Time Periodic Solutions for A Pseudo-parabolic Type Equation with Weakly Nonlinear Periodic Sources

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Abstract

In this paper, we prove the existence of nontrivial nonnegative time periodic solutions for a pseudo-parabolic type equation with weakly nonlinear periodic sources. Moreover, we investigate the asymptotic behavior of solutions as the viscous coefficient k tends to zero.

Keywords: pseudo-parabolic equation; periodic solution; existence; asymptotic behavior

1 Introduction

This paper is concerned with the following one spatial dimensional viscous diffusion equation

$$\frac{\partial u}{\partial t} - k\frac{\partial D^2 u}{\partial t} = D^2u + m(x,t)u^q + f(x,t), \qquad (x,t) \in Q \tag{1.1}$$

subject to the boundary value conditions

$$u(0,t) = u(1,t) = 0, \qquad t \geq 0 \tag{1.2}$$

and the time periodic condition

$$u(x,t+\omega) = u(x,t), \qquad (x,t) \in Q. \tag{1.3}$$

where $Q = (0,1) \times \mathbb{R}^+$, $D = \partial/\partial x$, $0 < q < 1$, $k > 0$ denotes the viscous coefficient, $m(x,t)$ and $f(x,t)$ are positive time periodic functions in $C^{\alpha,\alpha/2}(\overline{Q})$ with period $\omega > 0$ and $\alpha \in (0,1)$. The purpose of this paper is to investigate the existence and asymptotic behavior of the time periodic solutions of the problem (1.1)–(1.3).

Equations of type (1.1) with a one time derivative appearing in the highest order term are called pseudo-parabolic or Sobolev equations, and arise in many areas of mathematics and physics. They have been used, for instance, to model fluid flow in fissured porous media [1], two phase flow in porous media with dynamical capillary pressure [2], heat conduction involving a thermodynamic temperature $\theta = u - k\Delta u$ and a conductive temperature u [3], flow of some non-Newtonian fluids [4], etc. Third order mixed derivatives terms also appear as regularization of forward-backward diffusion equations as in [5], and in the viscous Cahn-Hilliard equation [6].

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Mathematical study of pseudo-parabolic equations goes back to works of Showalter in the seventies [7], since then, numerous of interesting results about linear and nonlinear pseudo-parabolic equations have been obtained. Existence and uniqueness of solutions to nonlinear pseudo-parabolic equations are proved in [8, 9, 10, 11], error estimates for an Euler implicit time discretization scheme for nonlinear pseudo-parabolic equations are also given in [11], and the research for asymptotic behavior of solutions can be found in [12, 13], whereas the existence and uniqueness of solutions for pseudo-parabolic Burgers' equations, including the long time behavior are considered in [14, 15]. Recently, considerable attentions have been paid to the study of propagation problems for pseudo-parabolic equations. In [16], existence, monotonicity and stability of global traveling waves are studied for a pseudo-parabolic Burgers' equation which models non-static groundwater flow. Traveling wave solutions and their relation to non-standard shock solutions to hyperbolic conservation laws are investigated in [17, 18] for the extension Buckley-Leverett equation with a third order mixed derivatives term. Moreover, the small and waiting time behavior of interfaces is analyzed in [19]. Besides above, periodic problems of pseudo-parabolic equations have also been investigated, but most works are devoted to periodic boundary value problems of [20, 21, 22, 23].

For time periodic problems of pseudo-parabolic equations, according to our survey, expect the early works of Matahashi and Tsutsumi, and the recent researches of authors of this paper, there are no other investigations. In [24] and [25], Matahashi and Tsutsumi have established the existence theorems of time periodic solutions for the linear case

$$\frac{\partial u}{\partial t} - \frac{\partial \Delta u}{\partial t} = \Delta u + f(x,t),$$

and the semilinear case

$$\frac{\partial u}{\partial t} - \frac{\partial \Delta u}{\partial t} = \Delta u - |u|^{p-1}u + f(x,t)$$

for $1 < p < 1 + \frac{4}{N}$ with $N = 2, 3, 4$ or $0 < p < 3$ with $N = 1$, respectively. In [26] and [27], the authors investigate the viscous diffusion equation (1.1) with $f = 0$, for one-dimensional case and N-dimensional case, respectively. They obtain a rather complete classification of the exponent q, in terms of the existence and nonexistence of nontrivial and nonnegative periodic solutions. Further, they also investigate the asymptotic behavior of time periodic solutions when $k \to 0$.

In this paper, we are going to deal with the existence of nontrivial time periodic solutions of (1.1) for the case $0 < q < 1$ with positive f. In fact, considering results and technique for both parabolic and pseudo-parabolic equations, there are essential differences between weakly nonlinear sources u^q $(0 < q < 1)$ and strongly nonlinear sources u^q $(q > 1)$. For initial-boundary value problem or Cauchy problem, there exists at least one initial datum such that the solution blows up in a finite time in the case $q > 1$, while there exist global solutions for each initial datum in the case $0 < q < 1$ [28, 29]. For time periodic problem, there exists at least one nontrivial periodic solution in the case $0 < q < 1$, but there may have no nontrivial periodic solution when $q > 1$ [30, 31, 27]. Moreover the discussion using topological degree method for strongly nonlinear source in [26] strongly depends on the assumption $q > 1$ and is invalid for the case $0 < q < 1$. Here we apply Leray-Schauder's fixed point theorem, and prove that there exists a fixed point in a "ball" $\{u;\ \|u\|_{C^{\alpha,\alpha/2}(\overline{Q}_\omega)} \le C\}$ including the original point $u \equiv 0$. Only from the above result, we cannot exclude that the fixed point we obtained is just the trivial one. Besides, it is evident that when $f \equiv 0$, then $u \equiv 0$ is a trivial periodic solution. However due to the positivity of f, we can eliminate the trivial periodic solution.

Actually, in many physical reality, the positive source term f does exist. At this case, the problem is known as inhomogeneous problem, which has significantly difference with homogeneous one, namely $f \equiv 0$. Taking its influence on the asymptotic behavior of solutions for example, inhomogeneous term usually makes the critical Fujita exponent bigger than that of the corresponding homogeneous problem, see [32, 33]. That is mainly caused by the reason that the inhomogeneous term f will strength the energy aggregation which promote blow-up of solutions. To the best of our knowledge, there are few investigations devoted to inhomogeneous semilinear pseudo-parabolic equation. Then the existence of nontrivial nonnegative periodic solutions obtained here reveals that, at least for semilinear pseudo-parabolic equation with weakly nonlinear source u^q $(0 < q < 1)$, the inhomogeneous term f is not strong enough to cause blow-up.

This paper is organized as follows. In Section 2, we prove the existence of time periodic solutions to the problem (1.1)–(1.3). In Section 3, we discuss the asymptotic behavior of the solutions as the viscosity coefficient k tends to zero.

2 Existence of Periodic Solutions

This section is devoted to the solvability of the time periodic problem (1.1)–(1.3). Due to the time periodicity of solutions under consideration, we need only to consider the problem on $Q_\omega = (0,1) \times (0,\omega)$. Throughout the paper, we use standard notations.

Notations (1) Let k be a nonnegative integer and $1 \le p < +\infty$. The set

$$\{u; D^\beta D_t^r u \in L^p(Q_\omega), \text{ for any } \beta \text{ and } r \text{ such that } |\beta| + 2r \le 2k\}$$

endowed with the norm

$$\|u\|_{W_p^{2k,k}(Q_\omega)} = \left(\iint_{Q_\omega} \sum_{|\beta|+2r\le 2k} |D^\beta D_t^r u|^p dxdt \right)^{1/p}$$

is denoted by $W_p^{2k,k}(Q_\omega)$. Specially, $H^{2k,k}(Q_\omega) = W_2^{2k,k}(Q_\omega)$.

(2) For any points $P_1(x_1,t_1), P_2(x_2,t_2) \in Q_\omega$, define the parabolic distance between them as

$$d(P_1,P_2) = (|x_1 - x_2|^2 + |t_1 - t_2|)^{1/2}.$$

Let $v(x,t)$ be a function on Q_ω. For $0 < \alpha < 1$, define

$$[v]_{\alpha,\alpha/2;Q_\omega} = \sup_{P_1,P_2\in Q_\omega, P_1\ne P_2} \frac{|v(P_1) - v(P_2)|}{d^\alpha(P_1,P_2)},$$

which is a semi-norm, and denote by $C^{\alpha,\alpha/2}(\overline{Q}_\omega)$ the set of all functions on Q_ω such that $[v]_{\alpha,\alpha/2;Q_\omega} < +\infty$, endowed with the norm

$$\|v\|_{C^{\alpha,\alpha/2}(\overline{Q}_\omega)} = \|v\|_{0;Q_\omega} + [v]_{\alpha,\alpha/2;Q_\omega},$$

where $\|v\|_{0;Q_\omega}$ is the maximum norm of $v(x,t)$ on Q_ω. For any integer $k \ge 0$, denote the Hölder space

$$C^{2k+\alpha,k+\alpha/2}(\overline{Q}_\omega) = \{u;\ \partial_x^\beta \partial_t^r u \in C^{\alpha,\alpha/2}(\overline{Q}_\omega), \text{ for any } \beta, r \text{ such that } |\beta| + 2r \le 2k\}.$$

The main result of this section is as follows.

Theorem 2.1 *The problem* (1.1)–(1.3) *admits a nontrivial nonnegative time periodic solution* $u \in C^{2+\alpha,1+\alpha/2}(\overline{Q}_\omega)$ *with* $D^2u_t \in C^{\alpha,\alpha/2}(\overline{Q}_\omega)$.

The existence results we obtained are finally for the classical solutions, but due to the proof procedure, we need first to discuss the strong solutions of the problem (1.1)–(1.3).

Definition 2.1 *A function* u *is said to be a strong time periodic solution of the problem* (1.1)–(1.3), *if* $u \in C^{\alpha,\alpha/2}(\overline{Q}_\omega) \cap H^{2,1}(Q_\omega)$ *with* Du_t *and* D^2u_t *in* $L^2(Q_\omega)$, *and satisfies* (1.2), (1.3) *and the following equation*

$$\iint_{Q_\omega} \frac{\partial u}{\partial t}\varphi dxdt - k\iint_{Q_\omega} \frac{\partial D^2 u}{\partial t}\varphi dxdt = \iint_{Q_\omega} D^2u\varphi dxdt + \iint_{Q_\omega} m(x,t)u^q\varphi dxdt + \iint_{Q_\omega} f(x,t)\varphi dxdt,$$

for any $\varphi \in C(\overline{Q}_\omega)$ *with* $\varphi(x,0) = \varphi(x,\omega)$ *and* $\varphi(0,t) = \varphi(1,t) = 0$ *for any* $t \in [0,\omega]$.

Proposition 2.1 *The problem* (1.1)–(1.3) *admits a nontrivial strong time periodic solution.*

To prove the existence of strong solutions, we employ the following Leray-Schauder's fixed point theorem.

Theorem 2.2 (Leray-Schauder's Fixed Point Theorem) *Let* X *be a Banach space,* $F(u,\sigma)$ *be a mapping from* $X \times [0,1]$ *to* X *satisfying the following conditions:*

(i) F *is a compact mapping;*

(ii) $F(u,0) = 0$, $\forall\ u \in X$;

(iii) There exists a constant $M > 0$, *such that for any* $u \in X$, *if* $u = F(u,\sigma)$ *holds for some* $\sigma \in [0,1]$, *then* $\|u\|_X \le M$.

Then the mapping $F(\cdot,1)$ *has a fixed point, that is, there exists* $u \in X$, *such that* $u = F(u,1)$.

In terms of the above theorem, we can study the problem (1.1)–(1.3) by considering the following equation

$$\frac{\partial u}{\partial t} - k\frac{\partial D^2 u}{\partial t} = D^2u + \sigma m(x,t)v^q + \sigma f(x,t), \qquad (x,t) \in Q_\omega \tag{2.1}$$

subject to the conditions (1.2)–(1.3), where σ is a parameter taking value on the interval $[0,1]$, and $v \in C^{\alpha,\alpha/2}(\overline{Q}_\omega)$ is periodic in time t with period ω. Recalling $m(x,t), f(x,t) \in C^{\alpha,\alpha/2}(\overline{Q}_\omega)$, by virtue of the results in [24], [26], we know that the linear equation (2.1) with conditions (1.2) and (1.3) admits a unique classical solution $u \in C^{2+\alpha,1+\alpha/2}(\overline{Q}_\omega)$ with $D^2u_t \in C^{\alpha,\alpha/2}(\overline{Q}_\omega)$, and hence we can define a map F as follows

$$F : C^{\alpha,\alpha/2}(\overline{Q}_\omega) \times [0,1] \to C^{\alpha,\alpha/2}(\overline{Q}_\omega), \quad (v,\sigma) \mapsto u.$$

Since $C^{2+\alpha,1+\alpha/2}(\overline{Q}_\omega)$ can be compactly embedded into $C^{\alpha,\alpha/2}(\overline{Q}_\omega)$, the map F is compact. Obviously, for any given $v \in C^{\alpha,\alpha/2}(\overline{Q}_\omega)$, $F(v,0) = 0$. By virtue of Leray-Schauder's Fixed Point Theorem, to prove the existence of solutions to the problem (1.1)–(1.3), we only need to show that if $u = F(v,\sigma)$ admits a fixed point u_σ in the space $C^{\alpha,\alpha/2}(\overline{Q}_\omega)$ for some $\sigma \in [0,1]$, then $\|u_\sigma\|_{C^{\alpha,\alpha/2}(\overline{Q}_\omega)} \le C$ with C being a constant independent of u_σ and σ. In the following we omit the subscript of u_σ for convenience.

Lemma 2.1 *Let u be a time periodic solution of the equation*

$$\frac{\partial u}{\partial t} - k\frac{\partial D^2 u}{\partial t} = D^2 u + \sigma m(x,t)u^q + \sigma f(x,t), \qquad (x,t) \in Q_\omega \tag{2.2}$$

subject to the conditions (1.2), (1.3), where $\sigma \in [0,1]$. Then

$$\|u\|_{C^{\alpha,\alpha/2}(\overline{Q}_\omega)} \le C, \tag{2.3}$$

where C is a constant independent of the solution u and σ. Moreover, we also have

$$\|u\|_{H^{2,1}(Q_\omega)} + \|Du_t\|_{L^2(Q_\omega)} + \|D^2 u_t\|_{L^2(Q_\omega)} \le C. \tag{2.4}$$

Proof. Multiplying (2.2) by u and integrating the result with respect to x over $(0,1)$, we have

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{dt}}\int_0^1 \left(u^2 + k|Du|^2\right) dx + \int_0^1 |Du|^2 dx =& \sigma \int_0^1 m(x,t)u^{q+1} dx + \sigma \int_0^1 f u dx \\ \le& \frac{1}{2}\int_0^1 |Du|^2 dx + C,\end{aligned} \tag{2.5}$$

where the fact $0 < q < 1$, Young's inequality and Poincaré's inequality have been used. From (2.5), we have that

$$\frac{\mathrm{d}}{\mathrm{dt}}\int_0^1 \left(u^2 + k|Du|^2\right) dx \le C, \qquad \forall t \in (0,\omega). \tag{2.6}$$

Integrating (2.5) over $(0,\omega)$ and noticing the periodicity of u, we get

$$\iint_{Q_\omega} |Du|^2 dxdt \le C,$$

which combining with Poincaré's inequality gives

$$\iint_{Q_\omega} \left(u^2 + k|Du|^2\right) dxdt \le C. \tag{2.7}$$

Set

$$F(t) = \int_0^1 (u^2(x,t) + k|Du(x,t)|^2) dx, \quad \forall t \in [0,\omega].$$

By (2.7), the mean value theorem implies that there exists a point $\tilde{t} \in (0,\omega)$ such that

$$F(\tilde{t}) = \frac{1}{\omega}\int_0^\omega F(t) dt \le C.$$

For any $t \in (\tilde{t},\omega]$, integrating (2.6) from $\tilde{t}$ to t gives

$$F(t) \le C + F(\tilde{t}) \le C, \qquad \forall t \in [\tilde{t},\omega].$$

Noticing the periodicity of $F(t)$, we have

$$F(0) = F(\omega) \le C.$$

Hence, integrating (2.6) over $(0,t)$, we get

$$F(t) \le C, \qquad \forall t \in [0,\omega].$$

Recalling the definition of $F(t)$ and $k>0$, we have

$$\int_0^1 |Du(x,t)|^2 dx \le C, \quad \forall t \in [0,\omega]. \tag{2.8}$$

Noticing that $u(0,t)=0$, there holds

$$|u(x,t)| = \left|\int_0^x Du(y,t)dy\right| \le \left(\int_0^1 |Du(x,t)|^2 dx\right)^{1/2} \le C, \qquad \forall (x,t) \in Q_\omega,$$

which implies that

$$\|u\|_{L^\infty(Q_\omega)} \le C. \tag{2.9}$$

Multiplying (2.2) with D^2u and integrating the result with respect to x over $(0,1)$, by (2.9) and Young's inequality, we have

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{dt}}\int_0^1 \left(|Du|^2 + k|D^2u|^2\right) dx + \int_0^1 |D^2u|^2 dx =& -\sigma\int_0^1 m(x,t)u^{q+1}D^2u dx - \sigma\int_0^1 fD^2u dx \\ \le& \frac{1}{2}\int_0^1 |D^2u|^2 dx + C,\end{aligned}$$

Similar to the above discussion, we can obtain

$$\iint_{Q_\omega} |D^2u|^2 dxdt \le C, \tag{2.10}$$

$$\int_0^1 |D^2u(x,t)|^2 dx \le C, \qquad \forall t \in [0,\omega]. \tag{2.11}$$

From (2.8), by the mean value theorem, we see that there exists a point $\hat{x} \in (0,1)$ such that

$$|Du(\hat{x},t)|^2 = \int_0^1 |Du(x,t)|^2 dx \le C.$$

Then, by (2.11) we have

$$\begin{aligned}|Du(x,t)| \le& |Du(x,t) - Du(\hat{x},t)| + |Du(\hat{x},t)| = \left|\int_{\hat{x}}^x D^2u(y,t)dy\right| + |Du(\hat{x},t)| \\ \le& \left(\int_0^1 |D^2u(x,t)|^2 dx\right)^{1/2} + |Du(\hat{x},t)| \le C, \quad \forall (x,t) \in Q_\omega,\end{aligned}$$

which implies that

$$\|Du\|_{L^\infty(Q_\omega)} \le C. \tag{2.12}$$

Multiplying (2.2) by $\frac{\partial u}{\partial t}$ and integrating the result over $(0,1)$ with respect to x, by (2.9), (2.12) and Young's inequality, we have

$$\begin{aligned}\int_0^1 \left|\frac{\partial u}{\partial t}\right|^2 dx + k\int_0^1 \left|\frac{\partial Du}{\partial t}\right|^2 dx =& -\int_0^1 Du\frac{\partial Du}{\partial t}dx + \sigma\int_0^1 m(x,t)u^q\frac{\partial u}{\partial t}dx + \sigma\int_0^1 f\frac{\partial u}{\partial t}dx\\ \le& \frac{k}{2}\int_0^1 \left|\frac{\partial Du}{\partial t}\right|^2 dx + \frac{1}{2}\int_0^1 \left|\frac{\partial u}{\partial t}\right|^2 dx + C,\end{aligned}$$

from which we have

$$\int_0^1 \left|\frac{\partial u}{\partial t}(x,t)\right|^2 dx \le C, \qquad \int_0^1 \left|\frac{\partial Du}{\partial t}(x,t)\right|^2 dx \le C, \qquad \forall t\in[0,\omega]. \tag{2.13}$$

We rewrite the equation (2.2) into the following form

$$\frac{\partial D^2 u}{\partial t} = \frac{1}{k}\frac{\partial u}{\partial t} - \frac{1}{k}D^2 u - \frac{\sigma}{k}m(x,t)u^q - \frac{\sigma}{k}f(x,t).$$

Recalling $k>0$ and using (2.9), (2.11) and (2.13), we get

$$\int_0^1 \left|\frac{\partial D^2 u}{\partial t}(x,t)\right|^2 dx \le C, \qquad \forall t\in[0,\omega]. \tag{2.14}$$

Now, we claim that

$$|u(x_1,t_1)-u(x_2,t_2)| \le C(|x_1-x_2| + |t_1-t_2|^{1/2}) \tag{2.15}$$

for all $(x_i,t_i)\in\overline{Q}_\omega (i=1,2)$. It is obvious that the above inequality is equivalent to

$$|u(x_1,t)-u(x_2,t)| \le C|x_1-x_2|, \quad \forall t\in[0,\omega], \quad x_1,x_2\in[0,1], \tag{2.16}$$

$$|u(x,t_1)-u(x,t_2)| \le C|t_1-t_2|^{1/2}, \quad \forall x\in[0,1], \quad t_1,t_2\in[0,\omega]. \tag{2.17}$$

In fact, (2.16) is a direct consequence of (2.8). To prove (2.17), it suffices to consider that $0\le x\le 1/2$, $\Delta t = t_2 - t_1 > 0$, $\Delta t \le 1/4$. For any $y\in(x,x+\Delta t)$, integrating the equation (2.2) over $(y,y+\Delta t)\times(t_1,t_2)$ yields

$$\begin{aligned}&\int_y^{y+\Delta t}(u(z,t_2)-u(z,t_1))dz\\ =&k\int_{t_1}^{t_2}\int_y^{y+\Delta t}\frac{\partial D^2 u}{\partial t}(z,t)dzdt + \int_{t_1}^{t_2}\int_y^{y+\Delta t}D^2 u(z,t)dzdt\\ &+\sigma\int_{t_1}^{t_2}\int_y^{y+\Delta t}m(z,t)u^q(z,t)dzdt + \sigma\int_{t_1}^{t_2}\int_y^{y+\Delta t}f(z,t)dzdt.\end{aligned}$$

From (2.9), (2.11), (2.14), it follows that

$$\begin{aligned}&\Delta t\int_0^1 [u(y+\theta\Delta t,t_2)-u(y+\theta\Delta t,t_1)]\,d\theta\\ \le& k\int_{t_1}^{t_2}\left[\left(\int_0^1\left|\frac{\partial D^2 u}{\partial t}\right|^2 dx\right)^{1/2}\left(\int_y^{y+\Delta t}1^2 dx\right)^{1/2}\right]dt\end{aligned}$$

$$+\int_{t_1}^{t_2}\left[\left(\int_0^1 |D^2u|^2dx\right)^{1/2}\left(\int_y^{y+\Delta t}1^2dx\right)^{1/2}\right]dt+C(\Delta t)^2$$
$$\leq C(\Delta t)^{3/2}.$$

Integrating the above equality with respect to y over $(x, x+\Delta t)$, by using the mean value theorem, we get

$$|u(x^*,t_2)-u(x^*,t_1)|\leq C|t_2-t_1|^{1/2},$$

where $x^* = y^* + \theta^*\Delta t$, $y^* \in (x, x+\Delta t)$, $\theta^* \in (0,1)$. Combining the above inequality with (2.16), we have

$$\begin{aligned}|u(x,t_1)-u(x,t_2)| &\leq |u(x,t_1)-u(x^*,t_1)|+|u(x^*,t_1)-u(x^*,t_2)|+|u(x^*,t_2)-u(x,t_2)|\\ &\leq C|t_1-t_2|^{1/2}.\end{aligned}$$

Thus, (2.17) holds. So the estimate (2.3) follows. Moreover, from (2.7), (2.10), (2.13) and (2.14), we also obtain the estimate (2.4). The proof of Lemma 2.1 is complete. □

Proof of Proposition 2.1 From Lemma 2.1, the condition (iii) of Leray-Schauder's Fixed Point Theorem 2.2 can be satisfied. By the arguments which are addressed before Lemma 2.1, we know that the conditions (i) and (ii) of Leray-Schauder's Fixed Point Theorem also hold. Therefore, Leray-Schauder's Fixed Point Theorem implies that the operator F has a fixed point u with $\sigma = 1$. In terms of (2.3) and (2.4) we see that $u \in C^{\alpha,\alpha/2}(\overline{Q}_\omega) \cap H^{2,1}(Q_\omega)$ with $Du_t, D^2u_t \in L^2(Q_\omega)$ is a strong time periodic solution of the problem (1.1)–(1.3). Moreover, by virtue of the positivity of $f(x,t)$, u is obviously nontrivial. The proof of this proposition is complete. □

Proposition 2.2 *If $u \in C^{\alpha,\alpha/2}(\overline{Q}_\omega)$ is a nontrivial strong time periodic solution of*

$$\frac{\partial u}{\partial t}-k\frac{\partial D^2u}{\partial t}=D^2u+m(x,t)|u|^q+f(x,t), \tag{2.18}$$

subject to (1.2), (1.3), *then it is just the nontrivial nonnegative time periodic solution $u \in C^{2+\alpha,1+\alpha/2}$ $(\overline{Q}_\omega)$ with $D^2u_t \in C^{\alpha,\alpha/2}(\overline{Q}_\omega)$.*

Proof. We rewrite the equation (2.18) into the following form

$$\frac{\partial u}{\partial t}+\frac{1}{k}u=(I-kD^2)^{-1}\left[\frac{1}{k}u+m(x,t)|u|^q+f(x,t)\right]. \tag{2.19}$$

As is well known, the operator $(I-kD^2)^{-1}$ is bounded from $C^{\alpha,\alpha/2}(\overline{Q}_\omega)$ to $C^{2+\alpha,\alpha/2}(\overline{Q}_\omega)$. Recalling the strong solution $u \in C^{\alpha,\alpha/2}(\overline{Q}_\omega)$ and the functions $m(x,t), f(x,t) \in C^{\alpha,\alpha/2}(\overline{Q}_\omega)$, we have

$$\frac{\partial u}{\partial t}+\frac{1}{k}u \in C^{2+\alpha,\alpha/2}(\overline{Q}_\omega). \tag{2.20}$$

Multiplying $e^{t/k}$ on both sides of the equation (2.19), we get

$$\frac{\partial}{\partial t}(e^{t/k}u)=e^{t/k}(I-kD^2)^{-1}\left[\frac{1}{k}u+m(x,t)|u|^q+f(x,t)\right].$$

For any $t \in [0, \omega]$, integrating the above equation over $[t, t+\omega]$ and noticing the periodicity of u yield

$$u(x,t) = (e^{(t+\omega)/k} - e^{t/k})^{-1} \int_t^{t+\omega} e^{s/k}(I - kD^2)^{-1} \left[\frac{1}{k}u(x,s) + m(x,s)|u|^q(x,s) + f(x,s)\right] \mathrm{d}s,$$

which with (2.20) imply that

$$u \in C^{2+\alpha, 1+\alpha/2}(\overline{Q}_\omega), \qquad \frac{\partial u}{\partial t} \in C^{2+\alpha, \alpha/2}(\overline{Q}_\omega).$$

Hence, u is a classical solution, which satisfies

$$\frac{\partial u}{\partial t} - (I - kD^2)^{-1}D^2u = (I - kD^2)^{-1}[m(x,t)|u|^q + f(x,t)].$$

Further, we conclude that $u \geq 0$. Suppose to the contrary, there exists a pair of points $(x_0, t_0) \in (0,1) \times (0, \omega)$ such that

$$u(x_0, t_0) < 0.$$

Since u is continuous, then there exists a interval (a, b) which contains x_0 such that $u(x, t_0) < 0$ in (a, b) and $u(a, t_0) = u(b, t_0) = 0$. Multiplying (2.18) by φ, which is the principle eigenfunction of $-D^2$ in (a, b) with homogeneous Dirichlet boundary condition, and integrating the result on (a, b), we can get

$$(1 + k\lambda_r) \int_a^b u_t \varphi dx + \lambda_r \int_a^b u\varphi dx = \int_a^b m(x,t)|u|^q \varphi dx + \int_a^b f(x,t)\varphi dx, \tag{2.21}$$

where λ_r is the first eigenvalue. Integrating the above inequality from 0 to ω and using the periodicity of u, we have

$$\lambda_r \int_0^\omega \int_a^b u\varphi dx dt > 0.$$

By the mean value theorem, there exists a point $t^* \in (0, \omega)$ such that

$$\int_a^b u(x, t^*)\varphi dx > 0.$$

Actually (2.21) is equivalent to

$$\int_a^b \frac{\partial e^{t\lambda_r/(1+k\lambda_r)}u}{\partial t} \varphi dx = \frac{1}{1 + k\lambda_r} \int_a^b e^{t\lambda_r/(1+k\lambda_r)}(m(x,t)|u|^q + f(x,t))\varphi dx. \tag{2.22}$$

Integrating the above inequality from t^* to ω implies that

$$\int_a^b e^{\omega\lambda_r/(1+k\lambda_r)} u(x, \omega)\varphi dx > 0.$$

Recalling the periodicity of u, we see that

$$\int_a^b u(x, 0)\varphi dx > 0.$$

Then integrating (2.22) over $(0,t)$ implies that

$$\int_a^b e^{t\lambda_r/(1+k\lambda_r)}u(x,t)\varphi dx > 0, \qquad t\in(0,\omega),$$

which is contradict with $u(x,t_0)<0$ in (a,b). The proof is complete. □

Proof of Theorem 2.1 Actually, Proposition 2.1 is also valid for (2.18) subject to (1.2) and (1.3). Then using Proposition 2.2, we know that (2.18) subject to (1.2) and (1.3) admits a nontrivial nonnegative classical time periodic solution, namely, we only need to consider the problem throw off the symbol of absolute value of $|u|$. Thus, Theorem 2.1 is just the deduction of Proposition 2.1 and 2.2.

3 Asymptotic Behavior

In this section, we discuss the asymptotic behavior of time periodic solutions as $k\to 0$. Here, we denote by C a constant independent of u and k, $C(k)$ a constant independent of u.

Theorem 3.1 *If u_k is a nontrivial nonnegative time periodic solution of the problem* (1.1)–(1.3), *then $u_k\to u$ uniformly in Q_ω as $k\to 0$, and the limit function $u\in C^{1/2,1/4}(\overline{Q}_\omega)\cap H^{2,1}(Q_\omega)$ is a nontrivial nonnegative weak periodic solution of the following problem*

$$\frac{\partial u}{\partial t}=D^2u+m(x,t)u^q+f(x,t), \qquad (x,t)\in Q_\omega, \tag{3.1}$$

$$u(0,t)=u(1,t)=0, \qquad t\in[0,\omega], \tag{3.2}$$

$$u(x,\omega)=u(x,0), \qquad x\in[0,1]. \tag{3.3}$$

Proof. Multiplying u_k for (1.1) and integrating the result with respect to x over $(0,1)$ yield

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_0^1 (u_k^2+k|Du_k|^2)dx\le C, \quad \forall t\in(0,\omega), \tag{3.4}$$

$$\iint_{Q_\omega}|Du_k|^2dxdt\le C. \tag{3.5}$$

From (3.5), the Poincaré inequality gives

$$\iint_{Q_\omega}(u_k^2+k|Du_k|^2)dxdt\le C+C(k). \tag{3.6}$$

Combining (3.4) with (3.6), we can deduce that

$$\int_0^1 u_k^2(x,t)dx\le C+C(k), \quad \forall t\in[0,\omega]. \tag{3.7}$$

Multiplying (1.1) with D^2u_k and integrating the result with respect to x over $(0,1)$, by using (3.7), we get

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_0^1 (|Du_k|^2+k|D^2u_k|^2)dx\le C+C(k), \quad \forall t\in(0,\omega), \tag{3.8}$$

$$\iint_{Q_\omega} |D^2 u_k|^2 dxdt \le C + C(k). \tag{3.9}$$

From (3.5) and (3.9), we have

$$\iint_{Q_\omega} (|Du_k|^2 + k|D^2 u_k|^2) dxdt \le C + C(k).$$

Combining the above inequality with (3.8), we can deduce that

$$\int_0^1 |Du_k(x,t)|^2 dx \le C + C(k), \qquad \forall t \in [0,\omega],$$

from which and the boundary value conditions (1.2), we can obtain

$$\|u_k\|_{L^\infty(Q_\omega)} \le C + C(k). \tag{3.10}$$

Multiplying (1.1) by $\dfrac{\partial u_k}{\partial t}$ and integrating the result over Q_ω, by (3.10) and noticing the periodicity of u_k, we have

$$\begin{aligned}\iint_{Q_\omega} \left|\frac{\partial u_k}{\partial t}\right| dxdt + k\iint_{Q_\omega} \left|\frac{\partial Du_k}{\partial t}\right| dxdt =& \iint_{Q_\omega} [m(x,t)u_k^q + f(x,t)]\frac{\partial u_k}{\partial t} dxdt \\ \le& \frac{1}{2}\iint_{Q_\omega} \left|\frac{\partial u_k}{\partial t}\right| dxdt + C + C(k).\end{aligned}$$

It follows that

$$\iint_{Q_\omega} \left|\frac{\partial u_k}{\partial t}\right|^2 dxdt \le C + C(k). \tag{3.11}$$

Recalling the equation (1.1), by (3.9)–(3.12), we get

$$\iint_{Q_\omega} \left|k\frac{\partial D^2 u_k}{\partial t}\right|^2 dxdt \le C + C(k). \tag{3.12}$$

Similar to the proof in Proposition 2.1, we can prove that

$$|u_k(x_1,t_1) - u_k(x_2,t_2)| \le (C + C(k))(|x_1 - x_2|^{1/2} + |t_1 - t_2|^{1/4})$$

for all $(x_i,t_i) \in \overline{Q}_\omega (i = 1,2)$. Therefore, there exists a function $u \in C^{\mu,\mu/2}(\overline{Q}_\omega) \cap H^{2,1}(Q_\omega)$ with $\mu \in (0,1/2)$ such that

$$\begin{aligned} &u_k \to u && \text{uniformly in } \ Q_\omega,\\ &\frac{\partial u_k}{\partial t} \rightharpoonup \frac{\partial u}{\partial t}, \quad D^2 u_k \rightharpoonup D^2 u && \text{weakly in } L^2(Q_\omega),\end{aligned}$$

as $k \to 0$. Recalling the equation (1.1), for any $\varphi \in C^{2,0}(\overline{Q}_\omega)$ satisfying $\varphi(x,\omega) = \varphi(x,0)$ and $\varphi(0,t) = \varphi(1,t) = 0$ for any $t \in [0,\omega]$, we have

$$\iint_{Q_\omega} \frac{\partial u_k}{\partial t}\varphi dxdt - k\iint_{Q_\omega} \frac{\partial u_k}{\partial t} D^2\varphi dxdt = \iint_{Q_\omega} D^2 u_k \varphi dxdt + \iint_{Q_\omega} m(x,t)u_k^q \varphi dxdt.$$

Taking $k \to 0$, we have

$$\iint_{Q_\omega} \frac{\partial u}{\partial t}\varphi dxdt = \iint_{Q_\omega} D^2 u\varphi dxdt + \iint_{Q_\omega} m(x,t)u^q\varphi dxdt,$$

which implies that $u \in C^{\mu,\mu/2}(\overline{Q}_\omega) \cap H^{2,1}(Q_\omega)$ satisfies the equation (3.1) in the sense of distribution. It is evident that u satisfies (3.2) and (3.3). Furthermore, noticing that f is positive and u_k is nonnegative, u is obviously nontrivial and nonnegative. The proof is complete. □

Final Remark. In fact, when D^2u is replaced by a p-Laplacian term $D(|Du|^{p-2}Du)$, the global existence and asymptotic behavior of solutions for initial-boundary value problem are obtained in [36]. A more interesting problem is that is the periodicity preserved for such nonlinear pseudo-parabolic model with periodic coefficients? We guess it is true, but to prove it, there needs to get more precise estimates and find suitable space for solutions.

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