}(g)\|^2_{L^\infty_{\nu_{k,k'}}(\mathbb{R}_+)}||f||^2_{L^2_{A_{k,k'}}(\mathbb{R})}.
\end{aligned}$$

So, we obtain

$$\|\mathcal{J}_{k,k'}^{-s}(f)\|_{L^2_{A_{k,k'}}(\mathbb{R})} \leq ||(1+\lambda^2)^{-\frac{s}{2}}||_{L^\infty_{\nu_{k,k'}}(\mathbb{R})}\|f\|_{L^2_{A_{k,k'}}(\mathbb{R})} \leq C\|f\|_{L^2_{A_{k,k'}}(\mathbb{R})}.$$

ii) For $p \in [1,2)$, $g \in L^2_{A_{k,k'}}(\mathbb{R})$ and $f \in L^p_{A_{k,k'}}(\mathbb{R})$. Using Proposition 4 and Proposition 7 ii), we prove

$$\mathcal{J}_{k,k'}^{-s}(f) = g *_{k,k'} f.$$

Finally, applying Proposition 6 ii), we obtain

$$\|\mathcal{J}_{k,k'}^{-s}f\|_{L^2_{A_{k,k'}}(\mathbb{R})} \leq C||f||_{L^p_{A_{k,k'}}(\mathbb{R})}.$$

This completes the proof of theorem. □

Proposition 19. *For $s > k + \frac{1}{2} + t$ and $1 \leq p \leq 2$, then, we have*

$$H^{s,p}_{k,k'}(\mathbb{R}) \subset H^{t,2}_{k,k'}(\mathbb{R}).$$

Moreover, there exits a positive constant C, such that for all $u \in H^{s,p}_{k,k'}(\mathbb{R})$

$$||u||_{H^{t,2}_{k,k'}(\mathbb{R})} \leq C||u||_{H^{s,p}_{k,k'}(\mathbb{R})}.$$

Proof. Let $u \in H^{s,p}_{k,k'}(\mathbb{R})$. Then, we have $\mathcal{J}^{-s}_{k,k'}(u) = f$, where $f \in L^p_{A_{k,k'}}(\mathbb{R})$. From Lemma 5 and Proposition 18 we can write

$$\mathcal{J}^{-t}_{k,k'}(u) = \mathcal{J}^{-t+s}_{k,k'}(\mathcal{J}^{-s}_{k,k'}(u)) = f *_{k,k'} g,$$

where g is such that

$$(1+\lambda^2)^{-\frac{s-t}{2}} = \mathcal{F}(g)(\lambda).$$

Furthermore, we have

$$||u||_{H^{t,2}_{k,k'}(\mathbb{R})} = ||f *_{k,k'} g||_{L^2_{A_{k,k'}}(\mathbb{R})} \leq C||f||_{L^p_{A_{k,k'}}(\mathbb{R})} \leq C||u||_{H^{s,p}_{k,k'}(\mathbb{R})}.$$

□

In the last of this section we assume that the Jacobi-Cherednik operator is the unidimensionnel Cherednik operators with the root systems $R = \{-2\alpha, -\alpha, \alpha, 2\alpha\}$ with α the positive root (cf. [4, 5]).

Now, our purpose is obtain the analogue of the Caldérons theorem for the Opdam-Cherednik transform. For this, we need the following proposition.

Proposition 20. *Let* $0 \leq m \leq s/2,\ m \in \mathbb{N},\ 1 \leq p < \infty$. *Then,* $\triangle^m_{k,k'}\mathcal{J}^s_{k,k'}$ *is a continuous linear mapping of* $L^p_{A_{k,k'},e}(\mathbb{R}) := \Big\{f \in L^p_{A_{k,k'}}(\mathbb{R}) : \quad f \text{ is even}\Big\}$ *into itself.*

Proof. Applying the multiplier theorem given in [5] with

$$m(\lambda) = (-1)^m\lambda^{2m}(\lambda^2+1)^{-s/2},$$

the desired result is established. □

We denote by

$$\mathcal{W}^{n,p}_{k,k',e}(\mathbb{R}) := \Big\{f \in \mathcal{W}^{n,p}_{k,k'}(\mathbb{R}) : \quad f \text{ is even}\Big\}$$

and

$$H^{2n,p}_{k,k',e}(\mathbb{R}) := \Big\{f \in H^{2n,p}_{k,k'}(\mathbb{R}) : \quad f \text{ is even}\Big\}.$$

Now, we are in conditions to demonstrate the Caldérón's theorem, that is exposed as follows.

Theorem 1. *Let* $n \in \mathbb{N}$ *and* $1 < p < \infty$. *Then,* $f \in \mathcal{W}^{n,p}_{k,k',e}(\mathbb{R})$ *if and only if* $f \in H^{2n,p}_{k,k',e}(\mathbb{R})$.

Proof. Let $f \in H^{2n,p}_{k,k',e}(\mathbb{R})$, then by definition, $f = \mathcal{J}^{2n}_{k,k'}(g),\ g \in L^p_{A_{k,k'},e}(\mathbb{R})$.
Moreover, if $m \leq n,\ m \in \mathbb{N}$, by Proposition 20, $\triangle^m_{k,k'}f = \triangle^m_{k,k'}\mathcal{J}^{2n}_{k,k'}(g) \in L^p_{A_{k,k'},e}(\mathbb{R})$ and

$$\begin{aligned}\|\triangle^m_{k,k'}f\|_{L^p_{A_{k,k'}}(\mathbb{R})} &= \|\triangle^m_{k,k'}\mathcal{J}^{2n}_{k,k'}(g)\|_{L^p_{A_{k,k'}}(\mathbb{R})} \\ &\leq C\|g\|_{L^p_{A_{k,k'}}(\mathbb{R})} = C\|\mathcal{J}^{-2n}_{k,k'}(f)\|_{L^p_{A_{k,k'}}(\mathbb{R})} = C\|f\|_{H^{2n,p}_{k,k'}(\mathbb{R})}.\end{aligned}$$

Hence,

$$\|f\|_{\mathcal{W}^{n,p}_{k,k'}(\mathbb{R})} := \sum_{0\leq m\leq n}\|\triangle^m_{k,k'}f\|_{L^p_{A_{k,k'}}(\mathbb{R})} \leq C\|f\|_{H^{2n,p}_{k,k'}(\mathbb{R})}$$

and therefore $f \in \mathcal{W}^{n,p}_{k,k',e}(\mathbb{R})$.

Conversely, we consider $f \in \mathcal{W}^{n,p}_{k,k',e}(\mathbb{R})$. Then $\triangle^m_{k,k'}f \in L^p_{A_{k,k'}}(\mathbb{R})$, for all $m \in \mathbb{N}$, $0 \leq m \leq n$. By definition we have $\mathcal{J}^{-2n}_{k,k'}(f) = (1-\triangle_{k,k'})^n f$ and then taking norms we obtain

$$\|f\|_{H^{2n,p}_{k,k'}(\mathbb{R})} = \|\mathcal{J}^{-2n}_{k,k'}(f)\|_{L^p_{A_{k,k'}}(\mathbb{R})} = \|(1-\triangle_{k,k'})^n f\|_{L^p_{A_{k,k'}}(\mathbb{R})} \leq C\sum_{0\leq m\leq n}\|\triangle^m_{k,k'}f\|_{L^p_{A_{k,k'}}(\mathbb{R})} = C\|f\|_{\mathcal{W}^{n,p}_{k,k'}(\mathbb{R})}.$$

□

4. $B^{s,k,k'}_{p,q}$, $F^{s,k,k'}_{p,q}$ SPACES AND BASIC PROPERTIES

Let ψ be a non-negative function in $D(\mathbb{R})$ even, satisfying $\psi(\xi) \equiv 1$ for $|\xi| \leq \frac{1}{2}$ and $\psi(\xi) \equiv 0$ for $|\xi| \geq 1$. We define a function φ on $\mathbb{R}$ by

$$\varphi(\xi) = \psi(\frac{\xi}{2}) - \psi(\xi).$$

Definition 8. *For $j = 0, 1, 2, \cdots$, the operators S_j and Δ_j on $\mathcal{S}'_{k,k'}(\mathbb{R})$ are defined by*

$$\mathcal{F}(S_j f) = \psi(\frac{\xi}{2^j})\mathcal{F}(f), \quad j = 0, 1, 2, \cdots$$
$$\mathcal{F}(\Delta_j f) = \varphi(\frac{\xi}{2^j})\mathcal{F}(f), \quad j = 1, 2, \cdots$$

and put $\Delta_0 = S_0$. We call $\Delta_j f$ the j-th dyadic block of the generalized Littlewood-Paley decomposition of f.

In this section we define analogues of the Besov and Triebel-Lizorkin spaces associated with the Jacobi-Cherednik operator on $\mathbb{R}$ and obtain their basic properties.

4.1. **Definitions.** From now, we make the convention that for all non-negative sequence $\{a_q\}_{q\in\mathbb{Z}}$, the notation $(\sum_q a_q^r)^{\frac{1}{r}}$ stands for $\sup_q a_q$ in the case $r = \infty$. Let $s \in \mathbb{R}$ and $1 \leq p \leq \infty$. For a sequence $\{u_j\}$ of functions on $\mathbb{R}$, we define

$$\|\{u_j\}\|_{l_q^s(L^p_{A_{k,k'}}(\mathbb{R}))} = \|u_0\|_{L^p_{A_{k,k'}}(\mathbb{R})} + (\sum_{j>0}(2^{js}\|u_j\|_{L^p_{A_{k,k'}}(\mathbb{R})})^q)^{\frac{1}{q}},$$
$$\|\{u_j\}\|_{L^p_{A_{k,k'}}(\mathbb{R})(l_q^s)} = \|u_0\|_{L^p_{A_{k,k'}}(\mathbb{R})} + \|\sum_{j>0}(2^{js}|u_j(x)|)^q)^{\frac{1}{q}}\|_{L^p_{A_{k,k'}}(\mathbb{R})}.$$

Definition 9. *For $s \in \mathbb{R}$ and $1 \leq p, q \leq \infty$, the inhomogeneous generalized Besov space $B^{s,k,k'}_{p,q}(\mathbb{R})$ is defined by*

$$B^{s,k,k'}_{p,q}(\mathbb{R}) := \Big\{f \in \mathcal{S}'_{k,k'}(\mathbb{R}) : \|f\|_{B^{s,k,k'}_{p,q}(\mathbb{R})} = \|\{\Delta_j f\}\|_{l_q^s(L^p_{A_{k,k'}}(\mathbb{R}))} < \infty\Big\}.$$

Definition 10. *Let $s \in \mathbb{R}$ and $1 \leq p, q \leq \infty$, the inhomogeneous generalized Triebel-Lizorkin space $F^{s,k,k'}_{p,q}(\mathbb{R})$ is defined by*

$$F^{s,k,k'}_{p,q}(\mathbb{R}) := \Big\{f \in \mathcal{S}'_{k,k'}(\mathbb{R}) : \|f\|_{F^{s,k,k'}_{p,q}(\mathbb{R})} = \|\{\Delta_j f\}\|_{L^p_{A_{k,k'}}(\mathbb{R})(l_q^s)} < \infty\Big\}.$$

As in [16], §6, we can obtain.

Proposition 21. *Let $s \in \mathbb{R}$ and $1 \leq p, q < \infty$. Then $D(\mathbb{R})$ is dense in $B^{s,k,k'}_{p,q}(\mathbb{R})$ and $F^{s,k,k'}_{p,q}(\mathbb{R})$.*

4.2. **Embeddings.**

Theorem 2. *(1) If $s_1 < s_2$ and $1 \leq p, q \leq \infty$, then*

$$B^{s_2,k,k'}_{p,q}(\mathbb{R}) \hookrightarrow B^{s_1,k,k'}_{p,q}(\mathbb{R}),$$
$$F^{s_2,k,k'}_{p,q}(\mathbb{R}) \hookrightarrow F^{s_1,k,k'}_{p,q}(\mathbb{R}).$$

(2) If $s \in \mathbb{R}, 1 \leq p \leq \infty$ and $1 \leq q_1 < q_2 \leq \infty$, then

$$B^{s,k,k'}_{p,q_1}(\mathbb{R}) \hookrightarrow B^{s,k,k'}_{p,q_2}(\mathbb{R}),$$
$$F^{s,k,k'}_{p,q_1}(\mathbb{R}) \hookrightarrow F^{s,k,k'}_{p,q_2}(\mathbb{R}).$$

(3) For $s \in \mathbb{R}$ and $1 \leq p, q \leq \infty$, let $r = \min\{p, q\}, t = \max\{p, q\}$. Then

$$B^{s,k,k'}_{p,r}(\mathbb{R}) \hookrightarrow F^{s,k,k'}_{p,q}(\mathbb{R}) \hookrightarrow B^{s,k,k'}_{p,t}(\mathbb{R}). \tag{4.35}$$

Proof. The monotone character of l_q-spaces and Minkowski's inequality yield (1) and (2).

On the follow we want to prove (3). We must prove that

$$B^{s,k,k'}_{p,p}(\mathbb{R}) \subset F^{s,k,k'}_{p,q}(\mathbb{R}) \subset B^{s,k,k'}_{p,q}(\mathbb{R}), \tag{4.36}$$

if $p \leq q$, and

$$B^{s,k,k'}_{p,q}(\mathbb{R}) \subset F^{s,k,k'}_{p,q}(\mathbb{R}) \subset B^{s,k,k'}_{p,p}(\mathbb{R}), \tag{4.37}$$

if $q \leq p$.

To prove the previous embedding we will use the monotony of the l^s_q spaces:

$$l^s_q := \Big\{\xi :\ \xi = (\xi_j)_{j=0}^{\infty},\ \xi_j \text{ complex},\ ||\xi_j||_{l^s_q} := \Big(\sum_{j=0}^{\infty} 2^{sjq}|\xi_j|^q\Big)^{\frac{1}{q}} < \infty\Big\}, \quad 1 \leq q < \infty$$

and

$$l^s_\infty := \Big\{\xi :\ \xi = (\xi_j)_{j=0}^{\infty},\ \xi_j \text{ complex},\ ||\xi_j||_{l^s_\infty} := \sup_{j\in\mathbb{N}}(2^{sj}|\xi_j|) < \infty\Big\},$$

and the trivial equality $B^{s,k,k'}_{p,p}(\mathbb{R}) = F^{s,k,k'}_{p,p}(\mathbb{R})$.

First, we will prove (4.36). Let $f \in F^{s,k,k'}_{p,q}(\mathbb{R})$

$$\begin{aligned} ||f||_{B^{s,k,k'}_{p,q}(\mathbb{R})} &= ||S_0 f||_{L^p_{A_{k,k'}}(\mathbb{R})} + \Big(\sum_{j=0}^{\infty} 2^{jqs}||\Delta_j f||^q_{L^p_{A_{k,k'}}(\mathbb{R})}\Big)^{\frac{1}{q}} \\ &\leq ||S_0 f||_{L^p_{A_{k,k'}}(\mathbb{R})} + \Big|\Big|\Big(2^{jsp}||\Delta_j f||^p_{L^p_{A_{k,k'}}(\mathbb{R})}\Big)\Big|\Big|^{\frac{1}{p}}_{l^s_{\frac{q}{p}}}. \end{aligned}$$

Now, by using Minkowski's inequality we obtain:

$$\begin{aligned} ||f||_{B^{s,k,k'}_{p,q}(\mathbb{R})} &\leq ||S_0 f||_{L^p_{A_{k,k'}}(\mathbb{R})} + ||\Big(\sum_{j=0}^{\infty} 2^{js}|\Delta_j f|^q\Big)^{\frac{1}{q}}||_{L^p_{A_{k,k'}}(\mathbb{R})} = ||f||_{F^{s,k,k'}_{p,q}(\mathbb{R})} \\ &\leq ||S_0 f||_{L^p_{A_{k,k'}}(\mathbb{R})} + ||\Delta_j f||_{L^p_{A_{k,k'}}(\mathbb{R}, l^s_p)} \\ &= ||S_0 f||_{L^p_{A_{k,k'}}(\mathbb{R})} + ||\Delta_j f||_{l^s_p(L^p_{A_{k,k'}}(\mathbb{R}))} = ||f||_{B^{s,k,k'}_{p,p}(\mathbb{R})}. \end{aligned}$$

Then, to prove (4.37) we have

$$\begin{aligned} ||f||_{B^{s,k,k'}_{p,p}(\mathbb{R})} &= ||S_0 f||_{L^p_{A_{k,k'}}(\mathbb{R})} + ||\Delta_j f||_{L^p_{A_{k,k'}}(\mathbb{R}, l^s_p)} \\ &= ||S_0 f||_{L^p_{A_{k,k'}}(\mathbb{R})} + \Big|\Big|\Big(\sum_{j\geq 0} 2^{js}|\Delta_j f|^q\Big)\Big|\Big|^{\frac{1}{q}}_{L^{\frac{p}{q}}_{A_{k,k'}}(\mathbb{R})} \\ &\leq ||S_0 f||_{L^p_{A_{k,k'}}(\mathbb{R})} + \Big(\sum_{j\geq 0} 2^{jsq}||\,|\Delta_j f|^q||_{L^{\frac{p}{q}}_{A_{k,k'}}(\mathbb{R})}\Big)^{\frac{1}{p}} \\ &= ||S_0 f||_{L^p_{A_{k,k'}}(\mathbb{R})} + ||\Delta_j f||_{l^s_q(L^p_{A_{k,k'}}(\mathbb{R}))} = ||f||_{B^{s,k,k'}_{p,q}(\mathbb{R})}. \end{aligned}$$

□

4.3. **Lifting property.** We recall that for $f \in \mathcal{S}'_{k,k'}(\mathbb{R})$,

$$\mathcal{F}(\triangle_{k,k'}\Delta_n f)(\xi) = 2^{2n}\tilde{\phi}(\frac{\xi}{2^n})\mathcal{F}(f)(\xi), \quad \tilde{\phi}(\xi) = -\xi^2\phi(\xi).$$

Then we can obtain

Proposition 22. *Let $s \in \mathbb{R}$ and $1 \leq q \leq \infty$. The operator $\triangle_{k,k'}$ is a linear continuous operator from $B^{s,k,k'}_{2,q}(\mathbb{R})$ into $B^{s-2,k,k'}_{2,q}(\mathbb{R})$, and from $H^{s,k,k'}_{2,q}(\mathbb{R})$ into $H^{s-2,k,k'}_{2,q}(\mathbb{R})$.*

Proposition 23. *Let $s, t \in \mathbb{R}$ and $1 \leq q \leq \infty$. The operator $\mathcal{J}^t_{k,k'}$ is a linear continuous injective operator from $B^{s,k,k'}_{2,q}(\mathbb{R})$ onto $B^{s-t,k,k'}_{2,q}(\mathbb{R})$, and from $H^{s,k,k'}_{2,q}(\mathbb{R})$ onto $H^{s-t,k}_{2,q}(\mathbb{R})$.*

Proof. Since $\mathcal{F}$ satisfies (2.25), we can apply the same arguments used in the proof of Theorem 5.1.1 in [15]. □

Corollary 2. *(1) If $s \in \mathbb{R}$, then*

$$(H^{s,2}_{k,k'}(\mathbb{R}))' = H^{-s,2}_{k,k'}(\mathbb{R}).$$

(2) If $s \in \mathbb{R}$, and $1 \le q < \infty$, then

$$(B^{s,k,k'}_{2,q}(\mathbb{R}))' = B^{-s,k,k'}_{2,q'}(\mathbb{R}).$$

Proof. The first formula follows from Proposition 23 and the fact $\left(L^2_{A_{k,k'}}(\mathbb{R})\right)' = L^2_{A_{k,k'}}(\mathbb{R})$. The second formula is implied by definition of inhomogeneous generalized Besov space and the duality Theorem. □

4.4. **Interpolation.**

Theorem 3. *(1) Let $s_0 \neq s_1$, $0 < \theta < 1$, $s = (1-\theta)s_0 + \theta s_1$, $1 \le p, q, q_0, q_1 \le \infty$. Then*

$$(B^{s_0,k,k'}_{p,q_0}(\mathbb{R}), B^{s_1,k,k'}_{p,q_1}(\mathbb{R}))_{\theta,q} = B^{s,k,k'}_{p,q}(\mathbb{R}).$$

(2) Let $s \in \mathbb{R}$, $1 \le p_0, p_1 \le \infty$, $p_0 \neq p_1$, $0 < \theta < 1$, $\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$, then

$$(F^{s,k,k'}_{p_0,2}(\mathbb{R}), F^{s,k,k'}_{p_1,2}(\mathbb{R}))_{\theta,p} = F^{s,k,k'}_{p,2}(\mathbb{R}).$$

(3) Let $s_0, s_1 \in \mathbb{R}$, $s_0 \neq s_1$, $1 \le p_0, p_1 \le \infty$, $p_0 \neq p_1$, $0 < \theta < 1$, $s = (1-\theta)s_0 + \theta s_1$, $\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$, then

$$(F^{s_0,k,k'}_{p_0,2}(\mathbb{R}), F^{s_1,k,k'}_{p_1,2}(\mathbb{R}))_{\theta,p} = B^{s,k,k'}_{p,p}(\mathbb{R}).$$

(4) Let $s_0, s_1 \in \mathbb{R}$, $s_0 \neq s_1$, $0 < \theta < 1$, $s = (1-\theta)s_0 + \theta s_1$, $1 \le p, q, q_0, q_1 \le \infty$. Then

$$(F^{s_0,k,k'}_{p,q_0}(\mathbb{R}), F^{s_1,k,k'}_{p,q_1}(\mathbb{R}))_{\theta,q} = B^{s,k,k'}_{p,q}(\mathbb{R}). \tag{4.38}$$

Proof. (1), (2), (3) follows from the arguments used in Theorem 8.1.3 and Theorem 8.3.3 in [15]. (4) follows from (1) and (4.35). □

As a consequence of real and complex interpolations, we can deduce multiplicative inequalities, which will be needed in the theory of differential-difference operators.

Theorem 4. *(1) If u belongs to $B^{s,k,k'}_{p,q}(\mathbb{R}) \cap B^{t,k,k'}_{p,q}(\mathbb{R})$, then u belongs to $B^{\theta s+(1-\theta)t,k,k'}_{p,q}(\mathbb{R})$ for all $\theta \in [0,1]$ and*

$$\|u\|_{B^{\theta s+(1-\theta)t,k,k'}_{p,q}(\mathbb{R})} \le \|u\|^{\theta}_{B^{s,k,k'}_{p,q}(\mathbb{R})} \|u\|^{1-\theta}_{B^{t,k,k'}_{p,q}(\mathbb{R})}.$$

(2) If u belongs to $B^{s,k,k'}_{p,\infty}(\mathbb{R}) \cap B^{t,k,k'}_{p,\infty}(\mathbb{R})$ and $s < t$, then u belongs to $B^{\theta s+(1-\theta)t,k,k'}_{p,1}(\mathbb{R})$ for all $\theta \in (0,1)$ and there exists a positive constant $C(t,s)$ such that

$$\|u\|_{B^{\theta s+(1-\theta)t,k,k'}_{p,1}(\mathbb{R})} \le C(t,s) \|u\|^{\theta}_{B^{s,k,k'}_{p,\infty}(\mathbb{R})} \|u\|^{1-\theta}_{B^{t,k,k'}_{p,\infty}(\mathbb{R})}.$$

(3) If u belongs to $B^{s,k,k'}_{p,\infty}(\mathbb{R}) \cap B^{s+\varepsilon,k,k'}_{p,\infty}(\mathbb{R})$ and $\varepsilon > 0$, then u belongs to $B^{s,k,k'}_{p,1}(\mathbb{R})$ and there exists a positive constant C such that

$$\|u\|_{B^{s,k,k'}_{p,1}(\mathbb{R})} \le \frac{C}{\varepsilon} \|u\|_{B^{s,k,k'}_{p,\infty}(\mathbb{R})} \log_2 \Big(e + \frac{\|u\|_{B^{s+\varepsilon,k,k'}_{p,\infty}(\mathbb{R})}}{\|u\|_{B^{s,k,k'}_{p,\infty}(\mathbb{R})}} \Big).$$

Proof. (1) is obvious from Hölder's inequality. As for (2), we write $\|u\|_{B^{\theta s+(1-\theta)t,k,k'}_{p,1}(\mathbb{R})}$ as

$$\sum_{j \le N} 2^{j(\theta s+(1-\theta)t)} \|\Delta_j u\|_{L^p_{A_{k,k'}}(\mathbb{R})} + \sum_{j > N} 2^{j(\theta s+(1-\theta)t)} \|\Delta_j u\|_{L^p_{A_{k,k'}}(\mathbb{R})},$$

where N is chosen here after. By the definition of the generalized Besov norms, we see that

$$2^{j(\theta s+(1-\theta)t)}\|\Delta_j u\|_{L^p_{A_{k,k'}}(\mathbb{R})} \leq 2^{j(1-\theta)(t-s)}\|u\|_{B^{s,k,k'}_{p,\infty}(\mathbb{R})},$$
$$2^{j(\theta s+(1-\theta)t)}\|\Delta_j u\|_{L^p_{A_{k,k'}}(\mathbb{R})} \leq 2^{-j\theta(t-s)}\|u\|_{B^{t,k,k'}_{p,\infty}(\mathbb{R})}$$

and thus, $\|u\|_{B^{\theta s+(1-\theta)t,k,k'}_{p,1}(\mathbb{R})}$ is dominated by

$$\|u\|_{B^{s,k,k'}_{p,\infty}(\mathbb{R})}\sum_{j\leq N}2^{j(1-\theta)(t-s)}+\|u\|_{B^{t,k,k'}_{p,\infty}(\mathbb{R})}\sum_{j>N}2^{-j\theta(t-s)}$$
$$\leq C\|u\|_{B^{s,k,k'}_{p,\infty}(\mathbb{R})}\frac{2^{(N+1)(1-\theta)(t-s)}}{2^{(1-\theta)(t-s)}-1}+\|u\|_{B^{t,k,k'}_{p,\infty}(\mathbb{R})}\frac{2^{-N\theta(t-s)}}{1-2^{-\theta(t-s)}}.$$

Hence, in order to complete the proof of (2), it suffices to choose N such that

$$\frac{\|u\|_{B^{t,k,k'}_{p,\infty}(\mathbb{R})}}{\|u\|_{B^{s,k,k'}_{p,\infty}(\mathbb{R})}} \leq 2^{N(t-s)} < 2\frac{\|u\|_{B^{t,k,k'}_{p,\infty}(\mathbb{R})}}{\|u\|_{B^{s,k,k'}_{p,\infty}(\mathbb{R})}}.$$

As for (3), it is easy to see that $\|u\|_{B^{s,k,k'}_{p,1}(\mathbb{R})}$ is dominated as

$$\sum_{j\leq N-1}2^{js}\|\Delta_j u\|_{L^p_{A_{k,k'}}(\mathbb{R})}+\sum_{j\geq N}2^{js}\|\Delta_j u\|_{L^p_{A_{k,k'}}(\mathbb{R})}$$
$$\leq (N+1)\|u\|_{B^{s,k,k'}_{p,\infty}(\mathbb{R})}+\frac{2^{-(N-1)\varepsilon}}{2^{\varepsilon}-1}\|u\|_{B^{s+\varepsilon,k,k'}_{p,\infty}(\mathbb{R})}.$$

Hence, letting

$$N=1+\Big[\frac{1}{\varepsilon}\log_2\frac{\|u\|_{B^{s+\varepsilon,k,k'}_{p,\infty}(\mathbb{R})}}{\|u\|_{B^{s,k,k'}_{p,\infty}(\mathbb{R})}}\Big],$$

we can obtain the desired estimate. □

In the last of this section we assume that the Jacobi-Cherednik operator is the unidimensionnel Cherednik operators with the root systems $R=\{-2\alpha,-\alpha,\alpha,2\alpha\}$ with α the positive root (cf. [4, 5]).

To obtain a new characterization of the generalized Jacobi potentials spaces, we need to state the following lemma.

Lemma 6. *([5]). Let $s\in\mathbb{R}$ and let $\{r_j\}_{j\in\mathbb{N}}$ be the Rademacher functions (cf. [14]). Then, for every p with $1<p<\infty$ and for all $0<t<1$, there exists a positive constant C such that*

$$||\mathcal{F}^{-1}(n_i\mathcal{F}(g)))||_{L^p_{A_{k,k'},e}(\mathbb{R})} \leq C||g||_{L^p_{A_{k,k'},e}(\mathbb{R})},$$

where

$$n_1(\xi):=\sum_{j=0}^{\infty}2^{js}r_j(t)(1+|\xi|^2)^{-\frac{s}{2}}\varphi_j(\xi)$$

and

$$n_2(\xi):=\Big(\sum_{j=0}^{\infty}\varphi_j^2(\xi)\Big)^{-1}.$$

Theorem 5. *Let $1<p<\infty$ and $s\in\mathbb{R}$, we have*

$$F^{s,k,k'}_{p,2,e}(\mathbb{R})=H^{s,p}_{k,k',e}(\mathbb{R}),$$

where

$$F^{s,k,k'}_{p,2,e}(\mathbb{R}):=\Big\{f\in F^{s,k,k'}_{p,2}(\mathbb{R}): \quad f \text{ is even}\Big\}$$

and

$$H^{s,p}_{k,k',e}(\mathbb{R}) := \Big\{ f \in H^{s,p}_{k,k'}(\mathbb{R}) : \quad f \text{ is even} \Big\}.$$

Proof. We must see that exists a positive constant C such that

$$\frac{1}{C}||f||_{H^{s,p}_{k,k',e}(\mathbb{R})} \leq ||\Big(\sum_{j=0}^{\infty} 2^{2sj}|\Delta_j f|^2\Big)^{\frac{1}{2}}||_{L^p_{A_{k,k'},e}(\mathbb{R})} \leq C||f||_{H^{s,p}_{k,k',e}(\mathbb{R})}. \tag{4.39}$$

In a similar way to Proposition 17 we have that $\mathcal{S}_{k,k'}(\mathbb{R})$ is dense in $F^{s,k,k'}_{p,2,e}(\mathbb{R})$ for $s \in \mathbb{R}$, $1 < p < \infty$. Then, it is not difficult to obtain that the functions $f \in L^p_{A_{k,k'},e}(\mathbb{R})$ with $\text{supp}(\mathcal{F}(f))$ compact are dense both in $H^{s,p}_{k,k',e}(\mathbb{R})$ and in $F^{s,k,k'}_{p,2,e}(\mathbb{R})$, for $1 < p < \infty$. Therefore it is enough to prove (4.39) for a functions of this type. Moreover, we observe that in this case the infinite sum in (4.39) is actually finite.

We first prove the right hand estimate. Let $f \in H^{s,p}_{k,k',e}(\mathbb{R})$, then there exists $g \in L^p_{A_{k,k'},e}(\mathbb{R})$ such that $f = \mathcal{J}^s_{k,k'}(g)$. From Lemma 6, with n_1, we have for all $t \in (0,1)$

$$||\sum_{j=0}^{\infty} r_j(t)2^{sj}\Delta_j f||_{L^p_{A_{k,k'},e}(\mathbb{R})} \leq C||f||_{H^{s,p}_{k,k',e}(\mathbb{R})}.$$

Thus

$$\int_0^1 ||\sum_{j=0}^{\infty} r_j(t)2^{sj}\Delta_j f||_{L^p_{A_{k,k'},e}(\mathbb{R})} dt \leq C||f||_{H^{s,p}_{k,k',e}(\mathbb{R})}. \tag{4.40}$$

Using the Minkowski's inequality and the right hand inequality of ([14], Chapter V, Theorem 8.4, p. 213) with $p = 1$ we obtain

$$\begin{aligned} \Big|\Big|\Big(\sum_{j=0}^{\infty} 2^{2sj}|\Delta_j f|^2\Big)^{\frac{1}{2}}\Big|\Big|_{L^p_{A_{k,k'},e}(\mathbb{R})} &\leq C\Big|\Big|\int_0^1 \Big|\sum_{j\in\mathbb{N}} r_j(t)2^{sj}\Delta_j f\Big| dt\Big|\Big|_{L^p_{A_{k,k'},e}(\mathbb{R})} \\ &\leq C\int_0^1 \Big|\Big|\Big(\sum_{j\in\mathbb{N}} r_j(t)2^{sj}\Delta_j f\Big)\Big|\Big|_{L^p_{A_{k,k'},e}(\mathbb{R})} dt. \end{aligned}$$

Now, by (4.40) we have

$$\Big|\Big|\Big(\sum_{j=0}^{\infty} 2^{2sj}|\Delta_j f|^2\Big)^{\frac{1}{2}}\Big|\Big|_{L^p_{A_{k,k'},e}(\mathbb{R})} \leq C||f||_{H^{s,p}_{k,k',e}(\mathbb{R})}.$$

Therefore we achieve that $f \in F^{s,k,k'}_{p,2,e}(\mathbb{R})$.

For the converse inequality we will use duality. Let $f \in F^{s,k,k'}_{p,2,e}(\mathbb{R})$ and from Lemma 6 with $n_2(x) = (\sum_{j=0}^{\infty}\varphi_j^2)^{-1}$ gives

$$||\mathcal{J}^{-s}_{k,k'}(f)||_{L^p_{A_{k,k'},e}(\mathbb{R})} \leq C||h||_{L^p_{A_{k,k'},e}(\mathbb{R})}, \tag{4.41}$$

with

$$h = \mathcal{F}^{-1}\Big((1+|\xi|^2)^{\frac{s}{2}}\sum_{j=0}^{\infty}\varphi_j^2(\xi)\mathcal{F}(f)(\xi)\Big).$$

Now, consider $u \in L^q_{A_{k,k'},e}(\mathbb{R})$ be a function such that $\text{supp}(\mathcal{F}(u))$ is compact,

$$\int_{\mathbb{R}} |u(x)|^q A_{k,k'}(x)dx = 1, \tag{4.42}$$

$(\frac{1}{p} + \frac{1}{q} = 1)$ and

$$\int_{\mathbb{R}} u(x)h(x)A_{k,k'}(x)dx \geq \frac{1}{2}||h||_{L^p_{A_{k,k'}}(\mathbb{R})}. \tag{4.43}$$

Let v a even function defined by $\mathcal{F}(v)(\xi) = (1+\xi^2)^{\frac{s}{2}}\mathcal{F}(u)(\xi)$, then from (4.41),(4.43) and (2.16) we obtain

$$\begin{aligned} ||f||_{H^{s,p}_{k,k'}(\mathbb{R})} &= ||\mathcal{J}^{-s}_{k,k'}(f)||_{L^p_{A_{k,k'}}(\mathbb{R})} \\ &\leq C\int_{\mathbb{R}} u(x)h(x)A_{k,k'}(x)dx \leq C\int_{\mathbb{R}} |\mathcal{F}(u)(\xi)|\,|\mathcal{F}(h)(\xi)|d\nu_{k,k'}(\xi) \\ &\leq C\int_{\mathbb{R}} |\mathcal{F}(u)(\xi)|\sum_{j=0}^{\infty}(1+|\xi|^2)^{\frac{s}{2}}\varphi_j^2(\xi)|\mathcal{F}(f)(\xi)|d\nu_{k,k'}(\xi) \\ &\leq C\int_{\mathbb{R}}\sum_{j=0}^{\infty}\Big(|2^{js}\mathcal{F}(f)(\xi)\varphi_j(\xi)|\Big)\Big(2^{-js}|\mathcal{F}(v)(\xi)\varphi_j(\xi)|\Big)d\nu_{k,k'}(\xi). \end{aligned}$$

Therefore, by Plancherel formula and the Cauchy and Hölder inequalities we have

$$||f||_{H^{s,p}_{k,k'}(\mathbb{R})} \leq C||(\sum_{j=0}^{\infty}(2^{2js}|\Delta_j f|^2)^{\frac{1}{2}}||_{L^p_{A_{k,k'}}(\mathbb{R})}||(\sum_{j=0}^{\infty}(2^{-2js}|\Delta_j v|^2)^{\frac{1}{2}}||_{L^q_{A_{k,k'}}(\mathbb{R})}. \tag{4.44}$$

Then by the right hand inequality of (4.39) we achieve

$$||(\sum_{j=0}^{\infty}(2^{-2js}|\Delta_j v|^2)^{\frac{1}{2}}||_{L^q_{A_{k,k'}}(\mathbb{R})} \leq C||v||_{H^{-s,q}_{k,k'}(\mathbb{R})} = C\int_{\mathbb{R}}|u(x)|^q A_{k,k'}(x)dx. \tag{4.45}$$

Hence, combining (4.44), (4.45) and (4.42) the proof is finished. □

As a consequence of Theorem 5 and Theorem 1, we obtain the following result.

Corollary 3. *Let $s \in \mathbb{N}$ and $1 < p < \infty$ then $F^{2s,k,k'}_{p,2,e}(\mathbb{R}) = \mathcal{W}^{s,p}_{k,k',e}(\mathbb{R})$.*

5. Applications

In this Section we give some applications of the generalized potential and generalized Sobolev spaces.

5.1. Hypoellipticity of Jacobi-Cherednik operator.

Theorem 6. *Let $P(-\triangle_{k,k'}) = \sum_{j=0}^{n} a_j(-\triangle_{k,k'})^j,\ a_n \neq 0,$ a differential-difference operator with constant coefficients a_j and symbol $P(\lambda^2) = \sum_{j=0}^{n} a_j\lambda^{2j} \neq 0,\ \lambda \in \mathbb{R}$. If*

$$u \in L^2_{A_{k,k'}}(\mathbb{R}),\ P(-\triangle_{k,k'})u = f, \text{ and } f \in L^2_{A_{k,k'}}(\mathbb{R}), \quad \text{then} \quad u \in H^{2n,2}_{k,k'}(\mathbb{R}).$$

Proof. It is easy to see that there exists $R > 0$ and a positive constant C such that

$$|P(\lambda^2)| \geq C|\lambda|^{2n}, \quad |\lambda| \geq R. \tag{5.46}$$

We have

$$\|u\|^2_{H^{2n,2}_{k,k'}(\mathbb{R})} = \int_{\mathbb{R}_+}(\lambda^2+1)^{2n}\Big(|\mathcal{F}(u)(\lambda)|^2 + |\mathcal{F}(\check{u})(\lambda)|^2\Big)d\nu_{k,k'}(\lambda).$$

If we consider $R \geq 1$, we have

$$\begin{aligned} \|u\|^2_{H^{2n,2}_{k,k'}(\mathbb{R})} &= \int_0^R(\lambda^2+1)^{2n}(|\mathcal{F}(u)(\lambda)|^2 + |\mathcal{F}(\check{u})(\lambda)|^2)d\nu_{k,k'}(\lambda) \\ &+ \int_{\lambda\geq R}(\lambda^2+1)^{2n}(|\mathcal{F}(u)(\lambda)|^2 + |\mathcal{F}(\check{u})(\lambda)|^2)d\nu_{k,k'}(\lambda). \end{aligned}$$

Now, if $0 \leq \lambda \leq R$, we use that $(\lambda^2+1)^{2n} \leq (1+R^2)^{2n}$ and for $\lambda \geq R$, $(\lambda^2+1)^{2n} \leq C|\lambda|^{4n}$, obtaining

$$\begin{aligned}\|u\|^2_{H^{2n,2}_{k,k'}(\mathbb{R})} &\leq C(1+R^2)^{2n}\int_0^R (|\mathcal{F}(u)(\lambda)|^2 + |\mathcal{F}(\check{u})(\lambda)|^2)d\nu_{k,k'}(\lambda)\\ &+ C\int_{\lambda\geq R}|\lambda|^{4n}(|\mathcal{F}(u)(\lambda)|^2 + |\mathcal{F}(\check{u})(\lambda)|^2)d\nu_{k,k'}(\lambda).\end{aligned}$$

Using again Proposition 4, the relations (2.13) and (5.46) we have

$$\begin{aligned}\|u\|^2_{H^{2n,2}_{k,k'}(\mathbb{R})} &\leq C\left(\int_{\mathbb{R}}\big|u(x)\big|^2A_{k,k'}(x)dx + \int_{\mathbb{R}}|\lambda|^{4n}(|\mathcal{F}(u)(\lambda)|^2 + |\mathcal{F}(\check{u})(\lambda)|^2)d\nu_{k,k'}(\lambda)\right)\\ &\leq C\left(\|u\|^2_{L^2_{A_{k,k'}}(\mathbb{R})} + \int_{\mathbb{R}}|P(|\lambda|^2)|^2(|\mathcal{F}(u)(\lambda)|^2 + |\mathcal{F}(\check{u})(\lambda)|^2)d\nu_{k,k'}(\lambda)\right)\\ &\leq C\left(\|u\|^2_{L^2_{A_{k,k'}}(\mathbb{R})} + \int_{\mathbb{R}}\left[\left|\mathcal{F}\Big(P(-\triangle_{k,k'})u\Big)(\lambda)\right|^2 + \left|\mathcal{F}\Big(P(-\triangle_{k,k'})\check{u}\Big)(\lambda)\right|^2\right])d\nu_{k,k'}(\lambda)\right).\end{aligned}$$

Again, by using Plancharel formula we obtain

$$\|u\|^2_{H^{2n,2}_{k,k'}(\mathbb{R})} \leq C\left(\|u\|^2_{L^2_{A_{k,k'}}(\mathbb{R})} + \|P(-\triangle_{k,k'})u\|^2_{L^2_{A_{k,k'}}(\mathbb{R})}\right).$$

Now, we complete the proof using Proposition 17 that is, that $\mathcal{S}_{k,k'}(\mathbb{R})$ is dense in $H^{2n,2}_{k,k'}(\mathbb{R})$. □

Proposition 24. *Let $f \in B^{s,k,k'}_{2,q}(\mathbb{R})$ then exists $g \in S'_{k,k'}(\mathbb{R})$ such that*

$$(I - \triangle_{k,k'})^m g = f$$

where I is the identity operator and $m \in \mathbb{N}\backslash\{0\}$.

Proof. Let us consider $f \in B^{s,k,k'}_{2,q}(\mathbb{R})$. We want to obtain $g \in S'_{k,k'}(\mathbb{R})$ such that

$$(I - \triangle^m_{k,k'})g = f.$$

Applying the Opdam-Cherednik transform we have

$$(\lambda^2+1)^m\mathcal{F}(g) = \mathcal{F}(f).$$

Now, using the inverse transform we get

$$\mathcal{F}^{-1}\Big((\lambda^2+1)^{-m}\mathcal{F}(g)\Big) = \mathcal{J}^{2m}_{k,k'}(f).$$

On the other hand, by Proposition 23, we obtain that $g \in B^{s+2m,k,k'}_{2,q}(\mathbb{R})$. Thus the proof is complete. □

5.2. **Generalized wave equation.**

Lemma 7. i*) For all $p, q \in [1,\infty]$, $\frac{1}{r} = \frac{1}{p} + \frac{1}{q} - 1 \geq 0$, $s, s' \in \mathbb{R}$, $f \in H^{s,p}_{k,k'}(\mathbb{R})$ and $g \in H^{s',q}_{k,k'}(\mathbb{R})$ then*

$$f *_{k,k'} g \in H^{s+s',r}_{k,k'}(\mathbb{R}),$$

and we have

$$\|f *_{k,k'} g\|_{H^{s+s',r}_{k,k'}(\mathbb{R})} \leq C_{k,k'}\|f\|_{H^{s,p}_{k,k'}(\mathbb{R})}\|g\|_{H^{s',q}_{k,k'}(\mathbb{R})},$$

with $C_{k,k'}$ the constant given by the relation (2.19).

ii*) For all $1 \leq p < q \leq 2$, $s, s' \in \mathbb{R}$, $f \in H^{s,p}_{k,k'}(\mathbb{R})$ and $g \in H^{s',q}_{k,k'}(\mathbb{R})$ then*

$$f *_{k,k'} g \in H^{s+s',q}_{k,k'}(\mathbb{R}),$$

and we have

$$\|f *_{k,k'} g\|_{H^{s+s',q}_{k,k'}(\mathbb{R})} \leq C\|f\|_{H^{s,p}_{k,k'}(\mathbb{R})}\|g\|_{H^{s',q}_{k,k'}(\mathbb{R})},$$

with C is a positive constant.

iii*) Let $2 < p,q < \infty$ such that $\frac{q}{2} \leq p < q$, $s,s' \in \mathbb{R}$, $f \in H^{s,p}_{k,k'}(\mathbb{R})$ and $g \in H^{s',q'}_{k,k'}(\mathbb{R})$ then*

$$f *_{k,k'} g \in H^{s+s',q}_{k,k'}(\mathbb{R}),$$

and we have

$$\|f *_{k,k'} g\|_{H^{s+s',q}_{k,k'}(\mathbb{R})} \leq C\|f\|_{H^{s,p}_{k,k'}(\mathbb{R})}\|g\|_{H^{s',q'}_{k,k'}(\mathbb{R})},$$

with C is a positive constant, and q' the conjugate composent of q.

iv*) Let $1 < p < 2$ and $p < q \leq \frac{p}{2-p}$, $s,s' \in \mathbb{R}$, $f \in H^{s,p}_{k,k'}(\mathbb{R})$ and $g \in H^{s',p}_{k,k'}(\mathbb{R})$ then*

$$f *_{k,k'} g \in H^{s+s',q}_{k,k'}(\mathbb{R}),$$

and we have

$$\|f *_{k,k'} g\|_{H^{s+s',q}_{k,k'}(\mathbb{R})} \leq C\|f\|_{H^{s,p}_{k,k'}(\mathbb{R})}\|g\|_{H^{s',p}_{k,k'}(\mathbb{R})},$$

with C is a positive constant.

Proof. The results are given by Proposition 6 and the definition of the generalized potential spaces. □

This is the generalized wave equation where the unknown is a function u (with real values) of $(t,x) \in \mathbb{R} \times \mathbb{R}$:

$$(W)\begin{cases} \partial_t^2 u - \triangle_{k,k'} u &= 0 \\ u_{t|=0} &= u_0 \in H^{s,p}_{k,k'}(\mathbb{R}) \\ \partial_t u_{t|=0} &= u_1 \in H^{s',q}_{k,k'}(\mathbb{R}). \end{cases}$$

Corollary 4. *Let us define $\mathcal{C} := \Big\{\xi \in \mathbb{R},\ r \leq |\xi| \leq R\Big\}$ for some positive r and R such that $r < R$. We assume that u_0 and u_1 are two functions such that*

$$supp\ \mathcal{F}(u_j) \subset \mathcal{C}.$$

i*) For $p = q = 2$, $u \in H^{a+s,\infty}_{k,k'}(\mathbb{R}) + H^{b+s',\infty}_{k,k'}(\mathbb{R})$, where $a,b \in \mathbb{R}$. For $a+s = b+s' = c$,*

$$\|u\|_{H^{c,\infty}_{k,k'}(\mathbb{R})} \leq C\Big(\|u_0\|_{H^{s,2}_{k,k'}(\mathbb{R})} + \|u_1\|_{H^{s',2}_{k,k'}(\mathbb{R})}\Big).$$

ii*) For $p \neq 2$ and $q \neq 2$, $u \in H^{a+s,\frac{2p}{2-p}}_{k,k'}(\mathbb{R}) + H^{b+s',\frac{2q}{2-q}}_{k,k'}(\mathbb{R})$, where $a,b \in \mathbb{R}$. For $a+s = s'+b = c$*

$$\|u\|_{H^{c,\frac{2p}{2-p}}_{k,k'}(\mathbb{R})} \leq C\Big(\|u_0\|_{H^{s,p}_{k,k'}(\mathbb{R})} + \|u_1\|_{H^{s',p}_{k,k'}(\mathbb{R})}\Big).$$

iii*) For all $1 \leq p,q < r \leq 2$, $s,s' \in \mathbb{R}$, $u \in H^{a+s,r}_{k,k'}(\mathbb{R}) + H^{b+s',r}_{k,k'}(\mathbb{R})$, where $a,b \in \mathbb{R}$. For $a+s = s'+b = c$*

$$\|u\|_{H^{c,r}_{k,k'}(\mathbb{R})} \leq C\Big(\|u_0\|_{H^{s,p}_{k,k'}(\mathbb{R})} + \|u_1\|_{H^{s',q}_{k,k'}(\mathbb{R})}\Big).$$

iv*) Let $2 < p,q,r < \infty$ such that $\frac{r}{2} \leq p,q < r$, $s,s' \in \mathbb{R}$, $u \in H^{a+s,r}_{k,k'}(\mathbb{R}) + H^{b+s',r}_{k,k'}(\mathbb{R})$, where $a,b \in \mathbb{R}$. For $a+s = s'+b = c$*

$$\|u\|_{H^{c,r}_{k,k'}(\mathbb{R})} \leq C\Big(\|u_0\|_{H^{s,p}_{k,k'}(\mathbb{R})} + \|u_1\|_{H^{s',q}_{k,k'}(\mathbb{R})}\Big).$$

v*) Let $1 < p,q < 2$ and $p < r \leq \frac{p}{2-p}$, $q < r \leq \frac{q}{2-q}$, $s,s' \in \mathbb{R}$, $u \in H^{a+s,r}_{k,k'}(\mathbb{R}) + H^{b+s',r}_{k,k'}(\mathbb{R})$, where $a,b \in \mathbb{R}$. For $a+s = s'+b = c$*

$$\|u\|_{H^{c,r}_{k,k'}(\mathbb{R})} \leq C\Big(\|u_0\|_{H^{s,p}_{k,k'}(\mathbb{R})} + \|u_1\|_{H^{s',q}_{k,k'}(\mathbb{R})}\Big).$$

Proof. The results are immediately from Duhamel expression for the solution and Lemma 7. □

5.3. **Generalized Schrödinger equation.** We consider the following equation where the unknown is a function u (with complex value) of $(t,x)\in\mathbb{R}\times\mathbb{R}$

$$(S)\begin{cases}\partial_t u - i\triangle_{k,k'}u &= 0\\ u_{|t=0} &= g.\end{cases}$$

Theorem 7. *Let g be in ${\mathcal{S}'}_{k,k'}(\mathbb{R})$. There exists a unique solution $u\in\mathcal{E}(\mathbb{R};{\mathcal{S}'}_{k,k'}(\mathbb{R}))$ such that*

$$(S)\begin{cases}\partial_t u - i\triangle_{k,k'}u &= 0, \quad \text{in} \quad D'(\mathbb{R}\times\mathbb{R})\\ u_{|t=0} &= g.\end{cases}$$

Proof. Firstly we prove the existence. For $t\in\mathbb{R}$, we put

$$u_t = \mathcal{F}^{-1}(e^{-it|\xi|^2}\mathcal{F}(g)). \tag{5.47}$$

From (2.28) we have

$$\langle u_t,\varphi\rangle = \langle \mathcal{F}(g), e^{-it|\xi|^2}\mathcal{F}^{-1}(\varphi)\rangle.$$

Thus we deduce that $u_t\in\mathcal{E}(\mathbb{R};{\mathcal{S}'}_{k,k'}(\mathbb{R}))$, and $\mathcal{F}(u_t)\in\mathcal{E}(\mathbb{R};\mathcal{S}'(\mathbb{R}))$.
We recall that u is defined by

$$\langle u,\psi\rangle = \int_{\mathbb{R}}\langle u_t,\psi(t,.)\rangle dt,\ \ \psi\in\mathcal{S}(\mathbb{R},\mathcal{S}_{k,k'}(\mathbb{R})).$$

Then for any ψ in $\mathcal{S}(\mathbb{R},\mathcal{S}_{k,k'}(\mathbb{R}))$, we have from (2.29)

$$\begin{aligned}\langle\partial_t u - i\triangle_{k,k'}u,\psi\rangle &= -\langle u,\partial_t\psi + i\triangle_{k,k'}\psi\rangle = -\int_{\mathbb{R}}\langle u_t,\partial_t\psi(t,.) + i\triangle_{k,k'}\psi(t,.)\rangle dt\\ &= -\int_{\mathbb{R}}\langle\mathcal{F}(u_t),\mathcal{F}^{-1}\Big(\partial_t\psi(t,.) + i\triangle_{k,k'}\psi(t,.)\Big)\rangle dt\\ &= -\int_{\mathbb{R}}\langle e^{-it|.|^2}\mathcal{F}(g),(\partial_t - i|.|^2)\mathcal{F}^{-1}\psi(t,.)\rangle dt.\end{aligned}$$

But

$$\partial_t\Big(e^{-it|\xi|^2}\mathcal{F}^{-1}\psi(t,\xi)\Big) = \Big[(\partial_t - i|\xi|^2)\mathcal{F}^{-1}\psi(t,\xi)\Big]e^{-it|\xi|^2}.$$

Thus

$$\begin{aligned}\langle\partial_t u_t - i\triangle_{k,k'}u,\psi\rangle &= -\int_{\mathbb{R}}\langle\mathcal{F}(g),\partial_t\Big(e^{-it|.|^2}\mathcal{F}^{-1}\psi(t,.)\Big)\rangle dt\\ &= -\int_{\mathbb{R}}\partial_t\langle\mathcal{F}(g),e^{-it|.|^2}\mathcal{F}^{-1}\psi(t,.)\rangle dt = 0.\end{aligned}$$

Thus we have proved that u is solution of (S).

Now we prove the uniqueness, which equivalently proves that $u\equiv 0$ is the solution of problem

$$\begin{cases}\partial_t u - i\triangle_{k,k'}u &= 0, \quad \text{in} \quad \mathcal{E}(\mathbb{R};{\mathcal{S}'}_{k,k'}(\mathbb{R}))\\ u_{|t=0} &= 0.\end{cases}$$

Indeed for all ψ in $\mathcal{S}(\mathbb{R},\mathcal{S}_{k,k'}(\mathbb{R}))$ we have

$$0 = \langle\partial_t u - i\triangle_{k,k'}u,\psi\rangle = -\int_{\mathbb{R}}\langle u_t,(\partial_t + i\triangle_{k,k'})\psi(t,.)\rangle dt.$$

But

$$\frac{d}{dt}\langle u_t,\psi(t,.)\rangle = \langle u_t^{(1)},\psi(t,.)\rangle + \langle u_t,\partial_t\psi(t,.)\rangle,$$

hence

$$0 = -\int_{\mathbb{R}}\frac{d}{dt}\langle u_t,\psi(t,.)\rangle dt + \int_{\mathbb{R}}\Big[\langle u_t^{(1)},\psi(t,.)\rangle - i\langle u_t,\triangle_{k,k'}\psi(t,.)\rangle\Big]dt. \tag{5.48}$$

As $\psi(-\infty,.) = \psi(\infty,.) = 0$, then we obtain

$$\int_{\mathbb{R}}\Big[\langle u_t^{(1)},\psi(t,.)\rangle - i\langle u_t,\triangle_{k,k'}\psi(t,.)\rangle\Big]dt = 0. \tag{5.49}$$

Moreover, using that $\mathcal{F}(u_t^{(1)}) = (\mathcal{F}(u_t))^{(1)}$ and the relations (5.49),(2.13) we deduce

$$\int_{\mathbb{R}} \Big[\langle(\mathcal{F}(u_t))^{(1)}, \mathcal{F}^{-1}\psi(t,.)\rangle + i\langle\mathcal{F}(u_t), |.|^2\mathcal{F}^{-1}\psi(t,.)\rangle\Big]dt = 0,\ \forall\,\psi \in \mathcal{S}(\mathbb{R}, \mathcal{S}_{k,k'}(\mathbb{R})). \tag{5.50}$$

If we take ψ such that $\mathcal{F}^{-1}\psi(t,\xi) = e^{it|\xi|^2}\varphi(\xi)\chi(t)$ with φ in $\mathcal{S}_{k,k'}(\mathbb{R})$, χ in $\mathcal{S}(\mathbb{R})$, we obtain

$$\int_{\mathbb{R}} \Big[\langle(\mathcal{F}(u_t))^{(1)}, e^{it|.|^2}\varphi\rangle + i\langle\mathcal{F}(u_t), |.|^2 e^{it|.|^2}\varphi\rangle\Big]\chi(t)dt = 0,\ \forall\,\chi \in \mathcal{S}(\mathbb{R}). \tag{5.51}$$

Thus we deduce that

$$\frac{d}{dt}\langle\mathcal{F}(u_t), e^{it|.|^2}\varphi\rangle = \langle(\mathcal{F}(u_t))^{(1)}, e^{it|.|^2}\varphi\rangle + i\langle\mathcal{F}(u_t), |.|^2 e^{it|.|^2}\varphi\rangle = 0,\ \forall\,\varphi \in \mathcal{S}_{k,k'}(\mathbb{R}). \tag{5.52}$$

Hence for all φ in $\mathcal{S}_{k,k'}(\mathbb{R})$, the function $t \mapsto \langle\mathcal{F}(u_t), e^{it|.|^2}\varphi\rangle$ is constant.
Finally, since $u_0 = 0$ then

$$\langle\mathcal{F}(u_t), e^{it|.|^2}\varphi\rangle = \langle\mathcal{F}(u_0), \varphi\rangle = 0,\ \forall\,t \in \mathbb{R};\ \forall\,\varphi \in \mathcal{S}_{k,k'}(\mathbb{R}).$$

From this we deduce that $u = 0$. □

Proposition 25. *Let g be in $W^{s,p}_{k,k'}(\mathbb{R})$, $s \in \mathbb{R}$ and $1 \le p < \infty$, the solution u given by the Theorem 7 belongs to $C(\mathbb{R}; W^{s,p}_{k,k'}(\mathbb{R}))$. For m in $\mathbb{N}$, $(u_t^{(m)}) \in C(\mathbb{R}; W^{s-m,p}_{k,k'}(\mathbb{R}))$ and we have*

$$\begin{cases} ||u_t||_{W^{s,p}_{k,k'}(\mathbb{R})} &= ||g||_{W^{s,p}_{k,k'}(\mathbb{R})},\ \forall\,t \in \mathbb{R} \\ ||u_t^{(m)}||_{W^{s-m,p}_{k,k'}(\mathbb{R})} &\le C_m||g||_{W^{s,p}_{k,k'}(\mathbb{R})},\ \forall\,t \in \mathbb{R};\ \forall\,m \in \mathbb{N}^*. \end{cases} \tag{5.53}$$

Proof. The formula (5.47) give that, for all t in $\mathbb{R}$,

$$\mathcal{F}(u_t) = e^{-it|\xi|^2}\mathcal{F}(g).$$

Thus it is easy to deduce (5.53).

Now we will prove that for m in $\mathbb{N}$, $(u_t^{(m)})$ belongs to $C(\mathbb{R}; W^{s-m,p}_{k,k'}(\mathbb{R}))$. Indeed, let $(t_n)_n$ a sequence that converge to t_0 in $\mathbb{R}$, we have

$$||u_{t_n} - u_{t_0}||^2_{W^{s,p}_{k,k'}(\mathbb{R})} = \int_{\mathbb{R}} (1 + |\xi|^2)^{sp}|e^{-it_n|\xi|^2} - e^{-it_0|\xi|^2}|^p|\mathcal{F}(g)(\xi)|^p d\nu_{k,k'}(\xi).$$

The dominate convergence theorem gives that

$$\lim_{n\to\infty} ||u_{t_n} - u_{t_0}||^p_{W^{s,p}_{k,k'}(\mathbb{R})} = 0.$$

On the other hand, from (5.47) we have

$$\mathcal{F}(u_t^{(m)}) = (-i|\xi|^2)^m e^{-it|\xi|^2}\mathcal{F}(g).$$

From this we obtain

$$||u_{t_n}^{(m)} - u_{t_0}^{(m)}||^p_{W^{s,p}_{k,k'}(\mathbb{R})} = \int_{\mathbb{R}} (1 + |\xi|^2)^{sp}|e^{-it_n|\xi|^2} - e^{-it_0|\xi|^2}|^p|\xi|^{2mp}|\mathcal{F}(g)(\xi)|^p d\nu_{k,k'}(\xi).$$

Thus the dominate convergence theorem gives the result. □

5.4. **Practical real inversion formulas for the generalized wavelet transform $S_g^{(k,k')}$.** In this paragraph we give practical real inversion formulas. Using the harmonic analysis associated with the operator $T_{k,k'}$ we define and study in this subsection the generalized wavelet and the generalized continuous wavelet transform.

Definition 11. *A generalized wavelet on $\mathbb{R}$ is a measurable function h even on $\mathbb{R}$ satisfying for almost all $x \in \mathbb{R}$, the condition*

$$0 < C_h = \int_0^\infty |\mathcal{F}(h)(\lambda x)|^2 \frac{d\lambda}{\lambda} < \infty. \tag{5.54}$$

Example 1. *Let E_t, $t>0$, the heat kernel is defined on $\mathbb{R}$ by*

$$\forall x \in \mathbb{R}, \quad E_t(x) = \mathcal{F}^{-1}(e^{-t\lambda^2})(x). \tag{5.55}$$

The function $h(x) = -\frac{d}{dt}E_t(x)$ is a generalized wavelet on $\mathbb{R}$ in $\mathcal{S}(\mathbb{R})$, and we have $C_h = \frac{1}{8t^2}$.

Proposition 26. *Let $a>0$ and h be a generalized wavelet in $L^2_{A_{k,k'}}(\mathbb{R})$. Then there exists a function h_a in $L^2_{A_{k,k'}}(\mathbb{R})$ such that*

$$\forall\, y \in \mathbb{R},\ \mathcal{F}(h_a)(y) = \mathcal{F}(h)(ay). \tag{5.56}$$

This function is given by the relation

$$h_a = \frac{1}{\sqrt{a}}\mathcal{F}^{-1} \circ D_{a^{-1}} \circ \mathcal{F}(h) \tag{5.57}$$

and satisfies

$$||h_a||_{2,A_{k,k'}} \leq \frac{s(a)}{\sqrt{a}}||h||_{2,A_{k,k'}}, \tag{5.58}$$

where

$$s(a) = \sup_{\lambda\in\mathbb{R}} \frac{|c_{k,k'}(|\lambda|)|}{|c_{k,k'}(\frac{|\lambda|}{a})|}, \quad \text{and} \quad D_a(f)(x) = \frac{1}{\sqrt{a}}f(\frac{x}{a}).$$

Proof. We use a similar ideas as in [9]. □

Let $a>0$ and h be in $L^2_{A_{k,k'}}(\mathbb{R})$. We consider the family $h_{a,x}$, $x\in\mathbb{R}$, of functions on $\mathbb{R}$ in $L^2_{A_{k,k'}}(\mathbb{R})$ defined by

$$h_{a,x}(y) = \frac{a^{\frac{1}{2}}}{s(a)}\tau_x^{k,k'}h_a(-y),\ y\in\mathbb{R}, \tag{5.59}$$

where $\tau_x^{k,k'}$, $x\in\mathbb{R}$, are the generalized translation operators given by (2.17).

Definition 12. *Let h be a generalized wavelet on $\mathbb{R}$ in $L^2_{A_{k,k'}}(\mathbb{R})$. The generalized continuous wavelet transform $S_h^{(k,k')}$ on $\mathbb{R}$ is defined for regular functions f on $\mathbb{R}$ by*

$$S_h^{(k,k')}(f)(a,x) = \int_{\mathbb{R}} f(y)\overline{h_{a,x}(y)}A_{k,k'}(y)dy,\ a>0,\ x\in\mathbb{R}. \tag{5.60}$$

- Let h be a generalized wavelet on $\mathbb{R}$ in $L^2_{A_{k,k'}}(\mathbb{R})$ such that $\mathcal{F}(h)\in L^\infty_{\nu_{k,k'}}(\mathbb{R})$. It is easily to see that the generalized continuous wavelet transform $S_h^{(k,k')}$, is a bounded linear operator from $W^{2t,2}_{k,k'}(\mathbb{R})$, $t\geq 0$, into $L^2_{A_{k,k'}}(\mathbb{R})$, and we have

$$\|S_h^{(k,k')}f(a,.)\|_{L^2_{A_{k,k'}}(\mathbb{R})} \leq C(a)\|\mathcal{F}(h)\|_{L^\infty_{\nu_{k,k'}}(\mathbb{R})}\|f\|_{W^{2t,2}_{k,k'}(\mathbb{R})}, \quad f\in W^{2t,2}_{k,k'}(\mathbb{R}).$$

- Let $\lambda>0$, $t\geq 0$ and h be a generalized wavelet on $\mathbb{R}$ in $L^2_{A_{k,k'}}(\mathbb{R})$ such that $\mathcal{F}(h)\in L^\infty_{\nu_{k,k'}}(\mathbb{R})$. We define the Hilbert space $\mathcal{H}^{\lambda,t}_{h,A_{k,k'}}(\mathbb{R})$ as the subspace of $W^{2t,2}_{k,k'}(\mathbb{R})$ with the inner product:

$$\langle f,g\rangle_{\mathcal{H}^{\lambda,t}_{h,A_{k,k'}}} = \lambda\langle f,g\rangle_{W^{2t,2}_{k,k'}(\mathbb{R})} + \langle S_h^{(k,k')}f(a,.), S_h^{(k,k')}g(a,.)\rangle_{L^2_{A_{k,k'}}(\mathbb{R})}, \quad f,g\in W^{2t,2}_{k,k'}(\mathbb{R}).$$

The norm associated to the inner product is define by:

$$\|f\|^2_{\mathcal{H}^{\lambda,t}_{h,A_{k,k'}}} := \lambda\|f\|^2_{W^{2t,2}_{k,k'}(\mathbb{R})} + \|S_h^{(k,k')}f(a,.)\|^2_{L^2_{A_{k,k'}}(\mathbb{R})}.$$

We proceed as [9] we prove the following results.

Proposition 27. *Let* $t > \frac{2k+1}{4}$ *and* h *be a generalized wavelet on* $\mathbb{R}$ *in* $L^2_{A_{k,k'}}(\mathbb{R})$ *such that* $\mathcal{F}(h) \in L^\infty_{\nu_{k,k'}}(\mathbb{R})$. *Then the Hilbert space* $\mathcal{H}^{\lambda,t}_{h,A_{k,k'}}(\mathbb{R})$ *admits the following reproducing kernel*

$$\mathcal{W}_{\lambda,h}(x,y) = \int_{\mathbb{R}} \frac{G^{(k,k')}_\xi(x)G^{(k,k')}_\xi(-y)}{\lambda(1+|\xi|^2)^{2t} + [\,\mathcal{F}(h)(a\xi)\,]^2}(1-\frac{\rho}{i\xi})\frac{d\xi}{8\pi|c_{k,k'}(\xi)|^2}.$$

Theorem 8. *Let* $t > \frac{2k+1}{4}$ *and* h *be a generalized wavelet on* $\mathbb{R}$ *in* $L^2_{A_{k,k'}}(\mathbb{R})$ *such that* $\mathcal{F}(h) \in L^\infty_{\nu_{k,k'}}(\mathbb{R})$.

i) *For any* $g \in L^2_{A_{k,k'}}(\mathbb{R})$ *and for any* $\lambda > 0$, *the best approximate function* $f^*_{\lambda,g}$ *in the sense*

$$\inf_{f\in W^{2t,2}_{k,k'}(\mathbb{R})} \left\{\lambda\|f\|^2_{W^{2t,2}_{k,k'}(\mathbb{R})} + \|g - S^{(k,k')}_h f(a,.)\|^2_{L^2_{A_{k,k'}}(\mathbb{R})}\right\}$$
$$= \lambda\|f^*_{\lambda,g}\|^2_{W^{2t,2}_{k,k'}(\mathbb{R})} + \|g - S^{(k,k')}_h f^*_{\lambda,g}(a,.)\|^2_{L^2_{A_{k,k'}}(\mathbb{R})}$$

exists uniquely and $f^*_{\lambda,g}$ *is represented by*

$$f^*_{\lambda,g}(a,x) = \int_{\mathbb{R}} g(y)Q_{\lambda,h}(x,y)A_{k,k'}(y)dy,$$

where

$$Q_{\lambda,h}(x,y) = \int_{\mathbb{R}} \frac{\mathcal{F}(h)(a\xi)G^{(k,k')}_\xi(x)G^{(k,k')}_\xi(-y)}{\lambda(1+|\xi|^2)^{2t} + [\mathcal{F}(h)(a\xi)]^2}(1-\frac{\rho}{i\xi})\frac{d\xi}{8\pi|c_{k,k'}(\xi)|^2}.$$

ii) *If we take* $g = S^{(k,k')}_h f(a,.)$, *then*

$$f^*_{\lambda,g} \to f \quad as \quad \lambda \to 0, \quad uniformly\ .$$

iii) *Let* $\delta > 0$ *and let* g, g_δ *satisfy* $\|g - g_\delta\|_{L^2_{A_{k,k'}}(\mathbb{R})} \le \delta$. *Then*

$$\|f^*_{\lambda,g} - f^*_{\lambda,g_\delta}\|_{W^{2t,2}_{k,k'}(\mathbb{R})} \le \frac{\delta}{\sqrt{\lambda}}.$$

References

[1] **J.-PH. Anker, F. Ayadi and M. Sifi,** *Opdam's hypergeometric functions: product formula and convolution structure in dimension 1 ,* Adv. Pure Appl. Math. Vol. **3**, Issue 1, (2012), 11-44.

[2] **N. Ben Salem and A. Dachraoui**, Sobolev type spaces associated with Jacobi differential operators, Integ. Transf. and Special Funct. Vol. **9**, N^{o}3, (2000), 163-184.

[3] **I. Cherednik,** Aunification of Knizhnik-Zamolod chnikove quations and Dunkl operators via affine Hecke algebras, Invent. Math. **106** (1991), 411432.

[4] **L. Gallardo and K. Trimèche,** *Positivity of the Jacobi-Cherednik intertwining operator and its dual,* Adv. Pure Appl. Math. **1** (2010), no.2, 163-194.

[5] **T. R. Johansen**, On a class of non-integrable multipliers for the Jacobi transform, Integ. Transf. and Special Funct. DOI:10.1080/10652469.2012.708866. Version of record first published: 25 Jul 2012 or arXiv:1108.3760 [math.CA].

[6] **G.J. Heckmann and E.M. Opdam,** Root systems and hypergeometric functions I. Compositio Math. **64**, (1987), 329-352.

[7] **T.H. Koornwinder**, Jacobi functions and analysis on non compact semisimple Lie groups,in Special functions:group theoretical aspects and applications, R.A. Askey, T.H. Koornwinder and W. Schemp p,(eds.) Reidel (1984), 184.

[8] **H. Mejjaoli and K. Trimèche,** *Hypoellipticity and hypoanaliticity of the Dunkl Laplacian operator. Integ. Transf. and Special Funct. Vol.* **15**, *N^{o}6, (2004), 523-548.*

[9] **H. Mejjaoli and N. Sraieb,** *Uncertainty principles for the continuous Dunkl Gabor transform and the Dunkl continuous wavelet transform. Mediterr. J. Math.* **5** *(2008), no. 4, 443–466.*

[10] **M.A. Mourou,** *Transmutation operators and Paley-Wiener associated with a Cherednik type operator on the real line,* Anal. Appl. **8** (2010), 387-408.

[11] **E.M. Opdam**, Harmonic analysis for certain representations of graded Hecke algebras, Acta. Math. **175** (1995), 75121.

[12] **E.M. Opdam**, Lecture notes on Dunkl operators for real and complex reflection groups, Mem. Math. soc. Japon **8** (2000).

[13] **B. Schapira,** Contributions to the hypergeometric function theory of Heckman and Opdam: sharpe stimates, Schwartz spaces, heat kernel. Geom. Funct. Anal. **18**, (2008), vol.1, 222-250.

[14] **E.M. Stein,** *Singular Integrals and Differentiability Properties of Funxtions,* Princeton Univ.Press, 1970.

[15] **H. Triebel,** *Spaces of distributions of Besov type on Euclidean n-space. Duality, interpolation,* Ark. Mat. **11** (1973), 13-64.

[16] **H. Triebel,** *Interpolation theory, functions spaces differential operators. North Holland, Amesterdam, 1978.*

[17] **K. Trimèche,** *The trigonometric Dunkl intertwining operator and its dual associated with the Cherednik operators and the Heckman-Opdam theory,* Adv. Pure Appl. Math. Vol. **1**, Issue 3, (2010), 293-323.

Taiba University, College of Sciences, Department of Mathematics, PO BOX 30002 Al Madinah AL Munawara, Saudi Arabia.

E-mail address: *hatem.mejjaoli@ipest.rnu.tn or hatem.mejjaoli@yahoo.fr*