AN ITERATIVE METHOD FOR EQUILIBRIUM, VARIATIONAL INEQUALITY AND FIXED POINT PROBLEMS FOR A NONEXPANSIVE SEMIGROUP IN HILBERT SPACES

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Abstract. The purpose of this paper is to present a new iteration method based on the hybrid method in mathematical programming, extragradient method and Mann's method for finding a common element of the solution set of equilibrium problems, the solution set of variational inequality problems for a monotone, Lipschitz continuous mapping and the set of fixed points for a nonexpansive semigroup in Hilbert spaces. We obtain a strong convergence theorem for the sequences generated by this process. The results in this paper generalize and extend some well-known strong convergence theorems in the literature.

1. Introduction

Let H be a real Hilbert space with inner product $\langle ., . \rangle$ and norm $\|.\|$, respectively. Let C be a nonempty closed convex subset of H. Let A be a mapping of C into H and $G : C \times C \to \mathbb{R}$ be a bifunction, where $\mathbb{R}$ is the set of real numbers.

Recall that a mapping A is called monotone if

$$\langle Ax - Ay, x - y \rangle \geq 0 \text{ for all } x, y \in C,$$

strictly monotone if $\langle Ax-Ay, x-y \rangle > 0$ for all $x \neq y$, λ-inverse strongly monotone mapping if

$$\langle Ax - Ay, x - y \rangle \geq \lambda \|Ax - Ay\|^2 \text{ for all } x, y \in C, \ \lambda > 0,$$

and L-Lipschitz continuous if there exists a positive constant L such that

$$\|Ax - Ay\| \leq L\|x - y\| \text{ for all } x, y \in C.$$

It is clear that if A is λ-inverse strongly monotone, then A is monotone and Lipschitz continuous.

The variational inequality problem (for short, VI$(A; C)$) is to find $x^* \in C$ such that

$$\langle Ax^*, x - x^* \rangle \geq 0 \text{ for all } x \in C. \tag{1.1}$$

The set of solutions of the VI$(C; A)$ is denoted by Ω_A. Due to the many applications of the variational inequality problem to several branches of mathematics, but also to mechanics, economics etc, finding its solutions is a very important

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field of research. In some cases, as for strictly monotone operators A, the solution, if it exists, is unique. More generally the set of solutions Ω_A of a continuous monotone mapping A is a convex subset of C.

Numerous problems in physics, optimization, and economics reduce to find a solution of the equilibrium problem which is for a bifunction $G(u, v)$ defined on $C \times C$ to find $u^* \in C$ such that

$$G(u^*, v) \geq 0 \text{ for all } v \in C. \tag{1.2}$$

The set of solutions of (1.2) is denoted by $\mathrm{EP}(G)$. Given a mapping $B : C \to H$, let $G(u, v) = \langle Bu, v - u \rangle$ for all $u, v \in C$. Then, $w \in \mathrm{EP}(G)$ if and only if $\langle Bw, v - w \rangle \geq 0$ for all $v \in C$, i.e., w is a solution of the variational inequality. Some methods have been proposed to solve the equilibrium problem (1.2) (see [1]-[7]). Recently, Combettes and Hirstoaga [7] introduced an iterative scheme of finding the best approximation to the initial data when $\mathrm{EP}(G)$ is nonempty and proved a strong convergence theorem.

Let $T : C \to C$ be a mapping. Recall that T is nonexpansive if $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in C$. A point $x \in C$ is a fixed point of T provided $Tx = x$. Denote by $\mathrm{F}(T)$ the set of fixed points of T, that is, $\mathrm{F}(T) = \{x \in C : Tx = x\}$. We know that $\mathrm{F}(T)$ is nonempty if C is bounded (see [8]). We denote by $\mathcal{R}_+$ the set of nonegative numbers. Also, recall that a family $\{T(s) : s \in \mathcal{R}_+\}$ of mapping from C into itself is called a nonexpansive semigroup on C if it satisfies the following conditions:

(1) for each $s \in \mathcal{R}_+, T(s)$ is a nonexpansive mapping on C;
(2) $T(0)x = x$ for all $x \in C$;
(3) $T(s_1 + s_2) = T(s_1) \circ T(s_2)$ for all $s_1, s_2 \in \mathcal{R}_+$;
(4) for each $x \in C$, the mapping $T(.)x$ from $\mathcal{R}_+$ into C is continuous.

We denote by $\mathcal{F} = \cap_{s \geq 0} \mathrm{F}(T(s))$ the set of all common fixed points of $\{T(s) : s \in \mathcal{R}_+\}$. We know that $\mathcal{F}$ is nonempty if C is bounded (see [9]).

Takahashi and Toyoda [10] considered the problem of finding a solution of the variational inequality which is also a fixed point of some mapping. More precisely, given a nonempty closed convex subset C of H, a nonexpansive mapping $T : C \to C$ and an λ-inverse strongly monotone mapping $A : C \to H$, in order to find an element $p \in \mathrm{F}(T) \cap \Omega_A$ they introduced the following iterative scheme

$$\begin{aligned} &x_0 \in C \text{ chosen arbitrarily}, \\ &x_{k+1} = \alpha_k x_k + (1 - \alpha_k) T P_C (x_k - \lambda_k A x_k), \end{aligned} \tag{1.3}$$

for all $k \geq 0$, where $\{\alpha_k\}$ is a sequence in $(0, 1)$ and $\{\lambda_k\}$ is a sequence in $(0, 2\lambda)$ and P_C is the metric projection of H onto C. They proved that if $\mathrm{F}(T) \cap \Omega_A \neq \emptyset$, then the sequence $\{x_k\}$ defined by (1.3) converges weakly to same point $p \in \mathrm{F}(T) \cap \Omega_A$. Later on, in order to achieve strong convergence to an element of $\mathrm{F}(T) \cap \Omega_A$ under the same assumptions, Iiduka and Takahashi [11] modified the iterative scheme by using the hybrid method in mathematical programming as follows

$$\begin{aligned}
x_0 &\in C \text{ chosen arbitrarily},\\
y_k &= \alpha_k x_k + (1-\alpha_k) T P_C(x_k - \lambda_k A x_k),\\
C_k &= \{z \in C : \|z - y_k\| \le \|z - x_k\|\},\\
Q_k &= \{z \in C : \langle z - x_k, x_0 - x_k\rangle \le 0\},\\
x_{k+1} &= P_{C_k\cap Q_k}(x_0),
\end{aligned} \tag{1.4}$$

for all $k \ge 0$, where $0 \le \alpha_k \le c < 1$ and $0 < a \le \lambda_k \le b < 2\lambda$. They showed that if $\mathrm{F}(T) \cap \Omega_A \ne \emptyset$, then the sequence $\{x_k\}$ defined by (1.4) converges strongly to $P_{F(T)\cap\Omega_A}(x_0)$. To overcome the restriction of the above methods to the class of λ-inverse strongly monotone mappings, by combining a hybrid-type method with an extragradient-type method of Korpelevich [12], Nadezhkina and Takahashi [13] introduced the following iterative method for finding an element of $\mathrm{F}(T) \cap \Omega_A$ and established the strong convergence theorem under Lipschitz and monotone assumptions of the mapping A:

$$\begin{aligned}
x_0 &\in C \text{ chosen arbitrarily},\\
y_k &= P_C(x_k - \lambda_k A x_k),\\
z_k &= \alpha_k x_k + (1-\alpha_k) T P_C(x_k - \lambda_k A y_k),\\
C_k &= \{z \in C : \|z - z_k\| \le \|z - x_k\|\},\\
Q_k &= \{z \in C : \langle z - x_k, x_0 - x_k\rangle \le 0\},\\
x_{k+1} &= P_{C_k\cap Q_k}(x_0), k \ge 0,
\end{aligned} \tag{1.5}$$

where $k \ge 0$, $\{\alpha_k\} \subset [a, b]$ for some $a, b \in (0, 1/L)$ and $\{\lambda_k\} \subset [0, c]$ for some $c \in [0, 1)$. They proved that if $\mathrm{F}(T) \cap \Omega_A \ne \emptyset$, then the sequence $\{x_k\}$, $\{y_k\}$, $\{z_k\}$ defined by (1.5) converge strongly to the same point $z = P_{F(T)\cap\Omega_A}(x_0)$.

Tada and Takahashi [14] introduced the following iterative scheme by the hybrid method:

$$\begin{aligned}
x_0 &\in H \text{ chosen arbitrarily},\\
u_k &\in C,\ G(u_k, y) + \frac{1}{r_k}\langle u_k - x_k, y - u_k\rangle \ge 0,\ \forall y \in C,\\
y_k &= (1-\alpha_k)x_k + \alpha_k T u_k,\\
C_k &= \{z \in H : \|z_k - z\| \le \|x_k - z\|\},\\
Q_k &= \{z \in H : \langle x_k - z, x_0 - x_k\rangle \ge 0\},\\
x_{k+1} &= P_{C_k\cap Q_k}(x_0),\ k \ge 0,
\end{aligned} \tag{1.6}$$

for finding a common element of the set of solution (1.2) and the set of fixed points of a nonexpansive mapping in a Hilbert. They proved that under certain appropriate conditions imposed on $\{\alpha_k\}$ and $\{r_k\}$, the sequences $\{x_k\}$ and $\{u_k\}$ generated by (1.6) converge strongly to $P_{\mathrm{F}(T)\cap \mathrm{EP}(Q)}x_0$. Generally speaking, the algorithm suggested by Tada and Takahashi is based on two well-known types of methods, namely, on the Mann iterative methods and the so-called hybrid for solving fixed point problem.

In 2002, Suzuki [15] was the first to introduce the following implicit iteration process in Hilbert spaces:

$$
\begin{aligned}
&x_0 \in C \text{ chosen arbitrarily}, \\
&x_k = \alpha_k x_0 + (1-\alpha_k)T(t_k)x_k,\ k \geq 1,
\end{aligned} \tag{1.7}
$$

where $\{\alpha_k\}$ and $\{t_k\}$ are sequences of real numbers satisfying $0 < \alpha_k < 1$, $t_k > 0$ and $\lim_k t_k = \lim_k \alpha_k/t_k = 0$ for the nonexpansive semigroup. If $\mathcal{F} \neq \emptyset$, then the sequence $\{x_k\}$ defined by (1.7) converges strongly to the element of $\mathcal{F}$ nearest to x_0.

He and Chen [16] is proved a strong convergence theorem for nonexpansive semigroups in Hilbert spaces by hybrid method in the mathematical programming:

$$
\begin{aligned}
x_0 &\in C \text{ chosen arbitrarily}, \\
y_k &= \alpha_k x_k + (1-\alpha_k)T(t_k)x_k, \\
C_k &= \{z \in C : \|y_k - z\| \leq \|x_k - z\|\}, \\
Q_k &= \{z \in C : \langle x_k - z, x_0 - x_k \rangle \geq 0\}, \\
x_{k+1} &= P_{C_k \cap Q_k}(x_0),\ k \geq 0,
\end{aligned} \tag{1.8}
$$

where $\alpha_k \in [0, a)$ for some $a \in [0, 1)$ and $t_k \geq 0$, $\lim_{k\to\infty} t_k = 0$.

In 2008, Seajung [17] showed that the proof of the main result in [16] is very questionable and corrected this fact under some additional restriction on the parameter t_k:

$$
\liminf_k t_k = 0,\ \limsup_k t_k > 0, \quad \text{and} \quad \lim_k (t_{k+1} - t_k) = 0. \tag{1.9}
$$

In 2011, Buong [18] solved the problem of finding a common element of the set of solution (1.1) and the set of common fixed points of a nonexpansive semigroup $\{T(s),\ s \in \mathcal{R}_+\}$ on C for any monotone Lipschitz continuous mapping A by the following algorithm

$$
\begin{aligned}
x_0 &\in H \text{ chosen arbitrarily}, \\
y_k &= P_C(x_k - \lambda_k A P_C(x_k)), \\
z_k &= x_k - \mu_k[x_k - T_k P_C(x_k - \lambda_k A y_k)], \\
H_k &= \{z \in H : \|z_k - z\| \leq \|x_k - z\|\}, \\
W_k &= \{z \in H : \langle x_k - x_0, z - x_k \rangle \leq 0\}, \\
x_{k+1} &= P_{H_k \cap W_k}(x_0),\ k \geq 0,
\end{aligned} \tag{1.10}
$$

where $\{\lambda_k\} \subset [a, b]$ for some $a, b \in [0, 1/L]$, $\{\mu_k\} \subset [c, 1]$ for some $c \in (0, 1)$ and $\{t_k\}$ is a sequence of positive real numbers satisfying condition (1.9) and $T_k x = T(t_k)x$ for $x \in C$. If $\mathcal{F} \cap \Omega_A \neq \emptyset$, then the sequences $\{x_k\}$, $\{y_k\}$, $\{z_k\}$ defined by (1.10) converge strongly to the same point $z_0 = P_{\mathcal{F}\cap\Omega_A}(x_0)$.

On the other hand, in 2011, Buong and Duong [19] introduced a viscosity approximation method for finding a common element of the set of solution (1.1) and the set of solution (1.2). Ceng and Yao [20] investigated the problem of finding a common element of the set of solutions of a mixed equilibrium problem and the set of common fixed points of finitely many nonexpansive mappings in a Hilbert space. The authors's result is the improvements and extension of Takahashi and

Takahashi [21]. Plubtieng and Punpaeng [22] introduced an iterative process based on the extragradient method for finding the common element of the set of fixed points of a nonexpansive mapping, the set of an equilibrium problem and the set of solutions of variational inequality problem for an λ-inverse strongly monotone mapping. In [23] Penga and Yao introduced two iterative process by the extragradient-like methods for finding a common element of the set of solutions of a generalized equilibrium problem, the set of fixed points of an infinite family of nonexpansive mappings and the set of solutions of the variational inequality for a monotone, Lipschitz-continuous mapping in a Hilbert space.

In this paper, motivated the above results we give a new algorithm for finding a common element of the set of solutions of an equilibrium problem, the set of fixed points of a nonexpansive semigroup and the set of solutions of the variational inequality for a monotone, Lipschitz continuous mapping in a Hilbert space.

For this purpose, we consider the following algorithm

$$\begin{aligned} & x_0 \in H \text{ chosen arbitrarily}, \\ & u_k \in C: \ G(u_k, y) + \frac{1}{r_k}\langle u_k - x_k, y - u_k\rangle \geq 0, \ \forall y \in C, \\ & y_k = P_C(u_k - \lambda_k A u_k), \\ & z_k = (1-\mu_k)x_k + \mu_k T_k P_C(u_k - \lambda_k A y_k), \\ & H_k = \{z \in H : \|z_k - z\| \leq \|x_k - z\|\}, \\ & W_k = \{z \in H : \langle x_k - z, x_0 - x_k\rangle \geq 0\}, \\ & x_{k+1} = P_{H_k \cap W_k}(x_0), \ k \geq 0, \end{aligned} \tag{1.11}$$

where T_k is defined

$$\begin{aligned} & T_k x = T(s_k)x, \ \forall x \in C \text{ and} \\ & \liminf_k s_k = 0, \ \limsup_k s_k > 0, \ \lim_k (s_{k+1} - s_k) = 0, \end{aligned} \tag{1.12}$$

or T_k is defined by

$$T_k x = \frac{1}{s_k}\int_0^{s_k} T(s)x ds, \ \forall x \in C \text{ and } \lim_{k\to\infty} s_k = +\infty, \tag{1.13}$$

respectively. The strong convergence of (1.11) with (1.12) or (1.13) is proved in the Section 3. In Section 2, we give some preliminaries.

2. Preliminaries

Let H be a real Hilbert space with inner product $\langle .,. \rangle$ and norm $\|.\|$, respectively. When $\{x_n\}$ is a sequence in H, $x_n \rightharpoonup x$ implies that $\{x_n\}$ converges weakly to x and $x_n \to x$ means the strong convergence.

We need the following facts to prove our results.

Lemma 2.1. [24] *Let H be a real Hilbert space. Then:*
(i) $\|x+y\|^2 = \|x\|^2 + \|y\|^2 + 2\langle x, y\rangle$;
(ii) $\|tx+(1-t)y\|^2 = t\|x\|^2 + (1-t)\|y\|^2 - t(1-t)\|x-y\|^2, \ \forall t \in [0,1], \ \forall x, y \in H$;
(iii) $\|x-y\|^2 \geq \|x - P_C(x)\|^2 + \|y - P_C(x)\|^2$ *for any* $x \in H$ *and for all* $y \in C$, *where* C *is a nonempty closed convex subset in* H.

Lemma 2.2. [24] *Let C be a nonempty closed convex subset of a real Hilbert space H. For any $x \in H$, there exists a unique $z \in C$ such that $\|z-x\| \le \|y-x\|$ for all $y \in C$, and*

$$z \in P_C(x) \text{ if and only if } \langle z-x, y-z\rangle \ge 0 \text{ for all } y \in C, \tag{2.1}$$

where P_C is the metric projection of H onto C.

Let A be a monotone mapping of C into H. In the context of the variational inequality problem, the characterization of a projection in Lemma 2.2 implies the following:

$$u \in \Omega_A \Leftrightarrow u = P_C(u - kAu),\ k > 0.$$

Lemma 2.3. [25] *Every Hilbert space H has Randon-Riesz property or Kadec-Klee property, that is, for a sequence $\{x_n\} \subset H$ with $x_n \rightharpoonup x$ and $\|x_n\| \to \|x\|$, then there hodls $x_n \to x$.*

A set-valued mapping $B : H \to 2^H$ is called monotone if for all $x, y \in H$, $f \in Bx$ and $g \in By$ imply $\langle f-g, x-y\rangle \ge 0$. A monotone mapping $B : H \to 2^H$ is maximal if its graph $\mathrm{Gr}(B)$ of B is not properly contained in the graph of any other monotone mapping. It is known that a monotone mapping B is maximal if and only if for $(x, f) \in H \times H$, $\langle f-g, x-y\rangle \ge 0$ for every $(y, g) \in \mathrm{Gr}(B)$ implies $f \in Bx$. Let A be a monotone, λ-Lipschitz continuous mapping of C into H and let $N_C x$ be normal cone to C at $x \in C$, i.e. $N_C x = \{y \in H : \langle y, x-u\rangle \ge 0,\ \forall u \in C\}$. Define

$$Bx = \begin{cases} Ax + N_C x, & \text{if } x \in C \\ \emptyset, & \text{if } x \notin C. \end{cases}$$

Then B is maximal monotone and $0 \in Bx$ if and only if $x \in \Omega_A$ (see [26]).

For solving the equilibrium problem for a bifunction $G : C \times C \to \mathbb{R}$, assume that G satisfies the following set of standard properties:

$(A1)$ $G(u, u) = 0$ for all $u \in C$;

$(A2)$ G is monotone, i.e., $G(u, v) + G(v, u) \le 0$ for all $(u, v) \in C \times C$;

$(A3)$ For every $u \in C$, $G(u, .) : C \to \mathbb{R}$ is weakly lower semicontinuous and convex;

$(A4)$ $\overline{\lim}_{t\to+0} G((1-t)u + tz, v) \le G(u, v)$ for all $(u, z, v) \in C \times C \times C$.

The following lemma appears in [1].

Lemma 2.4. [1] *Let C be a nonempty closed convex subset of H and G be a bifunction of $C \times C$ into $\mathbb{R}$ satisfying conditions $(A1)$-$(A4)$. Let $r > 0$ and $x \in H$. Then, there exists $z \in C$ such that*

$$G(z, v) + \frac{1}{r}\langle z-x, v-z\rangle \ge 0 \text{ for all } v \in C. \tag{2.2}$$

The following lemma was also given in [7].

Lemma 2.5. [7] *Assume that $G : C \times C \to \mathbb{R}$ satisfies conditions $(A1)$-$(A4)$. For $r > 0$ and $x \in H$, define a mapping $T^r : H \to C$ as follows:*

$$T^r(x) = \Big\{u \in C : G(u, v) + \frac{1}{r}\langle u-x, v-u\rangle \ge 0,\ \forall v \in C\Big\}. \tag{2.3}$$

Then, the following statements hold:

(*i*) T^r *is single-valued;*
(*ii*) T^r *is firmly nonexpansive, i.e., for any* $x, y \in H$,

$$\|T^r(x) - T^r(y)\|^2 \leq \langle T^r(x) - T^r(y), x - y\rangle;$$

(*iii*) $\mathrm{F}(T^r) = \mathrm{EP}(G)$;
(*iv*) $\mathrm{EP}(G)$ *is closed and convex.*

Lemma 2.6. [27] *Let C be a nonempty bounded closed convex subset of H and let $\{T(t) : t \in \mathcal{R}_+\}$ be a nonexpansive semigroup on C. Then, for any $h \geq 0$*

$$\lim_{t\to\infty} \sup_{y\in C} \left\| T(h)\left(\frac{1}{t}\int_0^t T(s)yds\right) - \frac{1}{t}\int_0^t T(s)yds \right\| = 0. \tag{2.4}$$

Lemma 2.7. [25] (Demiclosedness Principle) *If C is a closed convex subset of H, T is a nonexpansive mapping on C, $\{x_n\}$ is a sequence in C such that $x_n \rightharpoonup x \in C$ and $x_n - Tx_n \to 0$, then $x - Tx = 0$.*

It is also known that H satisfies Opial's condition. See following definition in [28].

Definition 2.8. *A Banach space X is said to satisfy Opial's condition if whenever $\{x_k\}$ is a sequence in X which converges weakly to x, as $k \to \infty$, then*

$$\limsup_{k\to\infty} \|x_k - x\| < \limsup_{k\to\infty} \|x_k - y\|, \ \forall y \in X, \quad \text{with } x \neq y.$$

3. Main Results

Now, we are in a position to prove the following results.

Theorem 3.1. *Let C be a nonempty closed convex subset in a real Hilbert space H, $\{T(s) : s \in \mathcal{R}_+\}$ be a nonexpansive semigroup on C, G be a bifunction from $C \times C$ to $\mathbb{R}$ satisfying conditions (A1)-(A4), and $A : C \to H$ be a monotone L-Lipschitz continuous mapping such that $\mathcal{F} \cap \mathrm{EP}(G) \cap \Omega_A \neq \emptyset$. Let $\{x_k\}$, $\{u_k\}$, $\{y_k\}$ and $\{z_k\}$ be sequences generated by (1.11) for every $k \geq 0$, where $\{\mu_k\} \subset [a, 1]$ for some $a \in (0, 1)$, $\{r_k\} \subset (0, \infty)$ satisfies $\liminf_{k\to\infty} r_k > 0$, $\{\lambda_k\} \subset [b, c]$ for some $b, c \in (0, 1/\sqrt{2}L)$ and T_k with $\{s_k\}$ satisfies (1.12) or (1.13). Then, $\{x_k\}$, $\{u_k\}$, $\{y_k\}$ and $\{z_k\}$ converge strongly to an element $p \in \mathcal{F} \cap \mathrm{EP}(G) \cap \Omega_A$.*

Proof. First, we consider the case that T_k with $\{s_k\}$ satisfies (1.12).

It is obvious that H_k is closed and W_k is closed and convex for every $k \geq 0$. It follows that H_k is convex for every $k \geq 0$ because $\|z - z_k\| \leq \|z - x_k\|$ is equivalent to

$$\langle z_k - x_k, x_k - z\rangle \leq -\frac{1}{2}\|z_k - x_k\|^2, \tag{3.1}$$

so, $H_k \cap W_k$ is closed and convex for every $k \geq 0$. So that the $\{x_k\}$ is well defined for every $k \geq 0$.

We have $\mathcal{F} \cap \mathrm{EP}(G) \cap \Omega_A \subset H_k \cap W_k$ for every $k \geq 0$. Indeed, for each $u \in \mathcal{F} \cap \mathrm{EP}(G) \cap \Omega_A$, by putting $u_k = T^{r_k}x_k$ and using Lemma 2.5 we have that

$$\|u_k - u\| = \|T^{r_k}x_k - T^{r_k}u\| \leq \|x_k - u\|. \tag{3.2}$$

Putting $t_k = P_C(u_k - \lambda_k A y_k)$ for every $k \geq 0$. Using (i) in Lemma 2.1, (2.1) in Lemma 2.2 with $x = u_k - \lambda_k A y_k$ and $y = u$, it follows from the monotonicity of A and $u \in \Omega_A$ we obtain

$$\begin{aligned}
\|t_k - u\|^2 &\leq \|u_k - \lambda_k A y_k - u\|^2 - \|u_k - \lambda_k A y_k - t_k\|^2 \\
&\leq \|u_k - u\|^2 - \|u_k - t_k\|^2 + 2\lambda_k \langle A y_k, u - t_k \rangle \\
&= \|u_k - u\|^2 - \|u_k - t_k\|^2 + 2\lambda_k [\langle A y_k - Au, u - y_k \rangle \\
&\qquad + \langle Au, u - y_k \rangle + \langle A y_k, y_k - t_k \rangle] \\
&\leq \|u_k - u\|^2 - \|u_k - t_k\|^2 + 2\lambda_k \langle A y_k, y_k - t_k \rangle \\
&= \|u_k - u\|^2 - \|u_k - y_k\|^2 - 2\langle u_k - y_k, y_k - t_k \rangle \\
&\qquad - \|y_k - t_k\|^2 + 2\lambda_k \langle A y_k, y_k - t_k \rangle \\
&= \|u_k - u\|^2 - \|u_k - y_k\|^2 - \|y_k - t_k\|^2 \\
&\qquad + 2\langle u_k - \lambda_k A y_k - y_k, t_k - y_k \rangle.
\end{aligned} \tag{3.3}$$

Since $y_k = P_C(u_k - \lambda_k A u_k)$ in (1.11), A is L-Lipschitz continuous and (2.1) we have

$$\begin{aligned}
2\langle u_k - \lambda_k A y_k - y_k, t_k - y_k \rangle &= 2\langle u_k - \lambda_k A u_k - y_k, t_k - y_k \rangle \\
&\qquad + 2\lambda_k \langle A u_k - A y_k, t_k - y_k \rangle \\
&\leq 2\lambda_k \langle A u_k - A y_k, t_k - y_k \rangle \\
&\leq 2\lambda_k L \|u_k - y_k\| \|y_k - t_k\|.
\end{aligned} \tag{3.4}$$

Using monotonicity of A, $\{\lambda_k\} \in (0, 1/\sqrt{2}L)$ and P_C is a nonexpansive mapping, it follows from (3.3) and (3.4) that

$$\begin{aligned}
\|t_k - u\|^2 &\leq \|u_k - u\|^2 - \|u_k - y_k\|^2 - \|y_k - t_k\|^2 \\
&\qquad + 2\lambda_k L \|u_k - y_k\| \|y_k - t_k\| \\
&\leq \|u_k - u\|^2 - \|u_k - y_k\|^2 \\
&\qquad + 2\lambda_k L \|u_k - y_k\| \|P_C(u_k - \lambda_k A u_k) - P_C(u_k - \lambda_k A y_k)\| \\
&\leq \|u_k - u\|^2 + (2\lambda_k^2 L^2 - 1)\|u_k - y_k\|^2 \\
&\leq \|u_k - u\|^2.
\end{aligned} \tag{3.5}$$

By the convexity of $\|.\|^2$, properties of P_C and T_k, it follows from (3.2) and (3.5) that

$$\begin{aligned}
\|z_k - u\|^2 &= \|(1 - \mu_k)(x_k - u) + \mu_k (T_k t_k - u)\|^2 \\
&\leq (1 - \mu_k)\|x_k - u\|^2 + \mu_k \|T_k t_k - T_k u\|^2 \\
&\leq (1 - \mu_k)\|x_k - u\|^2 + \mu_k \|t_k - u\|^2 \\
&\leq (1 - \mu_k)\|x_k - u\|^2 + \mu_k \|u_k - u\|^2 \\
&\leq (1 - \mu_k)\|x_k - u\|^2 + \mu_k \|x_k - u\|^2 \\
&\leq \|x_k - u\|^2, \ \forall k \geq 0.
\end{aligned} \tag{3.6}$$

It follows from (3.6) that $\|z_k - u\| \leq \|x_k - u\|$, so $u \in H_k$. Hence $\mathcal{F} \cap \mathrm{EP}(G) \cap \Omega_A \subset H_k$ for all $k \geq 0$.

Next we show $\mathcal{F}\cap$ EP$(G)\cap\Omega_A \subset H_k\cap W_k$ for all $k \geq 0$. Indeed, in the case that $k = 0$, we have $x_0 \in C$ and $W_0 = H$. Consequently, $\mathcal{F}\cap$EP$(G) \cap \Omega_A \subset H_0 \cap W_0$. Suppose that x_i is given and $\mathcal{F}\cap$ EP$(G) \cap \Omega_A \subset H_i \cap W_i$ for some $i \geq 0$. We have to prove that $\mathcal{F}\cap$ EP$(G) \cap \Omega_A \subset H_{i+1} \cap W_{i+1}$. Since $\mathcal{F}\cap$ EP$(G) \cap \Omega_A$ is nonempty closed convex subset of H. So there exists a unique element $x_{i+1} \in \mathcal{F}\cap$ EP$(G)\cap\Omega_A$ such that $x_{i+1} = P_{\mathcal{F}\cap EP(G)\cap\Omega_A}(x_0)$. By Lemma 2.2, we have for every $z \in \mathcal{F}\cap$ EP$(G) \cap \Omega_A$ that

$$\langle x_{i+1} - z, x_0 - x_{i+1}\rangle \geq 0, \tag{3.7}$$

and hence $z \in W_{i+1}$. Finally, $z \in H_{i+1}\cap W_{i+1}$ and the $\mathcal{F}\cap$ EP$(G)\cap\Omega_A \subset H_k\cap W_k$ holds for all $k \geq 0$.

Next, we shall show that the $\{x_k\}$ generated by (1.11) is bounded.

Since $\mathcal{F}\cap$ EP$(G) \cap \Omega_A$ is a nonempty closed convex subset of C, there exists a unique element $z_0 \in \mathcal{F}\cap$ EP$(G) \cap \Omega_A$ such that $z_0 = P_{\mathcal{F}\cap EP(G)\cap\Omega_A}(x_0)$. Now, from $x_{k+1} = P_{H_k\cap W_k}(x_0)$ we obtain that

$$\|x_{k+1} - x_0\| \leq \|z - x_0\|, \ \forall z \in H_k \cap W_k. \tag{3.8}$$

As $z_0 \in \mathcal{F}\cap$ EP$(G) \cap \Omega_A \subset H_k \cap W_k$, we get

$$\|x_{k+1} - x_0\| \leq \|z_0 - x_0\|,$$

for each $k \geq 0$. Hence, the sequence $\{x_k\}$ is bounded.

We shall show that $\{x_k\}$, $\{u_k\}$, $\{y_k\}$ and $\{z_k\}$ converge strongly to an element $p \in \mathcal{F}\cap$ EP$(G)\cap\Omega_A$. Since $x_k = P_{W_k}(x_0)$ and $x_{k+1} \in W_k$, it follows from Lemma 2.2 that,

$$\|x_k - x_0\| \leq \|x_{k+1} - x_0\|, \tag{3.9}$$

for all $k \geq 0$. Then, there exists $\lim_{k\to\infty} \|x_k - x_0\| = c$. Since $x_k = P_{W_k}(x_0)$ and $x_{k+1} \in W_k$, from (ii) in Lemma 2.1 we have

$$\begin{aligned} \|x_k - x_0\|^2 &\leq \|\frac{x_k + x_{k+1}}{2} - x_0\|^2 \\ &\leq \|\frac{x_k - x_0}{2} + \frac{x_{k+1} - x_0}{2}\|^2 \\ &\leq \frac{\|x_k - x_0\|^2}{2} + \frac{\|x_{k+1} - x_0\|^2}{2} - \frac{\|x_k - x_{k+1}\|^2}{4}. \end{aligned} \tag{3.10}$$

So, we get

$$\|x_k - x_{k+1}\|^2 \leq 2(\|x_{k+1} - x_0\|^2 - \|x_k - x_0\|^2). \tag{3.11}$$

Since $\lim_{k\to\infty} \|x_k - x_0\| = c$, we obtain

$$\lim_{k\to\infty} \|x_k - x_{k+1}\| = 0. \tag{3.12}$$

From $x_{k+1} \in H_k$, we have

$$\|z_k - x_k\| \leq \|x_k - x_{k+1}\| + \|x_{k+1} - z_k\| \leq 2\|x_k - x_{k+1}\|. \tag{3.13}$$

It follows from (3.12) and (3.13) that

$$\lim_{k\to\infty} \|z_k - x_k\| = 0. \tag{3.14}$$

Now from (3.6) we can write

$$\|z_k - u\|^2 - \|x_k - u\|^2 \leq \mu_k[\|T_k t_k - u\|^2 - \|x_k - u\|^2] \leq 0. \tag{3.15}$$

On the other hand, by Lemma 2.1 we have

$$\|z_k - u\|^2 - \|x_k - u\|^2 = \|z_k - x_k\|^2 + 2\langle z_k - x_k, x_k - u\rangle. \tag{3.16}$$

It follows from (3.14)-(3.16) that

$$\lim_{k\to\infty} \mu_k[\|T_k t_k - u\|^2 - \|x_k - u\|^2] = 0. \tag{3.17}$$

Since $\{\mu_k\} \subset [a, 1]$ for some $a \in (0, 1)$, we have that

$$\lim_{k\to\infty} [\|T_k t_k - u\|^2 - \|x_k - u\|^2] = 0. \tag{3.18}$$

By (3.2), (3.5), (3.18) and the nonexpansive property of T_k, we get

$$0 = \lim_{k\to\infty} [\|T_k t_k - u\|^2 - \|x_k - u\|^2] \leq \lim_{k\to\infty} [\|t_k - u\|^2 - \|x_k - u\|^2] \leq 0. \tag{3.19}$$

Therefore,

$$\lim_{k\to\infty} [\|t_k - u\|^2 - \|x_k - u\|^2] = 0. \tag{3.20}$$

On the other hand, from (ii) in Lemma 2.5 we have for every $u \in \mathcal{F} \cap EP(G) \cap \Omega_A$ that

$$\begin{aligned} \|u_k - u\|^2 &= \|T^{r_k} x_k - T^{r_k} u\|^2 \\ &\leq \langle T^{r_k} x_k - T^{r_k} u, x_k - u\rangle \\ &= \langle u_k - u, x_k - u\rangle \\ &\leq \frac{1}{2}[\|u_k - u\|^2 + \|x_k - u\|^2 - \|u_k - x_k\|^2]. \end{aligned} \tag{3.21}$$

Thus,

$$\|u_k - u\|^2 \leq \|x_k - u\|^2 - \|u_k - x_k\|^2. \tag{3.22}$$

By the convexity of $\|.\|^2$, the properties of P_C and T_k, it follows from (3.6) and (3.22) that

$$\begin{aligned} \|z_k - u\|^2 &\leq (1 - \mu_k)\|x_k - u\|^2 + \mu_k\|u_k - u\|^2 \\ &\leq (1 - \mu_k)\|x_k - u\|^2 + \mu_k[\|x_k - u\|^2 - \|u_k - x_k\|^2] \\ &\leq \|x_k - u\|^2 - \mu_k\|u_k - x_k\|^2. \end{aligned} \tag{3.23}$$

Again, since $\mu_k \in [a, 1]$ for some $a \in (0, 1)$ and (3.23) we have

$$\begin{aligned} a\|u_k - x_k\|^2 &\leq \|x_k - u\|^2 - \|z_k - u\|^2 \\ &\leq (\|x_k - u\| + \|z_k - u\|)\|z_k - x_k\|. \end{aligned} \tag{3.24}$$

This together with (3.14) and the condition on $\{r_k\}$ implies that

$$\lim_{k\to\infty} \|u_k - x_k\| = 0 \text{ and } \lim_{k\to\infty} \frac{\|u_k - x_k\|}{r_k} = 0. \tag{3.25}$$

As $\{x_k\}$ is bounded, there exists a subsequence $\{x_{k_j}\}$ of $\{x_k\}$ converging weakly to some element p. From (3.25), we obtain also that $\{u_{k_j}\}$ converges weakly to p. Since $\{u_{k_j}\} \subset C$ and C is a closed convex subset in H, we obtain $p \in C$.

Now, we shall show that $p \in \mathcal{F} \cap$ EP$(G) \cap \Omega_A$.

First, we shall prove that $p \in \mathrm{EP}(G)$. By $u_k = T^{r_k} x_k$, we have

$$G(u_k, y) + \frac{1}{r_k}\langle u_k - x_k, y - u_k\rangle \geq 0, \ \forall y \in C. \tag{3.26}$$

It follows from condition $(A2)$ and (3.26) that

$$\frac{1}{r_k}\langle u_k - x_k, y - u_k\rangle \geq G(y, u_k), \ \forall y \in C. \tag{3.27}$$

Therefore,

$$\langle \frac{u_{k_j} - x_{k_j}}{r_{k_j}}, y - u_{k_j}\rangle \geq G(y, u_{k_j}), \ \forall y \in C. \tag{3.28}$$

From condition $(A3)$, (3.25) and (3.28), we have

$$0 \geq G(y, p), \ \forall y \in C. \tag{3.29}$$

So, $G(p, y) \geq 0$, for all $y \in C$. It means that $p \in \mathrm{EP}(G)$.

Further, we show that $p \in \Omega_A$. Set $Bv = Av + N_C v$ for $v \in C$ where

$$N_C v = \{w \in H : \langle v - u, w\rangle \geq 0, \ \forall u \in C\} \tag{3.30}$$

and $Bv = \emptyset$ for $v \notin C$. Then B is a maximal monotone mapping and $0 \in Bv$ if and only if $v \in \Omega_A$ (see [26]). Let $(v, w) \in G(B)$. Then we have $w \in Bv = Av + N_C v$ and $w - Av \in N_C v$ which is equivalent to

$$\langle v - u, w - Av\rangle \geq 0, \ \forall u \in C. \tag{3.31}$$

Consequently, from $t_k = P_C(u_k - \lambda_k A y_k)$, $v \in C$ and Lemma 2.2, we have that

$$\langle t_k - v, u_k - \lambda_k A y_k - t_k\rangle \geq 0. \tag{3.32}$$

Therefore,

$$\langle v - t_k, (t_k - u_k)/\lambda_k + A y_k\rangle \geq 0. \tag{3.33}$$

It follows from (3.31) and monotonicity of A that

$$\begin{aligned}
\langle v - t_{k_i}, w\rangle &\geq \langle v - t_{k_i}, Av\rangle \\
&\geq \langle v - t_{k_i}, Av\rangle - \langle v - t_{k_i}, (t_{k_i} - u_{k_i})/\lambda_{k_i} + A y_{k_i}\rangle \\
&\geq \langle v - t_{k_i}, Av - At_{k_i}\rangle + \langle v - t_{k_i}, At_{k_i} - Ay_{k_i}\rangle \\
&\quad - \langle v - t_{k_i}, (t_{k_i} - u_{k_i})/\lambda_{k_i}\rangle \\
&\geq \langle v - t_{k_i}, At_{k_i} - Ay_{k_i}\rangle - \langle v - t_{k_i}, (t_{k_i} - u_{k_i})/\lambda_{k_i}\rangle.
\end{aligned} \tag{3.34}$$

From (3.2) and (3.5) we obtain

$$(1 - 2\lambda_k^2 L^2)\|u_k - y_k\|^2 \leq \|x_k - u\|^2 - \|t_k - u\|^2. \tag{3.35}$$

It follows from (3.20), (3.35) and the condition $\{\lambda_k\} \subset (0, 1/\sqrt{2}L)$ that

$$\lim_{k\to\infty} \|u_k - y_k\| = 0. \tag{3.36}$$

Since $y_k = P_C(u_k - \lambda_k A u_k)$, $t_k = P_C(u_k - \lambda_k A y_k)$, it follows from (3.36) and properties of P_C and A that

$$\lim_{k\to\infty} \|y_k - t_k\| = 0, \tag{3.37}$$

and

$$\lim_{k\to\infty}\|Ay_k - At_k\| = \lim_{k\to\infty}\|y_k - t_k\| = 0. \tag{3.38}$$

Hence, after passing $i \to \infty$ in (3.34), using (3.36), (3.37) and (3.38) we obtain that $\langle v - p, w\rangle \geq 0$ for all $v \in C$. Since B is maximal monotone, $p \in B^{-1}0$. It means that $p \in \Omega_A$.

Next we show that $p \in \mathcal{F}$.

By using properties of P_C and T_k and $u_k \in C$, it follows from (1.11) that

$$\begin{aligned} a\|u_k - T_k u_k\| &\leq \mu_k\|u_k - T_k u_k\| \leq \mu_k\big(\|u_k - T_k t_k\| + \|T_k t_k - T_k u_k\|\big) \\ &= \|(1-\mu_k)P_C(x_k) + \mu_k P_C(u_k) - P_C(u_k) + u_k - z_k\| \\ &\quad + \mu_k\|t_k - u_k\| \\ &\leq (1-\mu_k)\|P_C(x_k) - P_C(u_k)\| + \|u_k - z_k\| \\ &\quad + \mu_k\|t_k - u_k\| \\ &\leq (1-\mu_k)\|x_k - u_k\| + \|u_k - z_k\| + \mu_k\|t_k - u_k\| \\ &\leq \|x_k - u_k\| + \|u_k - x_k\| + \|x_k - z_k\| + \mu_k\|t_k - u_k\| \\ &\leq 2\|x_k - u_k\| + \|x_k - z_k\| + \mu_k\|t_k - u_k\|. \end{aligned} \tag{3.39}$$

Therefore, from (3.14), (3.25), (3.36) and (3.37) it implies that

$$\lim_{k\to\infty}\|u_k - T_k u_k\| = 0. \tag{3.40}$$

From (3.40) and as in [16], without loss of generality, let

$$\lim_{j\to\infty} s_{k_j} = 0;\ \lim_{j\to\infty}\frac{\|u_{k_j} - T_{k_j}u_{k_j}\|}{s_{k_j}} = 0. \tag{3.41}$$

Now, we prove that $p = T(s)p$ for a fixed $s > 0$. It is easy to see that

$$\begin{aligned} \|u_{k_j} - T(s)p\| \leq & \sum_{l=0}^{[s/s_{k_j}]-1}\|T(ls_{k_j})u_{k_j} - T((l+1)s_{k_j})u_{k_j}\| \\ & + \left\|T\left(\left[\frac{s}{s_{k_j}}\right]s_{k_j}\right)u_{k_j} - T\left(\left[\frac{s}{s_{k_j}}\right]s_{k_j}\right)p\right\| + \left\|T\left(\left[\frac{s}{s_{k_j}}\right]s_{k_j}\right)p - T(s)p\right\| \\ & \leq \left[\frac{s}{s_{k_j}}\right]\|u_{k_j} - T(s_{k_j})u_{k_j}\| + \|u_{k_j} - p\| + \left\|T\left(s - \left[\frac{s}{s_{k_j}}\right]s_{k_j}\right)p - p\right\|. \end{aligned} \tag{3.42}$$

Therefore,

$$\begin{aligned} \|u_{k_j} - T(s)p\| \leq & \frac{s}{s_{k_j}}\|u_{k_j} - T(s_{k_j})u_{k_j}\| \\ & + \|u_{k_j} - p\| + \sup\{\|T(s)p - p\| : 0 \leq s \leq s_{k_j}\}. \end{aligned} \tag{3.43}$$

This fact and (3.41) imply that

$$\limsup_{j\to\infty}\|u_{k_j} - T(s)p\| \leq \limsup_{j\to\infty}\|u_{k_j} - p\|. \tag{3.44}$$

As every Hilbert space satisfies Opial's condition, we have $T(s)p = p$. Therefore, $p \in \mathcal{F}$. Thus, (3.8) with z replaced by $z_0 = P_{\mathcal{F}\cap EP(G)\cap\Omega_A}(x_0)$ and the weakly lower semicontinuity of the norm guarantee that

$$\begin{aligned}\|x_0 - z_0\| \le \|x_0 - p\| &\le \liminf_{j\to\infty} \|x_0 - x_{k_j}\| \\ &\le \limsup_{j\to\infty} \|x_0 - x_{k_j}\| \le \|x_0 - z_0\|.\end{aligned} \tag{3.45}$$

Hence, we obtain

$$\lim_{j\to\infty} \|x_{k_j} - x_0\| = \|x_0 - p\| = \|x_0 - z_0\|. \tag{3.46}$$

It means that

$$x_{k_j} \to p = z_0, \tag{3.47}$$

and all sequence $\{x_k\}$ converges strongly to p as $k \to \infty$. So, the strong convergence of the sequences $\{z_k\}$ and $\{u_k\}$ to z_0 is followed from (3.14) and (3.25), respectively. The strong convergence of the sequences $\{y_k\}$ is followed from the property of $\{u_k\}$ and (3.36).

For the case that T_k is defined by (1.13), we need only to prove $p \in \mathcal{F}$ from (3.40). For this purpose, we have for each $h > 0$ the following estimate:

$$\begin{aligned}\|T(h)u_k - u_k\| \le &\left\| T(h)u_k - T(h)\left(\frac{1}{s_k}\int_0^{s_k} T(s)u_k ds\right)\right\| \\ &+ \left\| T(h)\left(\frac{1}{s_k}\int_0^{s_k} T(s)u_k ds\right) - \frac{1}{s_k}\int_0^{s_k} T(s)u_k ds\right\| \\ &+ \left\| \frac{1}{s_k}\int_0^{s_k} T(s)u_k ds - u_k\right\| \\ \le 2&\left\| \frac{1}{s_k}\int_0^{s_k} T(s)u_k ds - u_k\right\| \\ &+ \left\| T(h)\left(\frac{1}{s_k}\int_0^{s_k} T(s)u_k ds\right) - \frac{1}{s_k}\int_0^{s_k} T(s)u_k ds\right\|.\end{aligned} \tag{3.48}$$

By Lemma 2.6, we get

$$\lim_{k\to\infty} \left\| T(h)\left(\frac{1}{s_k}\int_0^{s_k} T(s)u_k ds\right) - \frac{1}{s_k}\int_0^{s_k} T(s)u_k ds\right\| = 0, \tag{3.49}$$

for every $h \in (0, \infty)$ and hence, by (3.40), (3.48) and (3.49), we obtain

$$\lim_{k\to\infty} \|T(h)u_k - u_k\| = 0 \tag{3.50}$$

for each $h > 0$. By Lemma 2.7, this implies $p \in F(T(h))$ for all $h > 0$. As for the case (1.12), we also obtain that the sequences $\{x_k\}$, $\{u_k\}$, $\{y_k\}$ and $\{z_k\}$ defined by (1.11) with (1.13) converge strongly to p as $k \to \infty$. □

Putting $T(s) = T$ for all $s > 0$ we obtain the strong convergence for equilibrium problems, variational inequalities problems and fixed point problems for nonexpansive in Hilbert spaces.

Corollary 3.2. *Let C be a nonempty closed convex subset in a real Hilbert space H. Let T be a nonexpansive mapping on C, let G be a bifunction from $C \times C$ to $\mathbb{R}$ satisfying conditions (A1)-(A4), and let $A : C \to H$ be a monotone L-Lipschitz* *continuous mapping such that* $F(T)\cap\ EP(G)\cap\Omega_A\neq\emptyset$*. Let* $\{x_k\}$*,* $\{u_k\}$*,* $\{y_k\}$ *and* $\{z_k\}$ *be sequences generated by*

$$
\begin{aligned}
&x_0\in H \text{ chosen arbitrarily},\\
&u_k\in C:\ G(u_k,y)+\frac{1}{r_k}\langle u_k-x_k,y-u_k\rangle\geq 0,\ \forall y\in C,\\
&y_k=P_C(u_k-\lambda_k Au_k),\\
&z_k=(1-\mu_k)x_k+\mu_k TP_C(u_k-\lambda_k Ay_k),\\
&H_k=\{z\in H:\|z_k-z\|\leq\|x_k-z\|\},\\
&W_k=\{z\in H:\langle x_k-z,x_0-x_k\rangle\geq 0\},\\
&x_{k+1}=P_{H_k\cap W_k}(x_0),\ k\geq 0,
\end{aligned}
\tag{3.51}
$$

where $\{\mu_k\}\subset[a,1]$ *for some* $a\in(0,1)$ *and* $\{r_k\}\subset(0,\infty)$ *satisfies* $\liminf_{k\to\infty}r_k>0$*,* $\{\lambda_k\}\subset[b,c]$ *for some* $b,c\in(0;1/\sqrt{2}L)$*. Then, the sequences* $\{x_k\},\{u_k\},\{y_k\}$ *and* $\{z_k\}$ *converge strongly to an element* $p\in F(T)\cap\ EP(G)\cap\Omega_A$*.*

Corollary 3.3. *Let* C *be a nonempty closed convex subset in a real Hilbert space* H*. Let* $\{T(s):s\in\mathcal{R}_+\}$ *be a nonexpansive semigroup on* C *and let* $A:C\to H$ *be a monotone L-Lipschitz continuous mapping such that* $\mathcal{F}\cap\Omega_A\neq\emptyset$*. Let* $\{x_k\}$*,* $\{u_k\}$*,* $\{y_k\}$ *and* $\{z_k\}$ *be sequences generated by*

$$
\begin{aligned}
&x_0\in H \text{ chosen arbitrarily},\\
&u_k=P_C(x_k),\\
&y_k=P_C(u_k-\lambda_k Au_k),\\
&z_k=(1-\mu_k)u_k+\mu_k T_k P_C(u_k-\lambda_k Ay_k),\\
&H_k=\{z\in H:\|z_k-z\|\leq\|x_k-z\|\},\\
&W_k=\{z\in H:\langle x_k-z,x_0-x_k\rangle\geq 0\},\\
&x_{k+1}=P_{H_k\cap W_k}(x_0),\ k\geq 0,
\end{aligned}
\tag{3.52}
$$

where $\{\mu_k\}\subset[a,1]$ *for some* $a\in(0,1)$*,* $\{r_k\}\subset(0,\infty)$ *satisfies* $\liminf_{k\to\infty}r_k>0$*,* $\{\lambda_k\}\subset[b,c]$ *for some* $b,c\in(0;1/\sqrt{2}L)$ *and* T_k *is defined by (1.12) or (1.13). Then, the sequences* $\{x_k\}$*,* $\{u_k\}$*,* $\{y_k\}$ *and* $\{z_k\}$ *converge strongly to an element* $p\in\mathcal{F}\cap\Omega_A$*.*

Proof. Obviously, if $G(u,v)\equiv 0$ then u_k is defined by

$$\langle u_k-x_k,y-u_k\rangle\geq 0,\ \forall y\in C$$

which is equivalent to $u_k=P_C(x_k)$. So, the conclusion of Corollary 3.3 is proved similar Theorem 3.1. □

Putting $G(u,v)\equiv 0$ for all $u,v\in C$ and $A\equiv 0$, we obtain the following algorithm for finding a common fixed point of a nonexpansive semigroup $\{T(s):s\in\mathcal{R}_+\}$ on C.

Corollary 3.4. *Let* C *be a nonempty closed convex subset in a real Hilbert space* H*. Let* $\{T(s):s\in\mathcal{R}_+\}$ *be a nonexpansive semigroup on* C *such that* $\mathcal{F}\neq\emptyset$*. Let*

$\{x_k\}, \{u_k\}$ and $\{z_k\}$ be sequences generated by

$$(3.53)\qquad \begin{aligned} x_0 &\in H \text{ chosen arbitrarily,}\\ u_k &= P_C(x_k),\\ z_k &= (1-\mu_k)u_k + \mu_k T_k u_k,\\ H_k &= \{z \in H : \|z_k - z\| \le \|x_k - z\|\},\\ W_k &= \{z \in H : \langle x_k - z, x_0 - x_k\rangle \ge 0\},\\ x_{k+1} &= P_{H_k \cap W_k}(x_0), k \ge 0, \end{aligned}$$

where $\{\mu_k\} \subset [a,1]$ for some $a \in (0,1)$, $T_k = T(s_k)$ and $\{s_k\}$ satisfies condition (1.12). Then, the sequences $\{x_k\}, \{u_k\}$ and $\{z_k\}$ converge strongly to an element $p \in \mathcal{F}$.

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