Limit Theorems for a Galton-Watson Process with Immigration in Varying Environments*

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Abstract In this paper, we obtain the central limit theorem and the law of the iterated logarithm for Galton-Watson branching processes with time-dependent immigration in varying environments.

Keywords central limit theorem, the law of the iterated logarithm, Galton-Watson branching process with immigration, varying environment
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1 Introduction

Let $\{T_n; n \geq 0\}$ be a classical supercritical Galton-Watson branching process with offspring distribution $\{b_n; n \geq 0\}$ and mean $m := \sum_{n=0}^{\infty} nb_n, 1 < m < \infty$. Define $S_n = T_n/m^n$, it is well known that there exists a nonnegative random variable S such that $S = \lim_{n\to\infty} S_n$ *a.s.*. C.C.Heyde (cf. [10]-[12]) derived the central limit theorem (CLT) and the law of the iterated logarithm (LIL) for $\{T_n; n \geq 0\}$:

(I) Suppose that $\tau_1^2 := Var(T_1) < \infty$ and $P(S > 0) = 1$. Set $\tau_r^2 := Var(T_r), r \geq 1$, then

$$\tau_r^{-1} T_n^{-\frac{1}{2}} (T_{n+r} - m^r T_n) \xrightarrow{d} N(0,1) \ (n \to \infty),$$

$$(m^2 - m)^{\frac{1}{2}} \tau_1^{-1} T_n^{-\frac{1}{2}} m^n (S - S_n) \xrightarrow{d} N(0,1) \ (n \to \infty).$$

(II) Suppose that $E(T_1^3) < \infty$, then

$$\limsup_{n\to\infty} (\liminf_{n\to\infty}) \frac{T_{n+r} - m^r T_n}{(2\tau_r^2 T_n \log n)^{\frac{1}{2}}} = 1(-1) \quad a.s. \text{ on } \{S > 0\},$$

$$\limsup_{n\to\infty} (\liminf_{n\to\infty}) \frac{m^n S - T_n}{(2\tau_1^2 (m^2 - m)^{-1} T_n \log n)^{\frac{1}{2}}} = 1(-1) \quad a.s. \text{ on } \{S > 0\}.$$

Similar limit theorems for a classical Galton-Watson branching process with immigration were studied in [13].

A branching process in a random environment is a natural and important extension of the Galton-Watson process, it is a class of non-homegeneous Galton-Waltson process indexed by a time environment, which has been studied by many authors, see[1-3],[5-6],[14-15], [19-22].

In this article, we consider the Galton-Watson branching process with time-dependent immigration in the varying environments(IGWVE) defined as following:

Definition 1.1. Let $Z_0 \equiv 1$ and for any $n \geq 0$,

$$Z_{n+1} = \sum_{j=1}^{Z_n} \xi_{n,j} + Y_{n+1} \quad a.s.,$$

where $\{\xi_{n,j}; n \geq 0, j \geq 1\}$ are independent and have the same distribution in row, $\{Y_n, n \geq 1\}$ is a sequence of independent random variables taking values in $\mathbb{N}$ and independent of $\{\xi_{n,j}; n \geq$

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$0, j \geq 1\}$, then $\{Z_n, n \geq 0\}$ is said to be a Galton-Watson branching process with time-dependent immigration in varying environments. Particularly, if for any $n \geq 1$, $Y_n \equiv 0$, then $\{Z_n, n \geq 0\}$ is said to be a Galton-Watson branching process in varying environments(GWVE).

Throughout this paper we assume that the variances of $\{\xi_{n,j}; n \geq 0, j \geq 1\}$ and $\{Y_n, n \geq 1\}$ exist. Define $\mu_n = E(\xi_{n,j}), \delta_n^2 = Var(\xi_{n,j})$ and $\nu_n = E(Y_n)$, we assume that

$$P(\xi_{n,1} = 0) \equiv 0, \quad \infty > \mu_n > 1, \quad \infty > \nu_n, \delta_n^2 > 0.$$

Our first main result is the following growth rate of IGWVE.

Theorem 1.1. Let $\{Z_n, n \geq 0\}$ be an IGWVE. Define $m_n = \prod_{i=0}^{n-1} \mu_i, W_n = Z_n/m_n, n \geq 1$, then $\{W_n, n \geq 1\}$ is a nonnegative submartingale. If

$$a := \sum_{n=1}^{\infty} \frac{\nu_n}{m_n} < \infty, \tag{1}$$

then there exists a nonnegative random variable W with $E(W) < \infty$ such that $W_n \xrightarrow{a.s.} W$. Furthermore, if

$$b := \sup_{n \geq 1} \sum_{j=n}^{\infty} \frac{\delta_j^2}{\mu_j^2 m_{n,j-n}} < \infty, \tag{2}$$

where $m_{n,k} = \prod_{i=n}^{n+k-1} \mu_i$, then $W_n \xrightarrow{L^2} W$.

To prove the CLT and the LIL for IGWVE, we need the following two decomposition results.

(A) For any fixed $n \geq 0, r \geq 1$ one has

$$Z_{n+r} = \sum_{j=1}^{Z_n} X_{n,r}^{(j)} + Y_{n,r} \quad a.s.,$$

where $\{X_{n,r}^{(j)}, j \geq 1\}$ are i.i.d. and independent of $Y_{n,r}$. The variances of $\{X_{n,r}^{(j)}, j \geq 1\}$ and $Y_{n,r}$ exist. We define $m_{n,r} = E(X_{n,r}^{(1)}), \sigma_{n,r}^2 = Var(X_{n,r}^{(1)}), \pi_{n,r} = E(Y_{n,r}), \theta_{n,r}^2 = Var(Y_{n,r})$.

(B) For any fixed $n \geq 0$ one has

$$Z_n - m_n W = \sum_{j=1}^{Z_n} (1 - V_n^{(j)}) + I_n \quad a.s.,$$

where $\{V_n^{(j)}; j \geq 1\}$ are i.i.d. and independent of I_n. If (2) is satisfied, then the variances of $\{V_n^{(j)}, j \geq 1\}$ exist. Furthermore, if

$$c := \sup_{n \geq 0} \sum_{k=n+1}^{\infty} \frac{\nu_k}{m_{n,k-n}} < \infty, \tag{3}$$

then the variances of I_n exist. We define $\sigma_n^2 = Var(V_n^{(1)}), \pi_n = E(I_n), \theta_n^2 = Var(I_n)$.

Using the above two decomposition results we obtain the CLT and the LIL for IGWVE.

Theorem 1.2. Let $\{Z_n, n \geq 0\}$ be an IGWVE. If $Z_n \xrightarrow{P} \infty$, (2),(3) are satisfied and

$$0 < d = \inf_{n \geq 0} \frac{\delta_n^2}{\mu_n^2}, \tag{4}$$

then for any fixed $r \geq 1$, when $n \to \infty$ one has

$$\frac{Z_{n+r} - m_{n,r} Z_n}{\sigma_{n,r} \sqrt{Z_n}} \xrightarrow{d} N(0,1), \qquad \frac{Z_n - m_n W}{\sigma_n \sqrt{Z_n}} \xrightarrow{d} N(0,1).$$

Remark 1.1. Note that $Z_n \geq 1$, a necessary and sufficient condition for $Z_n \xrightarrow{a.s.} \infty$ is

$$\sum_{n=0}^{\infty}(1 - P(\xi_{n,1} = 1)) = +\infty,$$

which imply $Z_n \xrightarrow{P} \infty$(c.f.[17]).

Assume that there exists a constant $0 < \delta < 1$ such that for any $r \geq 1$,

$$\sup_n E\left(\left|\frac{X_{n,r}^{(j)} - m_{n,r}}{\sigma_{n,r}}\right|^{2+\delta}\right) < \infty, \qquad \sup_n E\left(\left|\frac{V_n^{(j)} - 1}{\sigma_n}\right|^{2+\delta}\right) < \infty. \tag{5}$$

For any $n \geq 1$ and $r \geq 1$, define

$$Q_{n,r}(x) = P\left(\frac{Z_{n+r} - m_{n,r}Z_n}{\sigma_{n,r}\sqrt{Z_n}} \leq x\right), \quad Q_n(x) = P\left(\frac{Z_n - m_n W}{\sigma_n\sqrt{Z_n}} \leq x\right), \quad -\infty < x < \infty. \tag{6}$$

Theorem 1.3. Let $\{Z_n, n \geq 0\}$ be an IGWVE. If $Z_n \xrightarrow{a.e.} +\infty$ and (2)-(5) are satisfied, then there exists constants $\{C_r, r \geq 1\}$, C and D such that

$$\sup_x |Q_{n,r}(x) - \Phi(x)| \leq C_r E\left(Z_n^{-\frac{\delta}{2}}\right) + C\left[E\left(Z_n^{-\frac{\delta}{2}}\right)\right]^{\frac{1}{2}}, \tag{7}$$

$$\sup_x |Q_n(x) - \Phi(x)| \leq DE\left(Z_n^{-\frac{\delta}{2}}\right) + C\left[E\left(Z_n^{-\frac{\delta}{2}}\right)\right]^{\frac{1}{2}}. \tag{8}$$

Theorem 1.4. Let $\{Z_n; n \geq 0\}$ be an IGWVE. Suppose that there exist five constants $\alpha, \beta, \tau, \gamma, \delta$ with $\beta > \alpha > 1, \tau > \gamma > 0$ and $0 < \delta < 1$ such that for any $n \geq 0$,

$$\alpha \leq \mu_n \leq \beta, \quad \gamma^2 \leq \delta_n^2 \leq \tau^2, \quad \sum_{n=0}^{\infty}\left[E\left(Z_n^{-\frac{\delta}{2}}\right)^{\frac{1}{2}}\right] < \infty. \tag{9}$$

If (3) and (5) are satisfied, then

$$\limsup_{n\to\infty}(\liminf_{n\to\infty})\frac{Z_{n+r} - m_{n,r}Z_n}{(2\sigma_{n,r}^2 Z_n \log n)^{\frac{1}{2}}} = 1(-1) \quad a.s.; \tag{10}$$

$$\limsup_{n\to\infty}(\liminf_{n\to\infty})\frac{Z_n - m_n W}{(2\sigma_n^2 Z_n \log n)^{\frac{1}{2}}} = 1(-1) \quad a.s.. \tag{11}$$

Remark 1.2. If (9) is satisfied, then $Z_n \xrightarrow{a.e.} +\infty$, (2) and (4) are true.

2 A growth rate for IGWVE

In order to prove Theorem 1.1 we need the following lemma:

Lemma 2.1. Let $\{X_n, n \geq 0\}$ be a GWVE. For any fixed $n \geq 0, r \geq 1$ one has

$$X_{n+r} = \sum_{j=1}^{X_n} X_{n,r}^{(j)},$$

where $\{X_{n,r}^{(j)}; j \geq 1\}$ are i.i.d. and independent of X_n. Furthermore,

$$m_{n,r} = E(X_{n,r}^{(j)}) = \prod_{j=n}^{n+r-1} \mu_j,$$

$$\sigma_{n,r}^2 = Var(X_{n,r}^{(j)}) = (m_{n,r})^2 \sum_{j=n}^{n+r-1} \frac{\delta_j^2}{\mu_j^2 m_{n,j-n}}.$$

Note that $m_{n,0} = 1, \quad m_{n,1} = \mu_n, \quad X_{n,1}^{(j)} = \xi_{n,j}$ and $\sigma_{n,1}^2 = \delta_n^2$.

Proof. Let $X_{n,r}^{(j)}$ be the size of the rth generation of GWVE starting with the jth particle at time n. The first result in Lemma 2.1 following from Definition 1.1. Similar to Proposition 4 and Proposition 6 in [8], the rest of Lemma 2.1 is true . □

Proof of Theorem 1.1. By our basic assumption it is obvious that W_n is integrable for all $n \geq 1$. By the independence of $\{Y_n, n \geq 1\}$ and $\{\xi_{n,j}, n \geq 0, j \geq 1\}$ one has

$$\begin{aligned} E(W_{n+1}|W_n, W_{n-1}, \cdots, W_1) &= E\left(\frac{Z_{n+1}}{m_{n+1}}|Z_n, Z_{n-1}, \cdots, Z_1\right) \\ &= \frac{1}{m_n} E\left[\frac{1}{\mu_n}\left(\sum_{i=1}^{Z_n} \xi_{n,j} + Y_{n+1}\right)|Z_n\right] \\ &= \frac{Z_n}{m_n} + \frac{v_{n+1}}{m_{n+1}} > \frac{Z_n}{m_n} = W_n, \end{aligned}$$

which means that $\{W_n, n \geq 1\}$ is a nonnegative submartingale.

Let X_n be the size of the original GWVE at time n and $U_{k,n}(k < n)$ be the number of descendants at time n of the particles that immigrated in generation k, then

$$Z_n = X_n + \sum_{k=1}^{n} U_{k,n}, \tag{12}$$

where $U_{n,n} = Y_n$. Let $\mathcal{G}$ be the $\sigma-$field generated by $\{Y_n, n \geq 1\}$, by the independence of $\{Y_n, n \geq 1\}$ and $\{\xi_{n,j}, n \geq 0, j \geq 1\}$ one has

$$E(W_n|\mathcal{G}) = E\left(\frac{Z_n}{m_n}|\mathcal{G}\right) = E\left(\frac{X_n}{m_n}\right) + E\left(\sum_{k=1}^{n} \frac{U_{k,n}}{m_n}|\mathcal{G}\right). \tag{13}$$

Now for $k \leq n$, the random variable $U_{k,n}$ is the size of the $(n-k)$th generation of ordinary GWVE starting, however, with Y_k particles at time k. Therefore, by the independence of $\{Y_n, n \geq 1\}$ and Lemma 2.1, its conditional expectation is just $Y_k m_{k,n-k}$. According to (13) and Lemma 2.1, one has

$$E(W_n|\mathcal{G}) = 1 + \sum_{k=1}^{n} \frac{Y_k}{m_k} \quad \text{and} \quad E(W_n) = 1 + \sum_{k=1}^{n} \frac{v_k}{m_k}.$$

Since (1) is satisfied we know that $\{W_n, n \geq 1\}$ is L^1-bounded. By the convergence theorem of submartingale (see [9] Theorem 2.5), there exists a nonnegative random variable W with $E(W) < \infty$ such that $W_n \xrightarrow{a.s.} W$.

In order to prove the last result in Theorem 1.1, we only need to prove that $\{W_n, n \geq 1\}$ is L^2-bounded (see Theorem 7.6.10 of [4]). In fact, by (12) and the independence of $\{Y_n, n \geq 1\}$

and $\{\xi_{n,j}, n \geq 0, j \geq 1\}$ one has

$$
\begin{aligned}
E(W_n^2|\mathcal{G}) &= E\left(\frac{1}{m_n^2}\left[X_n + \sum_{k=1}^{n} U_{k,n}\right]^2 |\mathcal{G}\right) \\
&= E\left\{\frac{1}{m_n^2}\left[X_n^2 + 2X_n\sum_{k=1}^{n} U_{k,n} + \left(\sum_{k=1}^{n} U_{k,n}\right)^2\right]|\mathcal{G}\right\} \\
&= \frac{1}{m_n^2}E(X_n^2) + 2\sum_{k=1}^{n}\frac{Y_k}{m_k} + \frac{1}{m_n^2}E\left[\left(\sum_{k=1}^{n}\left(\sum_{j=1+\sum_{i=1}^{k-1}Y_i}^{\sum_{i=1}^{k}Y_i} X_{k,n-k}^{(j)}\right)\right)^2 |\mathcal{G}\right] \\
&= \frac{1}{m_n^2}E(X_n^2) + 2\sum_{k=1}^{n}\frac{Y_k}{m_k} + \frac{1}{m_n^2}\left[\sum_{k=1}^{n}(Y_k Var(X_{k,n-k}^{(1)})) + \sum_{k=1}^{n} Y_k\left(E(X_{k,n-k}^{(1)})\right)^2\right],
\end{aligned}
$$

where $X_{k,n-k}^{(j)}$ is defined in Lemma 2.1, it is the size of the $(n-k)$th generation of GWVE starting with the jth particle immigrated at time k . By the Lemma 2.1 one has

$$
\begin{aligned}
E(W_n^2) &= \frac{1}{m_n^2}E(X_n^2) + 2\sum_{k=1}^{n}\frac{v_k}{m_k} + \frac{1}{m_n^2}\left\{\sum_{k=1}^{n}(v_k\sigma_{k,n-k}^2) + \sum_{k=1}^{n}(v_k m_{k,n-k}^2)\right\} \\
&= \frac{1}{m_n^2}E(X_n^2) + 2\sum_{k=1}^{n}\frac{v_k}{m_k} + \sum_{k=1}^{n}\left[\frac{v_k}{m_k^2}\cdot\left(\frac{\sigma_{k,n-k}^2}{m_{k,n-k}^2} + 1\right)\right] \\
&\leq b + 2a + ab + a < \infty,
\end{aligned}
$$

which means that $\{W_n, n \geq 1\}$ is L^2-bounded. We complete the proof. □

Remark 2.1. (1)-(3) are obviously satisfied for a classical supercritical Galton-Watson branching process with immigration which have finite second moments. In fact, in this case, $\{\xi_{n,j}, n \geq 0, j \geq 1\}$ are i.i.d. and $\{Y_n, n \geq 1\}$ are also i.i.d..

3 The CLT for IGWVE

In order to prove Theorem 1.2 we need the following lemmas.

Lemma 3.1. Let $\{Z_n, n \geq 0\}$ be an IGWVE. For any fixed $n \geq 0, r \geq 1$ one has

$$
Z_{n+r} = \sum_{j=1}^{Z_n} X_{n,r}^{(j)} + Y_{n,r} \qquad a.s., \tag{14}
$$

where $\{X_{n,r}^{(j)}, j \geq 1\}$ are i.i.d. and independent of $Y_{n,r}$. Furthermore,

$$
\pi_{n,r} = E(Y_{n,r}) = \sum_{i=n+1}^{n+r} \nu_i m_{i,n+r-i}, \quad \theta_{n,r}^2 = Var(Y_{n,r}) = \sum_{i=n+1}^{n+r} \nu_i \sigma_{i,n+r-i}^2. \tag{15}
$$

Proof. Similar to Lemma 2.1, let $X_{n,r}^{(j)}$ be the size of the rth generation of the ordinary GWVE staring with the jth particle at time n. Let $U_{k,n}$ be the number of descendants at time n of the particles that were immigrated in generation k. Define $Y_{n,r} = \sum_{k=n+1}^{n+r} U_{k,n+r}$. Then

$$
Z_{n+r} = \sum_{j=1}^{Z_n} X_{n,r}^{(j)} + Y_{n,r}, \qquad a.s..
$$

By the independence of $\{Y_n, n \geq 1\}$ and $\{\xi_{n,j}, n \geq 0, j \geq 1\}$ we know that $\{X_{n,r}^{(j)}, j \geq 1\}$ are i.i.d. and independent of $Y_{n,r}$.

Let $\mathcal{G}$ be the $\sigma-$field generated by $\{Y_n, n \geq 1\}$. Similar to the calculation of (13) we have

$$E(Y_{n,r}|\mathcal{G}) = E\left(\sum_{k=n+1}^{n+r} U_{k,n+r}|\mathcal{G}\right) = \sum_{k=n+1}^{n+r} Y_k E(X_{k,n+r-k}^{(1)}).$$

So by Lemma 2.1 we have $E(Y_{n,r}) = \sum_{k=n+1}^{n+r} \upsilon_k m_{k,n+r-k}$. On the other hand, by the independence of $\{Y_n, n \geq 1\}$ and $\{\xi_{n,j}, n \geq 0, j \geq 1\}$ one has

$$\begin{aligned} Var(Y_{n,r}|\mathcal{G}) &= Var\left(\sum_{k=n+1}^{n+r} U_{k,n+r}|\mathcal{G}\right) = \sum_{k=n+1}^{n+r} Var(U_{k,n+r}|\mathcal{G}) \\ &= \sum_{k=n+1}^{n+r} Var\left(\sum_{j=1+\sum_{i=n+1}^{k-1} Y_i}^{\sum_{i=n+1}^{k} Y_i} X_{k,n+r-k}^{(j)}|\mathcal{G}\right) = \sum_{k=n+1}^{n+r} Y_k Var(X_{k,n+r-k}^{(1)}). \end{aligned}$$

So by Lemma 2.1 we have $Var(Y_{n,r}) = \sum_{k=n+1}^{n+r} \upsilon_k \sigma_{k,n+r-k}^2$. We complete the proof. □

Lemma 3.2. **([8])** Let $\{X_n, n \geq 0\}$ be a GWVE. Define $V_n = X_n/m_n$ for all $n \geq 1$, then $\{V_n, n \geq 1\}$ is a nonnegative martingale and there exists a nonnegative random variable V such that $V_n \xrightarrow{a.s.} V$ when $n \to \infty$.

Lemma 3.3. Let $\{X_n, n \geq 0\}$ be a GWVE. Define $V_{n,r}^{(j)} = \frac{X_{n,r}^{(j)}}{m_{n,r}}$, where $X_{n,r}^{(j)}$ and $m_{n,r}$ are defined in Lemma 2.1, then $\{V_{n,r}^{(j)}, r \geq 1\}$ is a nonnegative martingale and $V_{n,r}^{(j)} \xrightarrow{a.s} V_n^{(j)}$ when n, j fixed and $r \to \infty$. Then for any fixed $n \geq 1$ one has

$$X_n - m_n V = \sum_{j=1}^{X_n} (1 - V_n^{(j)}) \quad a.s.,$$

where V is defined in Lemma 3.2, $\{V_n^{(j)}; j \geq 1\}$ are i.i.d. and independent of X_n.

Furthermore if (2) is satisfied, then $V_{n,r}^{(j)} \xrightarrow{a.s} V_n^{(j)}$ when n, j fixed and $r \to \infty$, $E(V_n^{(j)}) \equiv 1$,

$$Var(V_n^{(j)}) = \sigma_n^2 = \sum_{j=n}^{\infty} \frac{\delta_j^2}{\mu_j^2 m_{n,j-n}}.$$

Proof. Similar to Proposition 5 and Corollary 2 of [8], we know the first result in Lemma 3.3 is true. Then the second result following from Lemma 2.1. Using the same idea of Theorem 1 in [8], we can prove the last result. □

Lemma 3.4. Let $\{Z_n, n \geq 0\}$ be an IGWVE. For any fixed $n \geq 0$ one has

$$Z_n - m_n W = \sum_{j=1}^{Z_n} (1 - V_n^{(j)}) - I_n \quad a.s., \tag{16}$$

where $\{V_n^{(j)}, j \geq 1\}$ are i.i.d. and independent of I_n. Furthermore, if (2) and (3) are satisfied,

$$\pi_n = E(I_n) = \sum_{k=n+1}^{\infty} \frac{\nu_k}{m_{n,k-n}}, \qquad \theta_n^2 = Var(I_n) = \sum_{k=n+1}^{\infty} \frac{\nu_k \sigma_k^2}{m_{n,k-n}}. \tag{17}$$

Proof. Applying Theorem 1.1 one has that for each $n \geq 1$,

$$\lim_{r \to \infty} \frac{Z_{n+r}}{m_{n+r}} = \lim_{r \to \infty} W_{n+r} = W \quad a.s..$$

So by Lemma 3.1 one deduce that

$$
\begin{aligned}
Z_n - m_n W &= Z_n - m_n \lim_{r\to\infty} \frac{Z_{n+r}}{m_{n+r}} = \lim_{r\to\infty} \left[Z_n - \frac{m_n}{m_{n+r}} \left(\sum_{j=1}^{Z_n} X_{n,r}^{(j)} + Y_{n,r} \right) \right] \\
&= \lim_{r\to\infty} \left\{ \sum_{j=1}^{Z_n} \left[1 - \frac{X_{n,r}^{(j)}}{m_{n,r}} \right] - \frac{Y_{n,r}}{m_{n,r}} \right\} \quad a.s.. \qquad (18)
\end{aligned}
$$

By Lemma 3.3 one has

$$
\lim_{r\to\infty} \sum_{j=1}^{Z_n} \left[1 - \frac{X_{n,r}^{(j)}}{m_{n,r}} \right] = \sum_{j=1}^{Z_n} \left[1 - V_n^{(j)} \right] \quad a.s.. \qquad (19)
$$

One the other hand, define $T_{n,r} = \frac{Y_{n,r}}{m_{n,r}} = \sum_{k=n+1}^{n+r} U_{k,n+r}/m_{n,r}$ for each $r \geq 1$. Let $\mathcal{F}_{n,r}$ be the $\sigma-$field generated by $\{Y_n, \cdots, Y_{n+r}, \xi_{i,j}, n-1 \leq i \leq n+r-1, j \geq 1\}$. Note that by Lemma 3.1 one has

$$
E(T_{n,r}) = \frac{1}{m_{n,r}} \sum_{k=n+1}^{n+r} v_k m_{k,n+r-k} = \sum_{k=n+1}^{n+r} \frac{v_k}{m_{n,k-n}} \leq c,
$$

which means that $\{T_{n,r}, r \geq 1\}$ is L^1-bounded. Now

$$
\begin{aligned}
E(T_{n,r+1}|\mathcal{F}_{n,r}) &= E\left(\frac{Y_{n,r+1}}{m_{n,r+1}} |\mathcal{F}_{n,r} \right) = E\left(\frac{\sum_{k=n+1}^{n+r+1} U_{k,n+r+1}}{m_{n,r+1}} |\mathcal{F}_{n,r} \right) \\
&= \frac{1}{m_{n,r+1}} E\left\{ Y_{n+r+1} + \sum_{k=n+1}^{n+r} \left[\sum_{j=1+\sum_{i=n+1}^{k-1} Y_i}^{\sum_{i=n+1}^{k} Y_i} X_{k,n+r+1-k}^{(j)} \right] |\mathcal{F}_{n,r} \right\} \\
&= \frac{1}{m_{n,r+1}} \left\{ E(Y_{n+r+1}) + \sum_{k=n+1}^{n+r} \left[\sum_{j=1+\sum_{i=n+1}^{k-1} Y_i}^{\sum_{i=n+1}^{k} Y_i} X_{k,n+r-k}^{(j)} E(\xi_{n+r,1}) \right] \right\} \\
&= \frac{Y_{n,r}}{m_{n,r}} + \frac{v_{n+r+1}}{m_{n,r+1}} > T_{n,r},
\end{aligned}
$$

which means that $\{T_{n,r}, r \geq 1\}$ is a nonnegative submartingale with respect to $\mathcal{F}_{n,r}, r \geq 1$, then there exists a nonnegative random variable I_n with $E(I_n) < \infty$ such that

$$
\lim_{r\to\infty} T_{n,r} = I_n \quad a.s.. \qquad (20)
$$

By (18), (19) and (20) we obtain (16).

Now by Lemma 3.1 one has

$$
\begin{aligned}
E(T_{n,r})^2 &= Var(T_{n,r}) + (E(T_{n,r}))^2 \\
&= \frac{1}{m_{n,r}^2} \left[\sum_{k=n+1}^{n+r} \nu_k \sigma_{k,n+r-k}^2 + \left(\sum_{k=n+1}^{n+r} \nu_k m_{k,n+r-k} \right)^2 \right] \\
&\leq \sum_{k=n+1}^{n+r} \left[\frac{v_k}{m_{n,k-n}^2} \cdot \frac{\sigma_{k,n+r-k}^2}{m_{k,n+r-k}^2} \right] + \left(\sum_{k=n+1}^{n+r} \frac{v_k}{m_{n,k-n}} \right)^2 \\
&\leq bc + c^2 < \infty,
\end{aligned}
$$

which means that $\{T_{n,r}, r\geq 1\}$ is L^2-bounded, so $T_{n,r} \xrightarrow{L^2} I_n$ when $r\to\infty$. Thus,

$$E(I_n)=\lim_{r\to\infty}E(T_{n,r})=\sum_{k=n+1}^{\infty}\frac{v_k}{m_{n,k-n}},$$

$$Var(I_n)=\lim_{r\to\infty}Var(T_{n,r})=\lim_{r\to\infty}\sum_{k=n+1}^{n+r}\left[\frac{v_k}{m_{n,k-n}}\cdot\frac{\sigma^2_{k,n+r-k}}{m^2_{k,n+r-k}}\right]=\sum_{k=n+1}^{\infty}\left[\frac{v_k\sigma_k^2}{m_{n,k-n}}\right].$$

We complete the proof of Lemma 3.4. □

Now we consider a double sequence of random variables $\{\zeta_{n,j}, j\geq 1, n\geq 1\}$, where for any $n\geq 1$, $\{\zeta_{n,j}, j\geq 1\}$ are i.i.d.. Let $\{N_n, n\geq 1\}$ be a sequence of random variables taking values in $\mathbb{Z}_+ := \{1,2,\cdots\}$, where $N_n \xrightarrow{P} \infty$ and for any $n\geq 1$, $\{\zeta_{n,j}, j\geq 1\}$ and N_n are independent. Define

$$S_{N_n}=\sum_{j=1}^{N_n}\zeta_{n,j},$$

then we have the following result:

Lemma 3.5. If $E(\zeta_{n,j})\equiv 0$ and $Var(\zeta_{n,j})\equiv 1$, then one has

$$\frac{S_{N_n}}{\sqrt{N_n}}\xrightarrow{d} N(0,1), n\to\infty.$$

Proof. Since $\{\zeta_{n,j}, j\geq 1\}$ have the same distribution, we can set

$$\varphi_n(t)=E\left(exp\left(it\zeta_{n,j}\right)\right).$$

Note $E(\zeta_{n,j})\equiv 0$ and $Var(\zeta_{n,j})\equiv 1$, according to (3.8) of [18] P101, one obtains

$$\varphi_n(s)=\varphi_n(0)+\varphi_n'(0)s+\frac{\varphi_n''(0)}{2!}s^2+o(s^2)=1-\frac{s^2}{2}+o(\frac{s^2}{2}), (s\to 0).$$

For any fixed t and k large enough

$$\varphi_n\left(\frac{t}{\sqrt{k}}\right)=1-\frac{t^2}{2k}+o\left(\frac{1}{2k}\right). \tag{21}$$

Since for any $n\geq 1$, $\{N_n, \zeta_{n,j}, j\geq 1\}$ are independent, we have

$$\begin{aligned} E\left(\exp\left(it\frac{S_{N_n}}{\sqrt{N_n}}\right)\right) &= \sum_{k=1}^{\infty}E\left(\exp\left(it\frac{S_{N_n}}{\sqrt{N_n}}\right)|N_n=k\right)P(N_n=k)\\ &= \sum_{k=1}^{\infty}E\left(\exp\left(it\frac{S_k}{\sqrt{k}}\right)\right)P(N_n=k)\\ &= \sum_{k=1}^{\infty}\left(E\left(\exp\left(it\frac{\zeta_{n,1}}{\sqrt{k}}\right)\right)\right)^kP(N_n=k). \end{aligned} \tag{22}$$

Fix $t\in\mathbb{R}:=(-\infty,+\infty)$, note that $(1-\frac{x}{n}+o(\frac{1}{n}))^n\to e^{-x}$ $(n\to\infty)$, so there exists a constant $M=M(\varepsilon)>0$ such that for any $k\geq M$ one has

$$\left|\left(1-\frac{t^2}{2k}+o\left(\frac{1}{2k}\right)\right)^k-\exp\left(-\frac{t^2}{2}\right)\right|\leq\frac{\varepsilon}{4}. \tag{23}$$

Since $N_n \xrightarrow{P} \infty$, for any $\varepsilon > 0$, there exists a constant $N = N(\varepsilon) > 1$ such that for any $n \geq N$ one has $P(N_n \leq M) < \frac{\varepsilon}{4}$. Thus, when $n \geq N$, by (21),(22) and (23) one has

$$\begin{aligned}
\Delta &:= \left| E\left(\exp\left(it\frac{S_{N_n}}{\sqrt{N_n}}\right)\right) - \exp\left(-\frac{t^2}{2}\right) \right| \\
&= \left| \sum_{k=1}^{\infty} \left(E\left(\exp\left(it\frac{\zeta_{n,1}}{\sqrt{k}}\right)\right)\right)^k P(N_n = k) - \exp\left(-\frac{t^2}{2}\right) \right| \\
&\leq \left| \sum_{k=1}^{M} \left[\left(E\left(\exp\left(it\frac{\zeta_{n,1}}{\sqrt{k}}\right)\right)\right)^k - \exp\left(-\frac{t^2}{2}\right) \right] P(N_n = k) \right| \\
&+ \left| \sum_{k=M+1}^{\infty} \left[\left(1 - \frac{t^2}{2k} + o\left(\frac{1}{2k}\right)\right)^k - \exp\left(-\frac{t^2}{2}\right) \right] P(N_n = k) \right| \\
&\leq 2 \cdot \frac{\varepsilon}{4} + \frac{\varepsilon}{4} \cdot \sum_{k=M+1}^{\infty} P(N_n = k) < \varepsilon.
\end{aligned}$$

This means the characteristic function of $S_{N_n}/\sqrt{N_n}$ convergent to that of a standard normal random variable, so by Lévy continuous theorem we complete the proof of this lemma.□

Proof of Theorem 1.2. By Lemma 3.1, for any $n \geq 0, r \geq 1$ we have

$$Z_{n+r} - m_{n,r} Z_n = \sum_{j=1}^{Z_n} (X_{n,r}^{(j)} - m_{n,r}) + Y_{n,r} \quad a.s.,$$

where $\{X_{n,r}^{(j)}, j \geq 1\}$ are i.i.d. and independent of Z_n, $E(X_{n,r})^{(j)} \equiv m_{n,r}$, $Var(Z_{n,r}^{(j)}) \equiv \sigma_{n,r}^2$. According to Lemma 3.5, if we can prove that $Y_{n,r}/\sigma_{n,r}\sqrt{Z_n} \xrightarrow{P} 0$ when r fixed and $n \to \infty$, then we complete the proof of the first part of Theorem 1.2. In fact, for any $\epsilon > 0$,

$$P\left(\frac{Y_{n,r}}{\sigma_{n,r}\sqrt{Z_n}} \geq \epsilon\right) = E\left(P\left(\frac{Y_{n,r}}{\sigma_{n,r}\sqrt{Z_n}} \geq \epsilon | Z_n\right)\right) \leq \frac{\theta_{n,r}^2}{\epsilon^2 \sigma_{n,r}^2} \cdot E\left(\frac{1}{Z_n}\right).$$

Note that $Z_n \xrightarrow{P} \infty$ and $Z_n \geq 1$, for any $\epsilon > 0$, there exists two constants $N, M > 0$ such that $1/M < \epsilon/2$ and $P(Z_n < M) < \epsilon/2$ for all $n \geq N$. When $n \geq N$,

$$E\left(\frac{1}{Z_n}\right) = E\left(\frac{1}{Z_n} I_{[Z_n < M]}\right) + E\left(\frac{1}{Z_n} I_{[Z_n \geq M]}\right) \leq 1 \cdot P(Z_n < M) + \frac{1}{M} P(Z_n \geq M) < \epsilon.$$

Since (2),(3) and (4) are satisfied, then

$$P\left(\frac{Y_{n,r}}{\sigma_{n,r}\sqrt{Z_n}} \geq \epsilon\right) \leq \frac{bc}{\epsilon^2 d} \cdot E\left(\frac{1}{Z_n}\right) \to 0, \quad n \to \infty.$$

By Lemma 3.4, for any $n \geq 1$ one has

$$Z_n - m_n W = \sum_{j=1}^{Z_n} \left(1 - V_n^{(j)}\right) - I_n \quad a.s.,$$

where $\{V_n^{(j)}; j \geq 1\}$ are i.i.d.and independent of Z_n, $E(V_n^{(j)}) \equiv 1$, $Var(V_n^{(j)}) \equiv \sigma_n^2$. Similarly, according to Lemma 3.5, if we can prove that $I_n/\sigma_n\sqrt{Z_n} \xrightarrow{P} 0$ when $n \to \infty$, then we complete the proof of the second part of Theorem 1.2. In fact, for any $\epsilon > 0$,

$$P\left(\frac{I_n}{\sigma_n\sqrt{Z_n}} \geq \epsilon\right) = E\left(P\left(\frac{I_n}{\sigma_n\sqrt{Z_n}} \geq \epsilon | Z_n\right)\right) \leq \frac{\theta_n^2}{\epsilon^2 \sigma_n^2} \cdot E\left(\frac{1}{Z_n}\right).$$

Note that $Z_n \xrightarrow{P} \infty$, (2)-(4) are satisfied, then

$$P\left(\frac{I_n}{\sigma_n\sqrt{Z_n}} \geq \epsilon\right) \leq \frac{bc}{\epsilon^2 d} \cdot E\left(\frac{1}{Z_n}\right) \to 0, \quad n \to \infty.$$

We complete the proof of Theorem 1.2. □

4 Convergence rate in the CLT for IGWVE

In order to prove Theorem 1.3, we need following lemmas.

Lemma 4.1(c.f.[7]P322). Let $\{X_n, n \geq 1\}$ be a sequence of independent random variables with $E(X_n) = 0, n = 1, 2, \cdots$. Define

$$L_n = \sum_{j=1}^{n} X_j, \quad G_n(x) = P\left(\frac{L_n}{\sqrt{Var(L_n)}} \leq x\right), n \geq 1, x \in \mathbb{R}.$$

If there exists an arbitrary small constant $1 > \delta > 0$ such that

$$\Gamma_n^{2+\delta} := \sum_{j=1}^{n} E|X_j|^{2+\delta} < \infty,$$

then there exists a constant A such that

$$\sup_x |G_n(x) - \Phi(x)| \leq A\Gamma_n^{2+\delta}[Var(L_n)]^{-(1+\delta/2)},$$

where $\Phi(x)$ is the standard normal distribution.

Lemma 4.2. Let $\{\zeta_{n,j}, j \geq 1, n \geq 1\}$ be a double sequence of random variables with mean zero and variance 1, where for any $n \geq 1$, $\{\zeta_{n,j}, j \geq 1\}$ are i.i.d.. $\{N_n, n \geq 1\}$ is a sequence of integer valued random variables with $P(N_n \to \infty) = 1$, as $n \to \infty$. For any $n \geq 1$, N_n and $\{\zeta_{n,j}, j \geq 1\}$ are independent. $\{k_n, n \geq 1\}$ is a sequence of strictly increasing positive integers. Define

$$\widetilde{L}_n = \sum_{j=1}^{k_n} \zeta_{n,j}, \quad \widetilde{G}_n(x) = P\left(\frac{\widetilde{L}_n}{\sqrt{k_n}} \leq x\right).$$

$$\widehat{L}_n = \sum_{j=1}^{N_n} \zeta_{n,j}, \quad \widehat{G}_n(x) = P\left(\frac{\widehat{L}_n}{\sqrt{N_n}} \leq x\right).$$

If there exist two constants δ and $M > 0$ such that for any $n \geq 1$,

$$\gamma_n := E(|\zeta_{n,j}|^{2+\delta}) \leq M,$$

then there exists a constants C (not depend on $\{k_n\}$) such that

$$\sup_x |\widetilde{G}_n(x) - \Phi(x)| \leq Ck_n^{-\frac{\delta}{2}}, \quad \sup_x |\widehat{G}_n(x) - \Phi(x)| \leq CE(N_n^{-\frac{\delta}{2}}). \tag{24}$$

Proof. Let $C = AM$, where A is chosen in Lemma 4.1, then using Lemma 4.1 we have the first inequality of (24). As for the second inequality, since for any $n \geq 1$, N_n and $\{\xi_{n,j}, j \geq 1\}$ are independent, so

$$\begin{aligned}\widehat{G}_n(x) - \Phi(x) &= P\left(\frac{\sum_{i=1}^{N_n} \xi_{n,i}}{\sqrt{N_n}} \leq x\right) - \Phi(x) \\ &= \sum_{j=1}^{\infty}\left[P\left(\frac{\sum_{i=1}^{j} \xi_{n,i}}{\sqrt{j}} \leq x\right) - \Phi(x)\right] P(N_n = j).\end{aligned}$$

By the first inequality of (24), one has

$$\begin{aligned}
-CE\left(N_n^{-\frac{\delta}{2}}\right) &= -C\sum_{j=1}^{\infty} j^{-\frac{\delta}{2}} P(N_n=j) = \sum_{j=1}^{\infty}\left(-C\cdot j^{-\frac{\delta}{2}}\right)P(N_n=j) \\
&\le \sum_{j=1}^{\infty}\left(P\left(\frac{\sum_{i=1}^{j}\xi_{n,i}}{\sqrt{j}}\le x\right)-\Phi(x)\right)P(N_n=j) \le CE\left(N_n^{-\frac{\delta}{2}}\right).
\end{aligned}$$

We complete the proof. □

Lemma 4.3. Assume that $\{\zeta_{n,j}, j\ge 1, n\ge 1\}, \{N_n, n\ge 1\}$ satisfy the conditions of Lemma 4.2. Let $\{\eta_n, n\ge 1\}$ be a sequence of independent random variables with $E(|\eta_n|)<\infty$ and for any $n\ge 1$, η_n is independent of $\{\zeta_{n,j}, j\ge 1\}$ and N_n. Define

$$\widetilde{L}_n=\sum_{j=1}^{N_n}\zeta_{n,j}+\eta_n, \qquad \widetilde{H}_n(x)=P\left(\frac{\widetilde{L}_n}{\sqrt{N_n}}\le x\right),$$

then for any sequence ϵ_n of positive constants one has

$$\sup_x|\widetilde{H}_n(x)-\Phi(x)|\le CE(N_n^{-\frac{\delta}{2}})+\epsilon_n^{-1}E(|\eta_n|)E(N_n^{-\frac{\delta}{2}})+\frac{\epsilon_n}{2}. \tag{25}$$

Proof. Let $\Phi(x)$ be the distribution function of standard normal distribution. By Lemma 4.2 and the independence of η_n and $\{\zeta_{n,j}, j\ge 1\}$ one has

$$\begin{aligned}
-CE(N_n^{-\frac{\delta}{2}}) &= -C\sum_{j=1}^{\infty}j^{-\frac{\delta}{2}}P(N_n=j) \\
&= -C\sum_{j=1}^{\infty}j^{-\frac{\delta}{2}}\left[\int_{-\infty}^{\infty}dP(j^{-\frac{1}{2}}\eta_n\le y)\right]P(N_n=j) \\
&\le \sum_{j=1}^{\infty}\left[\int_{-\infty}^{\infty}\left\{P\left(j^{-\frac{1}{2}}\sum_{i=1}^{j}\zeta_{n,i}\le x-y\right)-\Phi(x-y)\right\}dP(j^{-\frac{1}{2}}\eta_n\le y)\right]P(N_n=j) \\
&\le CE(N_n^{-\frac{\delta}{2}}).
\end{aligned} \tag{26}$$

But,

$$\begin{aligned}
\nabla_n &:= \sum_{j=1}^{\infty}\left[\int_{-\infty}^{\infty}\left\{P\left(j^{-\frac{1}{2}}\sum_{i=1}^{j}\zeta_{n,i}\le x-y\right)-\Phi(x-y)\right\}dP(j^{-\frac{1}{2}}\eta_n\le y)\right]P(N_n=j) \\
&= \sum_{j=1}^{\infty}\left[P\left(j^{-\frac{1}{2}}\left(\sum_{i=1}^{j}\zeta_{n,i}+\eta_n\right)\le x\right)-P(\zeta+j^{-\frac{1}{2}}\eta_n\le x)\right]P(N_n=j) \\
&= \sum_{j=1}^{\infty}\left[P\left(j^{-\frac{1}{2}}\left(\sum_{i=1}^{j}\zeta_{n,i}+\eta_n\right)\le x|N_n=j\right)-P(\zeta+j^{-\frac{1}{2}}\eta_n\le x|N_n=j)\right]P(N_n=j) \\
&= P\left(N_n^{-\frac{1}{2}}\left(\sum_{i=1}^{N_n}\zeta_{n,i}+\eta_n\right)\le x\right)-P(\zeta+N_n^{-\frac{1}{2}}\eta_n\le x) \\
&= \widetilde{H}_n(x)-P(\zeta+N_n^{-\frac{1}{2}}\eta_n\le x),
\end{aligned} \tag{27}$$

where ζ has the standard normal distribution and is independent of η_n. By (26) and (27),

$$\sup_x |\widetilde{H}_n(x) - P(\zeta + N_n^{-\frac{1}{2}}\eta_n \le x)| \le CE(N_n^{-\frac{\delta}{2}}). \tag{28}$$

Now for any $\epsilon_n > 0$

$$\begin{aligned} P(\zeta + N_n^{-\frac{1}{2}}\eta_n \le x) &= P(\zeta + N_n^{-\frac{1}{2}}\eta_n \le x, N_n^{-\frac{1}{2}}|\eta_n| \le \epsilon_n) \\ &+ P(\zeta + N_n^{-\frac{1}{2}}\eta_n \le x, N_n^{-\frac{1}{2}}|\eta_n| > \epsilon_n) \\ &\le P(\zeta \le x + \epsilon_n) + P(N_n^{-\frac{1}{2}}|\eta_n| > \epsilon_n). \end{aligned} \tag{29}$$

Similarly, we have

$$P(\zeta + N_n^{-\frac{1}{2}}\eta_n > x) \le P(\zeta > x - \epsilon_n) + P(N_n^{-\frac{1}{2}}|\eta_n| > \epsilon_n),$$

or equivalently,

$$P(\zeta + N_n^{-\frac{1}{2}}\eta_n \le x) \ge P(\zeta \le x - \epsilon_n) - P(N_n^{-\frac{1}{2}}|\eta_n| > \epsilon_n). \tag{30}$$

Also, by mean value theorem, we know that

$$\begin{aligned} \sup_x |P(\zeta \le x - \epsilon_n) - P(\zeta \le x)| &= \sup_x |\Phi(x - \epsilon_n) - \Phi(x)| \\ &= \sup_x \Phi'(\alpha(x))\epsilon_n < \frac{\epsilon_n}{2}. \end{aligned} \tag{31}$$

Similarly,

$$\sup_x |P(\zeta \le x + \epsilon_n) - P(\zeta \le x)| < \frac{\epsilon_n}{2}. \tag{32}$$

By (29)-(32) and Markov inequality one has

$$\begin{aligned} \Lambda_n &:= \sup_x |P(\zeta + N_n^{-\frac{1}{2}}\eta_n \le x) - \Phi(x)| \le P(N_n^{-\frac{1}{2}}|\eta_n| > \epsilon_n) + \frac{\epsilon_n}{2} \\ &\le \epsilon_n^{-1}E(N_n^{-\frac{1}{2}}|\eta_n|) + \frac{\epsilon_n}{2} = \epsilon_n^{-1}E(N_n^{-\frac{1}{2}})E(|\eta_n|) + \frac{\epsilon_n}{2}. \end{aligned} \tag{33}$$

By (28) and (33) one obtain (25). We complete the proof of Lemma 4.3. □

Lemma 4.4. Let $\{Z_n, n \ge 0\}$ be an IGWVE. If (2)-(5) are satisfied, then there exists constants $\{C_r, r \ge 1\}$ and D such that for any sequence $\{\epsilon_n, n \ge 1\}$ of positive constants,

$$\sup_x |Q_{n,r}(x) - \Phi(x)| \le C_r E\left(Z_n^{-\frac{\delta}{2}}\right) + (\epsilon_n \sigma_{n,r})^{-1}\pi_{n,r} E\left(Z_n^{-\frac{\delta}{2}}\right) + \frac{\epsilon_n}{2}, \tag{34}$$

$$\sup_x |Q_n(x) - \Phi(x)| \le DE(Z_n^{-\frac{\delta}{2}}) + (\epsilon_n \sigma_n)^{-1}\pi_n E\left(Z_n^{-\frac{\delta}{2}}\right) + \frac{\epsilon_n}{2}. \tag{35}$$

Proof. By Lemma 3.1 one has

$$\frac{Z_{n+r} - m_{n,r}Z_n}{\sigma_{n,r}} = \sum_{j=1}^{Z_n} \frac{X_{n,r}^{(j)} - m_{n,r}}{\sigma_{n,r}} + \frac{Y_{n,r}}{\sigma_{n,r}} \quad a.s.,$$

where $\{X_{n,r}^{(j)}; j \ge 1\}$ are i.i.d. and are independent of Z_n and $Y_{n,r}$. Furthermore,

$$E\left(\frac{X_{n,r}^{(j)} - m_{n,r}}{\sigma_{n,r}}\right) \equiv 0, \quad Var\left(\frac{X_{n,r}^{(j)} - m_{n,r}}{\sigma_{n,r}}\right) \equiv 1.$$

Note that $Y_{n,r}$ is independent of Z_n, by Lemma 4.3 and (5) we obtain (34).

Similarly, by Lemma 3.4 one has

$$\frac{Z_n - m_n W}{\sigma_n} = \sum_{j=1}^{Z_n} \frac{1 - V_n^{(j)}}{\sigma_n} + \frac{I_n}{\sigma_n},$$

where $\{V_n^{(j)}; j \geq 1\}$ are i.i.d. and are independent of Z_n and I_n. Furthermore,

$$E\left(\frac{V_n^{(j)} - 1}{\sigma_n}\right) \equiv 0, \qquad Var\left(\frac{V_n^{(j)} - 1}{\sigma_n}\right) \equiv 1.$$

Note that I_n is independent of Z_n, by Lemma 4.3 and (5) we obtain (35). □

Proof of Theorem 1.3. Note that by Lemma 3.1, for any $n \geq 1, r \geq 1$, one has

$$\frac{\pi_{n,r}}{\sigma_{n,r}} = \frac{\sum_{i=n+1}^{n+r} \nu_i m_{i,n+r-i}}{m_{n,r}\sqrt{\sum_{j=n}^{n+r-1} \frac{\delta_j^2}{\mu_j^2 m_{n,j-n}}}} \leq \frac{1}{d} \sum_{i=n+1}^{n+r} \frac{\nu_i}{m_{n,i-n}} \leq \frac{c}{d}$$

and by Lemma 3.4, for any $n \geq 1$ we have

$$\frac{\pi_n}{\sigma_n} = \frac{\sum_{i=n+1}^{\infty} \frac{\nu_i}{m_{i,i-n}}}{\sqrt{\sum_{j=n}^{\infty} \frac{\delta_j^2}{\mu_j^2 m_{n,j-n}}}} \leq \frac{c}{d}.$$

Taking $\epsilon_n = \left[E\left(Z_n^{-\frac{\delta}{2}}\right)\right]^{\frac{1}{2}}$ in (34) and (35), $C = c/d + 1/2$, one obtains Theorem 1.3. □

5 The LIL for IGWVE

In order to prove Theorem 1.4 we need the following lemmas.

Lemma 5.1. For any $\varepsilon > 0$ one has

$$\sum_{n=2}^{\infty} [1 - \Phi((1+\varepsilon)(2\log n)^{\frac{1}{2}})] < \infty \quad \text{and} \quad \sum_{n=2}^{\infty} [1 - \Phi((1-\varepsilon)(2\log n)^{\frac{1}{2}})] = \infty. \tag{36}$$

Proof. For any $\varepsilon > 0$, one has

$$\begin{aligned} 1 - \Phi((1+\varepsilon)(2\log n)^{\frac{1}{2}}) &= \int_{(1+\varepsilon)(2\log n)^{\frac{1}{2}}}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{t^2}{2}} dt \leq \sum_{m=m_0}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{m^2}{2}} \\ &\leq \sum_{i=0}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{m_0(m_0+i)}{2}} = N \sum_{i=0}^{\infty} \left(\frac{1}{\sqrt{e^{m_0}}}\right)^i, \end{aligned} \tag{37}$$

where $m_0 = \left[(1+\varepsilon)(2\log n)^{\frac{1}{2}}\right] - 1$ and $N = \frac{1}{\sqrt{2\pi}} e^{-\frac{m_0^2}{2}}$. When n is large enough one has $\frac{1}{\sqrt{e^{m_0}}} < \frac{1}{2}$, so

$$1 - \Phi((1+\varepsilon)(2\log n)^{\frac{1}{2}}) \leq N \sum_{i=0}^{\infty} \left(\frac{1}{\sqrt{e^{m_0}}}\right)^i \leq 2N = O(n^{-(1+\varepsilon)^2}), \tag{38}$$

so we obtain the first formula of (36).

$$1 - \Phi((1-\varepsilon)(2\log n)^{\frac{1}{2}}) = \int_{(1-\varepsilon)(2\log n)^{\frac{1}{2}}}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{t^2}{2}} dt \geq cn^{-(1-\varepsilon)^2}, \tag{39}$$

where c is some positive constant. We obtain the second formula of (36) by (39). □

Lemma 5.2. Let $\{Z_n; n \geq 0\}$ be an IGWVE, then for any fixed $r \geq 1$, $\{Z_{rn}; n \geq 0\}$ is an IGWVE. If (2)-(5) are satisfied, then

$$\sup_x |Q_{rn,r}(x) - \Phi(x)| \leq C_r E\left(Z_{rn}^{-\frac{\delta}{2}}\right) + C\left[E\left(Z_{rn}^{-\frac{\delta}{2}}\right)\right]^{\frac{1}{2}}, \tag{40}$$

where C_r and C are defined in Theorem 1.3.

Proof. For any fixed $r \geq 1$, let $Z'_n = Z_{rn}$, by Lemma 3.1 one has

$$Z'_{n+1} = Z_{rn+r} = \sum_{j=1}^{Z_{rn}} X_{rn,r}^{(j)} + Y_{rn,r} = \sum_{j=1}^{Z'_n} X_{rn,r}^{(j)} + Y_{rn,r},$$

and for any $n \geq 0$, $\{X_{rn,r}^{(j)}, j \geq 1\}$ are i.i.d. By the proof of Lemma 3.1 one may see that $\{X_{rn,r}^{(j)}, j \geq 1\}$ are independent of $\{Z'_1, Z'_2, \cdots, Z'_n\}$ and $Y_{rn,r}$, $\{Y_{rn,r}, n \geq 0\}$ are independent and for any $n \geq 1$, $Y_{rn,r}$ is independent of Z'_n, so $\{Z_{rn}; n \geq 0\}$ is in fact an IGWVE.

(40) is derived from Theorem 1.3. □

Proof of Theorem 1.4. We only need to prove the lim sup part because of symmetry. Note that (2) and (4) are satisfied by (9). According to Theorem 1.3 and the assumption that $\sum_{n=0}^{\infty}\left[E\left(Z_n^{-\frac{\delta}{2}}\right)\right]^{\frac{1}{2}} < \infty$ one has

$$\sum_{n=1}^{\infty} \sup_x |Q_{n,r}(x) - \Phi(x)| < \infty. \tag{41}$$

By Lemma 5.1 one has that for any $\varepsilon > 0$,

$$\sum_{n=2}^{\infty}\left[1 - \Phi((1+\varepsilon)(2\log n)^{\frac{1}{2}})\right] < \infty,$$

so applying (41) one obtains

$$\sum_{n=2}^{\infty}\left[1 - Q_{n,r}\left((1+\varepsilon)(2\log n)^{\frac{1}{2}}\right)\right] < \infty. \tag{42}$$

Using (42) and Borel-Cantelli Lemma one has

$$\limsup_{n\to\infty} \frac{Z_{n+r} - m_{n,r}Z_n}{(2\sigma_{n,r}^2 Z_n \log n)^{\frac{1}{2}}} \leq 1 \quad a.s.. \tag{43}$$

and

$$\liminf_{n\to\infty} \frac{Z_{n+r} - m_{n,r}Z_n}{(2\sigma_{n,r}^2 Z_n \log n)^{\frac{1}{2}}} \geq -1 \quad a.s.. \tag{44}$$

Fix $r = 1$. For any $0 < \varepsilon < 1, n \geq 2$, define

$$A_n = \left\{Z_n - \mu_{n-1}Z_{n-1} > (1-\varepsilon)\delta_{n-1}(2Z_{n-1}\log(n-1))^{\frac{1}{2}}\right\},$$

we know that $A_n \in \sigma(Z_1, Z_2, \cdots, Z_n)$. Observe that

$$\begin{aligned} P(A_{n+1}|Z_1, Z_2, \cdots, Z_n) &= P\left(\sum_{i=1}^{Z_n}(\xi_{n,i} - \mu_n) + Y_{n+1} > (1-\varepsilon)\delta_n(2Z_n\log n)^{\frac{1}{2}}|Z_n\right) \\ &\geq P\left(\sum_{i=1}^{Z_n}(\xi_{n,i} - \mu_n) > (1-\varepsilon)\delta_n(2Z_n\log n)^{\frac{1}{2}}|Z_n\right) \\ &\geq 1 - \Phi\left((1-\varepsilon)(2\log n)^{\frac{1}{2}}\right) - CZ_n^{-\frac{\delta}{2}}, \end{aligned} \tag{45}$$

where the last inequality is derived from Lemma 4.3 when N_n is a constant. By Lemma 5.1,

$$\sum_{n=2}^{\infty}\left[1-\Phi((1-\varepsilon)(2\log n)^{\frac{1}{2}})\right]=\infty. \tag{46}$$

According to the assumption that $\sum_{n=0}^{\infty}\left[E\left(Z_n^{-\frac{\delta}{2}}\right)\right]^{\frac{1}{2}}<\infty$, (45) and (46) one has

$$\sum_{n=1}^{\infty}P(A_{n+1}|Z_1,Z_2,\cdots,Z_n)=\infty,$$

then by the extension of Borel-Cantelli lemma (cf.[16] Corollary 7.20) one has $P(A_n\ i.o.)=1$, which implies

$$\limsup_{n\to\infty}\frac{Z_{n+1}-\mu_n Z_n}{(2\delta_n^2 Z_n\log n)^{\frac{1}{2}}}\geq 1\quad a.s.. \tag{47}$$

Given $r>1$, we use Z'_n to denote Z_{rn}. Then by Lemma 5.2 one has that $\{Z'_n,n\geq 0\}$ is an IGWVE. Applying the method for the proof of the case $r=1$ to the process $\{Z'_n,n\geq 0\}$,

$$\limsup_{n\to\infty}\frac{Z'_{n+1}-m_{rn,r}Z'_n}{(2\sigma_{rn,r}^2 Z'_n\log n)^{\frac{1}{2}}}\geq 1\quad a.s.,$$

which means

$$\limsup_{n\to\infty}\frac{Z_{rn+r}-m_{rn,r}Z_{rn}}{(2\sigma_{rn,r}^2 Z_{rn}\log rn)^{\frac{1}{2}}}\geq 1\quad a.s..$$

But for any fixed $r>1$,

$$\limsup_{n\to\infty}\frac{Z_{n+r}-m_{n,r}Z_n}{(2\sigma_{n,r}^2 Z_n\log n)^{\frac{1}{2}}}\geq\limsup_{n\to\infty}\frac{Z_{rn+r}-m_{rn,r}Z_{rn}}{(2\sigma_{rn,r}^2 Z_{rn}\log rn)^{\frac{1}{2}}}\geq 1\quad a.s.. \tag{48}$$

Finally we derive (10) from (43) and (48).

According to Theorem 1.3 one has

$$\sum_{n=1}^{\infty}\sup_x|Q_n(x)-\Phi(x)|<\infty, \tag{49}$$

but Lemma 5.1 tells us for any $\varepsilon>0$

$$\sum_{n=2}^{\infty}\left[1-\Phi\left((1+\varepsilon)(2\log n)^{\frac{1}{2}}\right)\right]<\infty,$$

so by (49) one has

$$\sum_{n=2}^{\infty}\left[1-Q_n\left((1+\delta)(2\log n)^{\frac{1}{2}}\right)\right]<\infty.$$

Thus by the Borel-Cantelli Lemma,

$$\limsup_{n\to\infty}\frac{Z_n-m_nW}{(2\sigma_n^2 Z_n\log n)^{\frac{1}{2}}}\leq 1\quad a.s.\quad\text{and}\quad\liminf_{n\to\infty}\frac{Z_n-m_nW}{(2\sigma_n^2 Z_n\log n)^{\frac{1}{2}}}\geq -1\quad a.s.. \tag{50}$$

As for the lower bound, first note that

$$\liminf_{n\to\infty}\frac{m_nW-Z_n}{(2\sigma_n^2 Z_n\log n)^{\frac{1}{2}}}\leq\liminf_{n\to\infty}\frac{m_{n,r}^{-1}Z_{n+r}-Z_n}{(2\sigma_n^2 Z_n\log n)^{\frac{1}{2}}}-\liminf_{n\to\infty}\frac{m_{n,r}^{-1}Z_{n+r}-m_nW}{(2\sigma_n^2 Z_n\log n)^{\frac{1}{2}}}=:I_1-I_2, \tag{51}$$

but by (10) we have

$$I_1 \le -\inf_n \frac{\sigma_{n,r}}{\sigma_n m_{n,r}} \quad a.s., \tag{52}$$

recalling the assumptions that for any $k \ge 1$, $\alpha \le \mu_k \le \beta$, $\gamma^2 \le \delta_k^2 \le \tau^2$ and the definition of $\sigma_{n,r}$ and σ_n one derives

$$1 - K_2\beta^{-r} \ge \frac{\sigma_{n,r}}{\sigma_n m_{n,r}} = 1 - \frac{\sum_{j=n+r}^{\infty} \frac{\delta_j^2}{\mu_j^2 m_{n,j-n}}}{\sum_{j=n}^{\infty} \frac{\delta_j^2}{\mu_j^2 m_{n,j-n}}} \ge 1 - K_1\alpha^{-r}, \tag{53}$$

where K_1 and K_2 are two positive constants. Now we consider I_2. One sees that

$$\frac{m_{n,r}^{-1} Z_{n+r} - m_n W}{(2\sigma_n^2 Z_n \log n)^{\frac{1}{2}}} = \frac{Z_{n+r} - m_{n+r}W}{(2\sigma_{n+r}^2 Z_{n+r} \log(n+r))^{\frac{1}{2}}} \cdot \left[\frac{\frac{Z_{n+r}}{m_{n+r}}}{\frac{Z_n}{m_n}}\right]^{\frac{1}{2}} \cdot \left[\frac{\sigma_{n+r}^2}{m_{n,r}\sigma_n^2}\right]^{\frac{1}{2}}, \tag{54}$$

so according to (54) we have

$$I_2 \ge -\sup_n \left[\frac{\sigma_{n+r}^2}{m_{n,r}\sigma_n^2}\right]^{\frac{1}{2}} \quad a.s., \tag{55}$$

but $\sup_n \frac{1}{m_{n,r}} \le \alpha^{-r}$ and

$$\sup_n \frac{\sigma_{n+r}^2}{\sigma_n^2} \le \frac{\sum_{j=n+r}^{\infty} \frac{\tau^2}{\alpha^{j-n-r+2}}}{\sum_{j=n}^{\infty} \frac{\gamma^2}{\beta^{j-n+2}}} \le \frac{\tau^2}{\gamma^2} =: K_3. \tag{56}$$

Since r is arbitrary, by (51)-(56) one has

$$\liminf_{n\to\infty} \frac{m_n W - Z_n}{(2\sigma_n^2 Z_n \log n)^{\frac{1}{2}}} \le -1 \quad a.s.,$$

which implies

$$\limsup_{n\to\infty} \frac{Z_n - m_n W}{(2\sigma_n^2 Z_n \log n)^{\frac{1}{2}}} \ge 1 \qquad a.s.. \qquad \Box$$

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References

[1] V.I. Afanasyev, On the maximum of a subcritical branching process in a random environment. Stoch. Proc. Appl., 2001, 93:87–107.

[2] V.I. Afanasyev, J. Geiger, G.Kersting and V.A.Vatutin, Functional limit theorems for strongly subcritical branching process in random environment. Stoch. Proc. Appl., 2005, 115:1658–1676.

[3] V.I.Afanasyev, J.Geiger, G.Kersting and V.A.Vatutin, Criticality for branching process in random environment. Ann. Prob., 2005, 33(2):645–673.

[4] R. B. Ash, Real analysis and Probability, New York: Academic Press, 1972.

[5] K.B. Athreya and S.Karlin, Branching processes with random environments, I: extinction probabilities. Ann. Math. Statist., 1971, 42(5):1499–1520.

[6] K.B.Athreya and S. Karlin, Branching processes with random environments, II: limit theorems. Ann. Math. Statist., 1971, 42(6):1843–1858.

[7] Y.S.Chow, H.Teicher (1997). Probability theory (3rd ed.). *Spring-Verlag*, New York.

[8] D. H. Fearn, Galton-Watson processes with generation dependence. Proc. Sixth Berkeley Symp. Math. Statist. Probab., 1971, 2: 159–172.

[9] P.Hall and C. C. Heyde, Martingale limit theory and its application, New York: Academic Press, 1980.

[10] C.C. Heyde, A rate of convergence result for the supercritical Galton-Watson process, J. Appl. Probab., 1970, 7(2): 451–454.

[11] C. C. Heyde, Some central limit analogues for the supercritical Galton-Watson process, J. Appl. Probab., 1971, 8(1): 52–59.

[12] C. C. Heyde, Some almost sure convergence theorems for branching processes, Z. W.Verw.Geb. 1971, 20(1): 189–192.

[13] C. C. Heyde and E. Seneta, Analogues of classical limit theorems for the supercritical Galton-Watson Process with immigration, Math. Biosci., 1971, 11: 249–259.

[14] D.H.Hu, The existence and moments of canonical branching chain in random environment. Acta. Math. Sci., 2004, 24B(3): 499–506.

[15] D.H.Hu, Infinitely dimensional control Markov branching chains in random enviaronments. Science in China: Series A Math., 2006, 49(1): 27–53.

[16] O. Kallenberg, Foundations of modern probability, 2rd ed, Beijing: Science Press, 2006.

[17] T.Lindvall, Almost sure convergence of branching processes in varying and random environments. Ann.Prob. 1974,2(2): 344-346.

[18] D. Richard (2005). Probability: theory and examples (3rd. ed.). *Thomson Brooks/Cole.*

[19] W.G.Wang Bounds of deviation for branching chains in random environments.Acta. Math. Sinica, 2011,27(5): 897-904.

[20] W. G.Wang, Y.Li and D.H.Hu, Extinction of Population-Size-Dependent Branching Chains in Random Environments. Acta. Math. Scientia, 2010,30B(4):1065-1072.

[21] W. G. Wang, P.Lv and D. H.Hu, Harmonic Moments of Branching Processes in Random Environments.[J]Acta Mathematica Sinica, 2009, 25(7): 1087-1096.

[22] J.H.Yu and J.L. Pei, Extinction of branching process in varying enviaronments. Statist. Proba. Lett., 2009, 79: 1872–1877.