A new Extension of Generalized Hermite Matrix Polynomials

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Abstract

The Hermite matrix polynomials have been generalized in a number of ways and many of these generalizations have been shown to be important tools in applications. In this paper we introduce a new generalization of the Hermite matrix polynomials and present the recurrence relations and the expansions of these new generalized Hermite matrix polynomials. We also give new series expansions of the matrix functions $\exp(xB)$, $\sin(xB)$, $\cos(xB)$, $\cosh(xB)$ and $\sinh(xB)$ in terms of these generalized Hermite matrix polynomials and thus prove that many of the seemingly different generalizations of the Hermite matrix polynomials may be viewed as particular cases of the two-variable polynomials introduced here. The generalized Chebyshev and Legendre matrix polynomials have also been introduced in this paper in terms of these generalized Hermite matrix polynomials.

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1 Introduction

The theory of generalized matrix Hermite polynomials has earlier been developed by Batahan [1], Defez and Jódar [2], Jódar and Company [7], Jódar and Defez [11, 12], Khammash [14] and Sayyed, Metwally and Batahan [18] and more recently by the second author [15, 16]. Important connections between orthogonal matrix polynomials and matrix differential equations of second order appear in [3, 4, 7, 8, 9, 10, 13, 19], and their orthogonality properties in [20].

The aim of the present paper is to introduce a new generalization of the Hermite matrix polynomials. An explicit representation and an expansion of the matrix exponential in a series of these matrix polynomials are obtained. The properties of the generalized Hermite matrix polynomials of two variables such as the recurrence formulas, which permit an efficient computation of matrix functions, are also established. By exploiting this family of matrix polynomials, we give the definition of the generalized Chebyshev and Legendre matrix polynomials.

Throughout this paper for a matrix A in $\mathbb{C}^{N\times N}$, its spectrum $\sigma(A)$ denotes the set of all the eigenvalues of A. We say that A is a positive stable matrix [5, 11] if

$$Re(z) > 0, \quad \text{for all } z \in \sigma(A). \tag{1.1}$$

If $A(k,n)$ and $B(k,n)$ are matrices on $\mathbb{C}^{N\times N}$ for $n \geq 0$, $k \geq 0$, it follows in an analogous way to the

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proof of Lemma **11** and **10** of Rainville [17] that

$$\begin{aligned}\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}A(k,n)&=\sum_{n=0}^{\infty}\sum_{k=0}^{[\frac{n}{m}]}A(k,n-mk),\\ \sum_{n=0}^{\infty}\sum_{k=0}^{\infty}B(k,n)&=\sum_{n=0}^{\infty}\sum_{k=0}^{n}B(k,n-k)\end{aligned}\tag{1.2}$$

for a positive integer m and similarly, as for the above equation, we can write

$$\begin{aligned}\sum_{n=0}^{\infty}\sum_{k=0}^{[\frac{n}{m}]}A(k,n)&=\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}A(k,n+mk),\\ \sum_{n=0}^{\infty}\sum_{k=0}^{n}B(k,n)&=\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}B(k,n+k).\end{aligned}\tag{1.3}$$

The next section is devoted to the theory of generalized Hermite matrix polynomials, treated within the context of the point of view so far developed.

2 Generalized Hermite matrix polynomials

Let A be a positive stable matrix in $\mathbb{C}^{N\times N}$ satisfying (1.1). We define the generalized Hermite matrix polynomials of two variables by the generating function

$$F(x,y,t,A)=\sum_{n=0}^{\infty}\frac{t^n}{n!}H_{n,m,p}(x,y,A)=\exp\left(xt^p\sqrt{mA}-yt^mI\right),\quad |t|<\infty \text{ and } |x|<\infty \tag{2.1}$$

where $F(x,y,t,A)$ regarded as a function of the complex variable t is an entire matrix function, therefore has the Taylor series about $t=0$ and the series obtained converges for all values of x, y and t with p and m being relatively prime integers. Using the first equation in (1.2), and letting $n\to\frac{n}{p}+mk-\frac{mk}{p}$ we get

$$\begin{aligned}\exp\left(xt^p\sqrt{mA}-yt^mI\right)&=\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\frac{(-1)^ky^k(x\sqrt{mA})^n}{n!k!}t^{np+mk}\\ &=\sum_{n=0}^{\infty}\sum_{k=0}^{[\frac{n}{m}]}\frac{(-1)^ky^k(x\sqrt{mA})^{\frac{n-mk}{p}}}{k!\Gamma\left(\frac{n-mk}{p}+1\right)}t^n.\end{aligned}\tag{2.2}$$

Thus, from (2.1) and (2.2), we obtain an explicit representation for the generalized Hermite matrix polynomials of two variables in the form

$$H_{n,m,p}(x,y,A)=\quad n!\sum_{k=0}^{[\frac{n}{m}]}\frac{(-1)^ky^k}{k!\Gamma\left(\frac{n-mk}{p}+1\right)}(x\sqrt{mA})^{\frac{n-mk}{p}}\tag{2.3}$$

with the restriction that $\frac{n-mk}{p}>-1$. In addition, we can write

$$H_{n,m,p}(x,y,A)=y^{\frac{n}{m}}H_{n,m,p}\left(\frac{x}{y^{\frac{p}{m}}},1,A\right)=y^{\frac{n}{m}}H_{n,m,p}\left(\frac{x}{y^{\frac{p}{m}}},A\right).\tag{2.4}$$

Before getting into the main body of the paper, it is easily seen that some important properties of the generalized Hermite matrix polynomials $H_{n,m,p}(x,y,A)$ are as follows:

(A) Multiplication Properties:

$$H_{n,m,p}(x,\alpha y,A)=\quad \alpha^{\frac{n}{m}}H_{n,m,p}\left(\alpha^{-\frac{p}{m}}x,y,A\right),\tag{2.5}$$

$$\alpha^{\frac{n}{p}} H_{n,m,p}(x,y,A) = \quad H_{n,m,p}\Big(\alpha x, \alpha^{\frac{m}{p}} y, A\Big), \tag{2.6}$$

$$H_{n,m,p}(x,y,A) = n! \sum_{k=0}^{[\frac{n}{m}]} \frac{(1-y)^k H_{n-mk,m,p}(x,A)}{k!(n-mk)!}, \tag{2.7}$$

$$u^n H_{n,m,p}\Big(\frac{x}{u^p}, y, A\Big) = n! \sum_{k=0}^{[\frac{n}{m}]} \frac{y^k (1-u^m)^k H_{n-mk,m,p}(x,y,A)}{k!(n-mk)!}. \tag{2.8}$$

(B) Addition Properties:

$$H_{n,m,p}(x+z,y,A) = n! \sum_{k=0}^{[\frac{n}{p}]} \frac{(z\sqrt{mA})^k H_{n-kp,m,p}(x,y,A)}{k!(n-kp)!} \tag{2.9}$$

and

$$H_{n,m,p}(x,y+w,A) = n! \sum_{k=0}^{[\frac{n}{m}]} \frac{(-1)^k w^k H_{n-mk,m,p}(x,y,A)}{k!(n-mk)!} \tag{2.10}$$

where α and u are constants.

The above relations will be used, along with other relations to derive new properties of the family of functions generated by (2.1) as given by the following theorem.

Theorem 2.1. *The generalized Hermite matrix polynomials satisfy the following relations*

$$H_{n,m,p}(\alpha x, \beta y, A) = n! \sum_{k=0}^{n} \frac{H_{k,m,p}\Big(\frac{\alpha}{2}x, \frac{\beta}{2}y, A\Big) H_{n-k,m}\Big(\frac{\alpha}{2}x, \frac{\beta}{2}y, A\Big)}{k!(n-k)!}, \tag{2.11}$$

$$H_{n,m,p}(\alpha x + \beta z, \mu y + \nu w, A) = n! \sum_{k=0}^{n} \frac{H_{k,m,p}(\beta z, \nu w, A) H_{n-k,m,p}(\alpha x, \mu y, A)}{k!(n-k)!} \tag{2.12}$$

and

$$H_{n,m,p}(\alpha x + \beta z, \mu y + \nu w, A) = n! \sum_{k=0}^{n} \frac{H_{k,m,p}\Big(\beta z, \frac{\mu y + \nu w}{2}, A\Big) H_{n-k,m,p}\Big(\alpha x, \frac{\mu y + \nu w}{2}, A\Big)}{k!(n-k)!} \tag{2.13}$$

where α, β, μ and ν are constants.

Proof. By using (2.1), consider the series

$$\sum_{n=0}^{\infty}\sum_{k=0}^{n} \frac{H_{n-k,m,p}\Big(\frac{\alpha}{2}x, \frac{\beta}{2}y, A\Big) H_{k,m,p}\Big(\frac{\alpha}{2}x, \frac{\beta}{2}y, A\Big) t^n}{k!(n-k)!} = \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} \frac{H_{n,m,p}\Big(\frac{\alpha}{2}x, \frac{\beta}{2}y, A\Big) H_{k,m,p}\Big(\frac{\alpha}{2}x, \frac{\beta}{2}y, A\Big) t^{n+k}}{k!n!}$$

$$= \sum_{n=0}^{\infty} \frac{H_{n,m,p}\Big(\frac{\alpha}{2}x, \frac{\beta}{2}y, A\Big) t^n}{n!} \sum_{k=0}^{\infty} \frac{H_{k,m,p}\Big(\frac{\alpha}{2}x, \frac{\beta}{2}y, A\Big) t^k}{k!}$$

$$= \exp\Big(\alpha x t^p \sqrt{mA} - \beta y t^m I\Big) = \sum_{n=0}^{\infty} \frac{H_{n,m,p}(\alpha x, \beta y, A)}{n!} t^n$$

from which by comparing the coefficients of t^n on both sides of the identity, we get (2.11). Further by considering the series

$$\sum_{n=0}^{\infty}\sum_{k=0}^{n} \frac{H_{n-k,m,p}(\beta z, \nu w, A) H_{k,m,p}(\alpha x, \mu y, A) t^n}{k!(n-k)!} = \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} \frac{H_{n,m,p}(\beta z, \nu w, A) H_{k,m}(\alpha x, \mu y, A) t^{n+k}}{k!n!}$$

$$= \sum_{n=0}^{\infty} \frac{H_{n,m,p}(\alpha z, \nu w, A) t^n}{n!} \sum_{k=0}^{\infty} \frac{H_{k,m,p}(\alpha x, \mu y, A) t^k}{k!} = \exp\left[\beta z t^p \sqrt{mA} - \nu w t^m I\right] \exp\left[\alpha x t^p \sqrt{mA} - \mu y t^m I\right]$$

$$= \exp\left[(\alpha x + \beta z) t^p \sqrt{mA} - (\mu y + \nu w) t^m I\right] = \sum_{n=0}^{\infty} \frac{H_{n,m,p}(\alpha x + \beta z, \mu y + \nu w, A) t^n}{n!}$$

and comparing the coefficients of t^n, we get (2.12). Lastly by considering the series

$$\sum_{n=0}^{\infty}\sum_{k=0}^{n}\frac{H_{n-k,m,p}\left(\beta z,\frac{\mu y+\nu w}{2},A\right)H_{k,m,p}\left(\alpha x,\frac{\mu y+\nu w}{2},A\right)t^n}{k!(n-k)!}$$

$$=\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\frac{H_{n,m,p}\left(\beta z,\frac{\mu y+\nu w}{2},A\right)H_{k,m,p}\left(\alpha x,\frac{\mu y+\nu w}{2},A\right)t^{n+k}}{k!n!}$$

$$=\sum_{n=0}^{\infty}\frac{H_{n,m,p}\left(\beta z,\frac{\mu y+\nu w}{2},A\right)t^n}{n!}\sum_{k=0}^{\infty}\frac{H_{k,m,p}\left(\alpha x,\frac{\mu y+\nu w}{2},A\right)t^k}{k!}$$

$$=\exp\left((\alpha x+\beta z)t^p\sqrt{mA}-(\mu y+\nu w)t^mI\right)=\sum_{n=0}^{\infty}\frac{H_{n,m,p}(\alpha x+\beta z,\mu y+\nu w,A)}{n!}t^n$$

and by comparing the coefficients of t^n, we get (2.13), thereby establishing the Theorem **2.1**. □

In the following corollary, we obtain same properties of generalized Hermite matrix polynomials as follows.

Corollary 2.1. *The generalized Hermite matrix polynomials satisfy the following relation*

$$H_{n,m,p}\left(\frac{x+z}{\sqrt[m]{2}},y,A\right)=n!(\sqrt[m]{2})^{-n}\sum_{k=0}^{n}\frac{H_{k,m,p}(z,y,A)H_{n-k,m,p}(x,y,A)}{k!(n-k)!}. \tag{2.14}$$

Proof. Now, from the properties of exponential matrix (1.2) in addition (2.1), we can write

$$\exp\left(xt^p\sqrt{mA}-yt^mI\right)\exp\left(zt^p\sqrt{mA}-yt^mI\right)=\exp\left(\frac{x+z}{2^{\frac{p}{m}}}t^p2^{\frac{p}{m}}\sqrt{mA}-y(t\sqrt[m]{2})^mI\right)$$

$$=\sum_{n=0}^{\infty}\frac{(t\sqrt[m]{2})^n}{n!}H_{n,m,p}\left(\frac{x+z}{\sqrt[m]{2}},y,A\right)=\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\frac{1}{n!k!}H_{n,m,p}(x,y,A)H_{k,m,p}(z,y,A)t^{n+k}$$

$$=\sum_{n=0}^{\infty}\sum_{k=0}^{n}\frac{1}{k!(n-k)!}H_{n-k,m,p}(x,y,A)H_{k,m,p}(z,y,A)t^n$$

by comparing the coefficients of t^n, we get (2.14) and the proof of Corollary **2.1** is completed. □

3 Recurrence relations

Some recurrence relations have been deduced for the generalized Hermite matrix polynomials. At first, we record the following theorem.

Theorem 3.1. *The generalized Hermite matrix polynomials $H_{n,m,p}(x,y,A)$ satisfy the following relations*

$$\frac{\partial^r}{\partial x^r}H_{n,m,p}(x,y,A)=\frac{(\sqrt{mA})^rn!}{(n-rp)!}H_{n-rp,m,p}(x,y,A);\quad 0\le r\le[\frac{n}{p}] \tag{3.1}$$

and

$$\frac{\partial^r}{\partial y^r}H_{n,m,p}(x,y,A)=\frac{(-1)^rn!}{(n-mr)!}H_{n-mr,m,p}(x,y,A);\quad 0\le r\le[\frac{n}{m}]. \tag{3.2}$$

Proof. Differentiating the identity (2.1) with respect to x yields

$$t^p\sqrt{mA}\exp\left(xt^p\sqrt{mA}-yt^mI\right)=\sum_{n=0}^{\infty}\frac{1}{n!}\frac{\partial}{\partial x}H_{n,m,p}(x,y,A)t^n. \tag{3.3}$$

From (2.1) and (3.3), we have

$$\sqrt{mA}\sum_{n=0}^{\infty}\frac{1}{n!}H_{n,m,p}(x,y,A)t^{n+p}=\sum_{n=1}^{\infty}\frac{1}{n!}\frac{\partial}{\partial x}H_{n,m,p}(x,y,A)t^{n}.$$

Hence, by identifying the coefficients in t^n, it follows that

$$\frac{\partial}{\partial x}H_{n,m,p}(x,y,A)=\frac{n!}{(n-p)!}\sqrt{mA}H_{n-p,m,p}(x,y,A);\quad n\geq p. \tag{3.4}$$

Iteration of (3.4), for $0\leq r\leq[\frac{n}{p}]$, implies (3.1). The proof of equation (3.2) is similar to that of equation (3.1) and thus the proof of the Theorem **3.1** is completed. □

The following corollary is a consequence of the Theorem **3.1**.

Corollary 3.1. *The generalized Hermite matrix polynomials satisfy the following relations*

$$\frac{\partial^m}{\partial x^m}H_{n,m,p}(x,y,A)-(-1)^p(\sqrt{mA})^m\frac{\partial^p}{\partial y^p}H_{n,m,p}(x,y,A)=0. \tag{3.5}$$

Proof. By (3.1) and (3.2) the equation (3.5) follows directly. □

According to (3.5), it is clear that, for the special case, when $m=2$ and $p=1$ the $H_{n,m,p}(x,y,A)$ are the natural solutions of the heat partial differential equation [1, 15] . The above three terms recurrence relation will be used in the following theorem.

Theorem 3.2. *Let A be a matrix in $C^{N\times N}$ satisfying (1.1), then we have*

$$H_{n,m,p}(x,y,A)=(n-1)!\left[\frac{xp\sqrt{mA}}{(n-p)!}H_{n-p,m,p}(x,y,A)-\frac{my}{(n-m)!}H_{n-m,m,p}(x,y,A)\right]. \tag{3.6}$$

Proof. Differentiating (2.2) with respect to x and t, we find respectively

$$\frac{\partial}{\partial x}F(x,y,t,A)=t^p\sqrt{mA}\exp\left(xt^p\sqrt{mA}-yt^mI\right)=\sum_{n=0}^{\infty}\frac{1}{n!}\frac{\partial}{\partial x}H_{n,m,p}(x,y,A)t^n$$

and

$$\begin{aligned}\frac{\partial}{\partial t}F(x,y,t,A)=&(xp\,t^{p-1}\sqrt{mA}-myt^{m-1}I)\exp\left(xt^p\sqrt{mA}-yt^mI\right)\\ =&\sum_{n=1}^{\infty}\frac{1}{(n-1)!}H_{n,m,p}(x,y,A)t^{n-1}.\end{aligned}$$

Therefore, $F(x,y,t,A)$ satisfies the partial matrix differential equation

$$(xp\,t^{p-1}\sqrt{mA}-myt^{m-1}I)\frac{\partial F}{\partial x}-t^p\sqrt{mA}\frac{\partial F}{\partial t}=0$$

this, by using (2.2), becomes

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{\sqrt{mA}}{(n-1)!}H_{n,m,p}(x,y,A)t^{n+p-1}=&xp\sqrt{mA}\sum_{n=0}^{\infty}\frac{1}{n!}\frac{\partial}{\partial x}H_{n,m,p}(x,y,A)t^{n+p-1}\\ &-\sum_{n=0}^{\infty}\frac{my}{n!}\frac{\partial}{\partial x}H_{n,m,p}(x,y,A)t^{n+m-1}\end{aligned}$$

and identifying coefficients in t^{n+p-1}, we get

$$\begin{aligned}\frac{\sqrt{mA}}{(n-1)!}H_{n,m,p}(x,y,A)=&xp\sqrt{mA}\frac{1}{n!}\frac{\partial}{\partial x}H_{n,m,p}(x,y,A)\\ &-\frac{my}{(n-m+p)!}\frac{\partial}{\partial x}H_{n-m+p,m,p}(x,y,A).\end{aligned}$$

Then for $n \geq m$ and $n \geq p$, it follows

$$\begin{aligned}\frac{1}{(n-1)!}H_{n,m,p}(x,y,A) =& \frac{xp}{n!}\frac{\partial}{\partial x}H_{n,m,p}(x,y,A)\\ &- \frac{my(\sqrt{mA})^{-1}}{(n-m+p)!}\frac{\partial}{\partial x}H_{n-m+p,m,p}(x,y,A).\end{aligned} \tag{3.7}$$

Using (3.4) and (3.7), we get (3.6). Finally, the proof of Theorem **3.2** is completed. □

4 Expansions of some elementary matrix functions in terms of the generalized Hermite matrix polynomials

Now, we use the expansion of the generalized Hermite matrix polynomials together with their properties to prove the following result.

Theorem 4.1. *Let A be a positive stable matrix in $\mathbb{C}^{N\times N}$ satisfy (1.1), then, we have*

$$(x\sqrt{mA})^n = \sum_{k=0}^{[\frac{np}{m}]}\frac{n!}{k!(np-mk)!}y^k H_{np-mk,m,p}(x,y,A), \quad -\infty < x < \infty. \tag{4.1}$$

Proof. In view of (2.1), one gets

$$\exp\left(xt^p\sqrt{mA}\right) = \exp\left(yt^m I\right)\sum_{n=0}^{\infty}\frac{t^n}{n!}H_{n,m,p}(x,y,A)$$

which can be written, by applying (1.2), in the form:

$$\begin{aligned}\exp\left(xt^p\sqrt{mA}\right) &= \sum_{n=0}^{\infty}\frac{(x\sqrt{mA})^n}{n!}t^{np} = \sum_{k=0}^{\infty}\frac{(yt^mI)^k}{k!}\sum_{n=0}^{\infty}\frac{t^n}{n!}H_{n,m,p}(x,y,A)\\ &= \sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\frac{y^kH_{n,m,p}(x,y,A)}{n!k!}t^{n+mk}\\ &= \sum_{n=0}^{\infty}\sum_{k=0}^{[\frac{np}{m}]}\frac{y^kH_{np-mk,m,p}(x,y,A)}{k!(np-mk)!}t^{np}.\end{aligned} \tag{4.2}$$

Expanding the left-hand side of (4.2) into powers of t and identifying the coefficients of t^{np} on both sides gives (4.1). Therefore, the expression (4.1) is established and the proof of Theorem **4.1** is completed. □

We now propose to give the following new series expansions of some elementary matrix functions like $\exp(xB)$, $\sin(xB)$, $\cos(xB)$, $\cosh(xB)$ and $\sinh(xB)$ in terms of the generalized Hermite matrix polynomials for matrices satisfying the spectral property

$$|Re(\lambda)| > |Im(\lambda)|, \quad \text{for all} \quad \lambda \in \sigma(B). \tag{4.3}$$

(see [6]).

Theorem 4.2. *Let B be a positive stable matrix in $\mathbb{C}^{\mathbb{N}\times\mathbb{N}}$ satisfying (4.3), then*

$$\exp(xB) = \exp(y)\sum_{n=0}^{\infty}\frac{1}{n!}H_{n,m,p}\left(x,y,\frac{1}{m}B^2\right); \quad -\infty < x < \infty, \tag{4.4}$$

$$\cos(xB) = \exp\left((-1)^{\frac{mk}{2p}}y\right)\sum_{n=0}^{\infty}\frac{(-1)^n}{(2np)!}H_{2np,m,p}\left(x,y,\frac{1}{m}B^2\right); \quad -\infty < x < \infty, \tag{4.5}$$

$$\sin(xB) = \exp\left((-1)^{\frac{mk}{2p}}y\right)\sum_{n=0}^{\infty}\frac{(-1)^n}{(2np+1)!}H_{2np+1,m,p}\left(x,y,\frac{1}{m}B^2\right); \quad -\infty < x < \infty, \tag{4.6}$$

$$\cosh(xB) = \exp(y)\sum_{n=0}^{\infty}\frac{1}{(2n)!}H_{2np,m,p}\bigg(x, y, \frac{1}{m}B^2\bigg); \quad -\infty < x < \infty \tag{4.7}$$

and

$$\sinh(xB) = \exp(y)\sum_{n=0}^{\infty}\frac{1}{(2n+1)!}H_{2np+1,m,p}\bigg(x, y, \frac{1}{m}B^2\bigg); \quad -\infty < x < \infty. \tag{4.8}$$

Proof. Let $A = \frac{1}{m}B^2$. By the spectral mapping theorem [6] and (4.3), it follows that

$$\sigma(A) = \bigg\{\frac{1}{m}b^2; b \in \sigma(B)\bigg\}, Re\bigg(\frac{1}{m}b^2\bigg) = \frac{1}{m}\bigg\{\Big(Re(b)\Big)^2 - \Big(Im(b)\Big)^2\bigg\} > 0, b \in \sigma(B). \tag{4.9}$$

Thus A is a positive stable matrix and taking $t = 1$ in (2.1), $A = \frac{1}{m}B^2$ gives

$$\exp\big(xB - yI\big) = \sum_{n=0}^{\infty}\frac{1}{n!}H_{n,m,p}\bigg(x, y, \frac{1}{m}B^2\bigg). \tag{4.10}$$

Therefore, (4.4) follows.

Considering (4.1) for the positive stable matrix $A = \frac{1}{m}B^2$, it follows that

$$x^{2n}I = B^{-2n}\sum_{k=0}^{[\frac{2np}{m}]}\frac{(2n)!}{k!(2np-mk)!}y^k H_{2np-mk,m,p}\bigg(x, y, \frac{1}{m}B^2\bigg).$$

Taking into account the series expansion of $\cosh(xB)$ and (1.3), we can write

$$\begin{aligned}
\cosh(xB) &= \sum_{n=0}^{\infty}\frac{B^{2n}}{(2n)!}x^{2n} = \sum_{n=0}^{\infty}\sum_{k=0}^{[\frac{2np}{m}]}\frac{y^k}{k!(2np-mk)!}H_{2np-mk,m,p}\bigg(x, y, \frac{1}{m}B^2\bigg)\\
&= \sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\frac{y^k}{k!(2np)!}H_{2np,m,p}\bigg(x, y, \frac{1}{m}B^2\bigg)\\
&= \sum_{k=0}^{\infty}\frac{y^k}{k!}\sum_{n=0}^{\infty}\frac{1}{(2np)!}H_{2np,m,p}\bigg(x, y, \frac{1}{m}B^2\bigg)\\
&= \exp(y)\sum_{n=0}^{\infty}\frac{1}{(2np)!}H_{2np,m,p}\bigg(x, y, \frac{1}{m}B^2\bigg).
\end{aligned}$$

Therefore, (4.7) follows. By similar arguments we can prove the relations (4.5), (4.6) and (4.8).

Moreover, the convergence of the matrix series appearing in (4.4)-(4.8) to the respective matrix functions $\exp(xB)$, $\sin(xB)$, $\cos(xB)$, $\sinh(xB)$ and $\cosh(xB)$ are uniform in any bounded interval of the real axis. Therefore, the result is established. □

In the following theorem, we obtain another representation for the generalized Hermite matrix polynomials as follows.

Theorem 4.3. *Suppose that A is a matrix in $\mathbb{C}^{N\times N}$ satisfying (1.1). Then*

$$H_{np,mp,p}(x, y, A) = \frac{(np)!}{n!}\exp\bigg(-\frac{y}{(\sqrt{mpA})^m}\frac{\partial^m}{\partial x^m}\bigg)(x\sqrt{mpA})^n. \tag{4.11}$$

Proof. It is clear by (2.1) that

$$\begin{aligned}
\exp\bigg(-\frac{y}{(\sqrt{mpA})^m}\frac{\partial^m}{\partial x^m}\bigg)\exp\big(xt^p\sqrt{mpA}\big) &= \sum_{n=0}^{\infty}\frac{(-1)^n y^n}{n!(\sqrt{mpA})^{mn}}\frac{\partial^{mn}}{\partial x^{mn}}\exp\big(xt^p\sqrt{mpA}\big)\\
&= \sum_{n=0}^{\infty}\frac{(-1)^n y^n}{n!}t^{mnp}\exp\big(xt^p\sqrt{mpA}\big) = \exp\Big(xt^p\sqrt{mpA} - yt^{mp}I\Big)\\
&= \sum_{n=0}^{\infty}\frac{1}{n!}H_{n,mp,p}(x, y, A)t^n = \sum_{n=0}^{\infty}\frac{1}{(np)!}H_{np,mp,p}(x, y, A)t^{np}.
\end{aligned}$$

Thus by identification of the coefficients of t^{np} in both sides gives the representation (4.11). □

By recalling that the generalized Hermite matrix polynomials $H_{n,m,p}(x, y, A)$ are also defined through the operational identity, the inverse of (4.11) allows us to conclude that

$$(x\sqrt{mpA})^n = \frac{n!}{(np)!}\exp\left(\frac{y}{(\sqrt{mpA})^m}\frac{\partial^m}{\partial x^m}\right)H_{np,mp,p}(x, y, A). \tag{4.12}$$

In the following corollary, we obtain another expansion formula for the generalized Hermite matrix polynomials as follows.

Corollary 4.1. *For the generalized Hermite matrix polynomials the following identities hold*

$$H_{np+kp,m,p}(x, y, A) = \frac{(np + kp)!}{(n + k)!}\frac{n!}{(np)!}\frac{k!}{(kp)!}H_{np,mp,p}(x, y, A)\exp\left(\frac{y}{(\sqrt{mpA})^m}\frac{\partial^m}{\partial x^m}\right)H_{kp,mp,p}(x, y, A) \tag{4.13}$$

and

$$H_{np,mp,p}(x, y + z, A) = \exp\left(-\frac{z}{(\sqrt{mpA})^m}\frac{\partial^m}{\partial x^m}\right)H_{np,mp,p}(x, y, A). \tag{4.14}$$

Proof. From Theorem **4.3**, we get directly the equation (4.13) and (4.14). □

5 The generalized Chebyshev matrix polynomials

Next, the generalized Hermite matrix polynomials of two variables $H_{n,m,p}(x, y, A)$ will be exploited here to define a matrix version of the generalized Chebyshev polynomials of the second kind.

Suppose that A is a matrix in $\mathbb{C}^{N\times N}$ satisfying the condition (1.1). By (2.3) it follows that

$$\frac{1}{n!}\int_0^\infty \exp(-t)H_{n,m,p}(xt, yt, A)dt = \int_0^\infty \exp(-t)\sum_{k=0}^{[\frac{n}{m}]}\frac{(-1)^k(yt)^k}{k!\Gamma\left(\frac{n-mk}{p}+1\right)}(xt\sqrt{mA})^{\frac{n-mk}{p}}dt. \tag{5.1}$$

Since the summation in the right-hand side of the above equality is finite, then the series and the integral can be permuted. Also, in view of

$$\Gamma\left(\frac{n-(m-p)k}{p}+1\right) = \int_0^\infty \exp(-t)t^{\frac{n-(m-p)k}{p}}dt \tag{5.2}$$

we can write

$$\frac{1}{n!}\int_0^\infty \exp(-t)H_{n,m,p}(xt, yt, A)dt = \sum_{k=0}^{[\frac{n}{m}]}\frac{(-1)^k\Gamma\left(\frac{n-(m-p)k}{p}+1\right)y^k}{k!\Gamma\left(\frac{n-mk}{p}+1\right)}(x\sqrt{mA})^{\frac{n-mk}{p}}. \tag{5.3}$$

Hence, the generalized Chebyshev matrix polynomials of the second kind can be defined by

$$U_{n,m,p}(x, y, A) = \sum_{k=0}^{[\frac{n}{m}]}\frac{(-1)^k\Gamma\left(\frac{n-(m-p)k}{p}+1\right)y^k}{k!\Gamma\left(\frac{n-mk}{p}+1\right)}(x\sqrt{mA})^{\frac{n-mk}{p}} \tag{5.4}$$

or, by using the integral transform of the generalized Hermite matrix polynomials as below

$$U_{n,m,p}(x, y, A) = \frac{1}{n!}\int_0^\infty \exp(-t)H_{n,m,p}(xt, yt, A)dt. \tag{5.5}$$

In a similar way, we define the generalized Chebyshev matrix polynomials of the first kind as follows

$$T_{n,m,p}(x, y, A) = n\sum_{k=0}^{[\frac{n}{m}]}\frac{(-1)^k\Gamma\left(\frac{n-(m-p)k}{p}\right)y^k}{k!\Gamma\left(\frac{n-mk}{p}+1\right)}(x\sqrt{mA})^{\frac{n-mk}{p}} \tag{5.6}$$

and

$$T_{n,m,p}(x,y,A) = \frac{1}{(n-1)!}\int_0^\infty \exp(-t)t^{-1}H_{n,m,p}(xt,yt,A)dt; n \geq 1,\ T_{0,m,p}(x,y,A) = I. \quad (5.7)$$

6 The generalized Legendre matrix polynomials

The generalized Hermite matrix polynomials will be utilized here to define a matrix version of the classical Legendre polynomials. The Legendre polynomials [17, p. 157(2) and p. 161(3)] are defined by

$$P_n(x) = \sum_{k=0}^{[\frac{1}{2}n]} \frac{(-1)^k(\frac{1}{2})_{n-k}(2x)^{n-2k}}{k!(n-2k)!} = \sum_{k=0}^{[\frac{1}{2}n]} \frac{(-1)^k(2n-2k)!x^{n-2k}}{2^n k!(n-2k)!(n-k)!}. \quad (6.1)$$

Let A be a positive stable matrix in $\mathbb{C}^{N\times N}$ satisfying the condition (1.1). By using (2.3) it follows that

$$\begin{aligned}\frac{2}{n!\sqrt{\pi}}\int_0^\infty e^{-t^2}t^{\frac{n}{p}}H_{n,m,p}(xt,y,A)dt =& \frac{2}{n!\sqrt{\pi}}\int_0^\infty e^{-t^2}t^{\frac{n}{p}}n!\sum_{k=0}^{[\frac{n}{m}]}\frac{(-1)^k y^k(x\sqrt{mA})^{\frac{n-mk}{p}}}{k!\Gamma\left(\frac{n-mk}{p}+1\right)}t^{\frac{n-mk}{p}}dt\\ =&\frac{2}{\sqrt{\pi}}\sum_{k=0}^{[\frac{n}{m}]}\frac{(-1)^k y^k(x\sqrt{mA})^{\frac{n-mk}{p}}}{k!\Gamma\left(\frac{n-mk}{p}+1\right)}\int_0^\infty t^{\frac{n}{p}}e^{-t^2}t^{\frac{n-mk}{p}}dt \quad (6.2)\\ =&\frac{2}{\sqrt{\pi}}\sum_{k=0}^{[\frac{n}{m}]}\frac{(-1)^k y^k(x\sqrt{mA})^{\frac{n-mk}{p}}}{k!\Gamma\left(\frac{n-mk}{p}+1\right)}\int_0^\infty e^{-t^2}t^{\frac{2n-mk}{p}}dt.\end{aligned}$$

Since the summation on the right-hand side of the above equality is finite, then the series and the integral can be permuted. From the definition of Gamma function, we have

$$\int_0^\infty e^{-t^2}t^{\frac{2n-mk}{p}}dt = \frac{1}{2}\Gamma\left(\frac{2n-mk}{2p}+\frac{1}{2}\right) \quad (6.3)$$

and applying Legendre duplication formula by Srivastava and Karlsson [21]

$$\Gamma\left(\frac{2n-mk}{2p}+\frac{1}{2}\right) = \frac{\sqrt{\pi}\,\Gamma\left(\frac{2n-mk}{p}+1\right)}{2^{\frac{2n-mk}{p}}\Gamma\left(\frac{2n-mk}{2p}\right)} \quad (6.4)$$

or

$$\Gamma\left(\frac{2n-mk}{2p}+\frac{1}{2}\right) = \sqrt{\pi}\left(\frac{1}{2}\right)_{\frac{2n-mk}{2p}}. \quad (6.5)$$

Then

$$\begin{aligned}\frac{2}{n!\sqrt{\pi}}\int_0^\infty e^{-t^2}t^{\frac{n}{p}}H_n(xt,y,A)dt =& \frac{2}{\sqrt{\pi}}\sum_{k=0}^{[\frac{n}{m}]}\frac{(-1)^k y^k(x\sqrt{mA})^{\frac{n-mk}{p}}}{k!\Gamma\left(\frac{n-mk}{p}+1\right)}\int_0^\infty e^{-t^2}t^{\frac{2n-mk}{p}}dt\\ =& \frac{2}{\sqrt{\pi}}\sum_{k=0}^{[\frac{n}{m}]}\frac{(-1)^k y^k(x\sqrt{mA})^{\frac{n-mk}{p}}}{k!\Gamma\left(\frac{n-mk}{p}+1\right)}\frac{1}{2}\frac{\sqrt{\pi}\,\Gamma\left(\frac{2n-mk}{p}+1\right)}{2^{\frac{2n-mk}{p}}\Gamma\left(\frac{2n-mk}{2p}\right)} \quad (6.6)\\ =& \sum_{k=0}^{[\frac{n}{m}]}\frac{(-1)^k\Gamma\left(\frac{2n-mk}{p}+1\right)y^k(x\sqrt{mA})^{\frac{n-mk}{p}}}{2^{\frac{2n-mk}{p}}k!\,\Gamma\left(\frac{n-mk}{p}+1\right)\Gamma\left(\frac{2n-mk}{2p}\right)} = P_{n,m,p}(x,y,A).\end{aligned}$$

Hence, the Legendre matrix polynomials can be defined by

$$P_{n,m,p}(x,y,A) = \sum_{k=0}^{[\frac{n}{m}]} \frac{(-1)^k \Gamma\left(\frac{2n-mk}{p}+1\right) y^k (x\sqrt{mA})^{\frac{n-mk}{p}}}{2^{\frac{2n-mk}{p}} k!\, \Gamma\left(\frac{n-mk}{p}+1\right) \Gamma\left(\frac{2n-mk}{2p}\right)}$$

or

$$P_{n,m,p}(x,y,A) = \sum_{k=0}^{[\frac{n}{m}]} \frac{(-1)^k \left(\frac{1}{2}\right)_{\frac{2n-mk}{2p}} y^k (x\sqrt{mA})^{\frac{n-mk}{p}}}{k!\, \Gamma\left(\frac{n-mk}{p}+1\right)}$$

or by using the Hermite matrix polynomials of integral representation in the form

$$P_{n,m,p}(x,y,A) = \frac{2}{n!\sqrt{\pi}} \int_0^\infty e^{-t^2} t^{\frac{n}{p}} H_{n,m,p}(xt,y,A)dt.$$

There are many way of investigating the generalized classes of Hermite matrix polynomials. Starting from the modified forms of the generating function of ordinary Hermite matrix polynomials is one of these direct methods and clearly some directions to develop more researches and studies in that area. The results of this paper are original, variant, significant and so it is interesting and capable to develop its study in the future.

7 Open problem

One can use the same class of new integral representation, operational methods and orthogonality property for the new generalized Hermite matrix polynomials with p and m are integers or not integers. Hence, new results and further applications can be obtained. Further results and applications will be discussed in a forthcoming paper.

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