

Semidualizing and tilting adjoint pairs, applications to comodules

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Abstract

The aim of this paper is to introduce the concept of right and left semidualizing adjoint pair of functors and study its main properties. This concept generalizes the concept of semidualizing module and allows one to consider semidualizing comodules, graded modules, etc. We also study tilting adjoint pair of functors as a particular case. We show generalized tilting theorem in this general setting and give some applications to tilting theory in the category of comodules over a coalgebra.

2010 Mathematics Subject Classification. Primary 16T15. Secondary 18E15

Keywords: Semidualizing comodule, tilting object, adjoint functors

Introduction and preliminaries

Semidualizing modules were introduced by Foxby [19], Golod [17] and Vasconcelos [41] under different names (Foxby called them PG-modules of rank one, Vasconcelos called them Spherical modules and Golod suitable modules). They generalize dualizing modules over Cohen-Macaulay rings and projective modules of rank one. More recently Christensen [10] used the term “semidualizing” to describe this kind of modules (and complexes) for commutative noetherian rings. The theory has been developed quickly by other authors in the last two decades (see for example papers by Enochs, Jenda, Xu, Holm, Jorgensen [14], [16], [21], etc). Araya, Takahashi and Yoshino extend the theory to the non commutative noetherian setting and investigate semidualizing bimodules in [2]. They are studied by Holm and White in [22] for arbitrary associative rings. These modules extend dualizing modules over local

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noetherian commutative ring and have a great importance in the theory of Cohen-Macaulay rings. Associated to a (semi)dualizing module there exist two classes of modules, the Auslander class and the Bass class [14] (also called Foxby classes for dualizing modules), that are equivalent as full subcategories of modules.

In 1980, Brenner and Butler in [5] and Happel and Ringel in [20] generalized the classical Morita theory of equivalences by introducing the notion of a tilting module over a finite-dimensional Artin algebra. In order to obtain equivalences between subcategories of the module category, tilting modules were assumed to be of projective dimension at most one. Later, Miyashita [31] considered tilting modules of finite projective dimension and studied the equivalences induced by them. Colby and Fuller in [7] extended the setting to arbitrary rings. B. Keller in [23] introduced the notion of tilting adjoint pair of functors in module categories over associative rings and a description of 1-tilting in terms of adjoint functors was given in [35].

Takeuchi in [39] characterized equivalences of comodule categories over fields, dualizing Morita Theorem on equivalences of module categories. As a generalization of the Takeuchi result, cotilting comodules are studied in order to obtain equivalences between certain subcategories of comodules. Mingyi [29] proved the tilting theorem for classical cotilting comodules over semiperfect conoetherian coalgebras over fields. More recently, Liu and Zhang extend the results of [29] and get a Brenner-Butler Theorem for semiperfect coalgebras [26, Theorem 5.2]. Simson in [36] investigate cotilting comodules and get different interesting examples in coalgebras over quivers.

In this paper we unify the concept of semidualizing object by extending this to the framework of adjoint functors. The advantage of this new point of view is that we get results valid for different abelian categories like those of comodules, graded modules, etc. We also connect this concept with the concept of a tilting adjoint pair given by Keller in [23]. In this way we get Brenner-Butler Theorem for a large class of examples.

The paper is structured as follows. We start by introducing semidualizing adjoint pair in Section 2. We prove several results about the Auslander and Bass classes in this general setting, generalizing several results of Enochs and Yassemi in [16] on the classes \mathcal{U} and \mathcal{W} associated to a right and left semidualizing adjoint pair respectively. Under some conditions it is also proved that Gorenstein projective objects are in the Auslander class and Gorenstein injective in the Bass class.

Next, in Section 3 we investigate tilting adjoint pairs. We relate semidualizing and tilting adjoint pairs with a generalization of Brenner-Butler Theorem. Finally, in Sections 4 and 5, we apply this general theory to the concrete case of categories of comodules. We prove that if two coalgebras C and D have a semidualizing bicomodule of finite injective dimension, then the coalgebras have isomorphic Grothendieck groups and we also show that C is finite dimensional if and only if D is so. We also give a relation among the concepts of semidualizing comodule and n -cotilting comodule [36]. We show that over a quasi-co-Frobenius coalgebras C every n -cotilting comodules is trivial (that is, the coendomorphism coalgebra of any n -cotilting comodule is Morita-Takeuchi equivalent to C).

Now we give the terminology used in this paper. For concepts about categories

and derived functors we refer the reader to [37] or [42]. Concerning coalgebras and comodules we use the notation of [38], [1] or [11].

Let C be a k -coalgebra, k a field. For two right C -comodules M and N , the k -space of all comodules maps from M to N will be written as $Com_C(M, N)$ and \mathcal{M}^C will denote the category of right C -comodules. In the same way we can construct the category ${}^C\mathcal{M}$ of left C -comodules. The category \mathcal{M}^C is isomorphic to the category of rational left C^* -modules, $Rat(C^* - Mod)$. In fact we have a left exact functor $Rat : C^* - Mod \rightarrow C^* - Mod$ which is a preradical.

If C and D are two coalgebras, a (C, D) -bicomodule is a left C -comodule and a right D -comodule M , such that the C -comodule structure map $\rho_M^- : M \rightarrow C \otimes M$ is a D -comodule map, or equivalently the D -comodule structure map $\rho_M^+ : M \rightarrow M \otimes D$ is a C -comodule map.

It is well known that \mathcal{M}^C is an abelian category (see [38], [1], [11]). In fact, \mathcal{M}^C is a Grothendieck category.

Let C be a k -coalgebras and M a right C -comodule. We denote by $Cogen(M)$ the full subcategory of right C -comodules cogenerated by M , that is subcomodules of products of copies of M . By $Prop(M)$ (resp. $add(M)$) we denote the full subcategory of direct summands of an arbitrary products (resp. finite direct sums) of copies of M , and $Add(M)$ denote the full subcategory of direct summands of an arbitrary direct sum of copies of M .

We recall from [39] that a right comodule M over a coalgebra C is *quasi-finite* if it contains only a finite number of copies of each simple right C -comodule S or equivalently if $Com_C(S, M)$ is finite dimensional for every simple S .

Now let M be a quasi-finite right C -comodule and consider the covariant right additive functor $h_C(M, -)$ from \mathcal{M}^C to \mathcal{M}_k . If C is right semiperfect [24], then for every $N \in \mathcal{M}^C$ we may consider a projective resolution

$$\mathbf{P}_N : \cdots \rightarrow P_n \rightarrow P_{n-1} \rightarrow \cdots \rightarrow P_1 \rightarrow P_0 \rightarrow N \rightarrow 0.$$

Therefore we can obtain the left derived functors of $h_C(M, -)$ denoted by $ext_C^i(M, -)$. Their main properties are listed in [30, Proposition 4.1].

On the other hand, $M \in \mathcal{M}^C$ is quasi-finite if and only if its injective envelope $E(M)$ is quasi-finite. Now, let us suppose that C is right semiperfect and let S be a simple right C -comodule. Then there exists a finite-dimensional projective right C -comodule P such that $P \rightarrow S \rightarrow 0$ is exact. From the exact sequences

$$0 \rightarrow Com_C(S, E(M)/M) \rightarrow Com_C(P, E(M)/M)$$

and

$$Com_C(P, M) \rightarrow Com_C(P, (E(M)/M)) \rightarrow 0$$

it follows that $E(M)/M$ is quasi-finite, since $Com_C(P, M)$ is finite-dimensional. Therefore, if C is right semiperfect and $M \in \mathcal{M}^C$, then there exists an injective resolution

$$\mathbf{E}_M : 0 \rightarrow M \rightarrow E^0 \rightarrow E^1 \rightarrow \cdots \rightarrow E^{n-1} \rightarrow E^n \rightarrow \cdots$$

where every E^i $i \geq 0$ is quasi-finite.

Now let $N \in \mathcal{M}^C$ and consider the right additive contravariant functor $h_C(-, N)$ from the full subcategory of quasi-finite right C -comodules to \mathcal{M}_k . $h_C(-, N)$ is right exact and, by the above, if C is right semiperfect, we can consider left derived functors of $h_C(-, N)$ using injective resolutions on the first variable. These functors will be denoted by $\overline{ext}_C^i(-, N)$, $i \geq 0$.

Now, by [13, Remark 8.2.15] or simply, from the fact that

$$h_C(M, N)^* \cong Com_C(N, M)$$

we get the following result.

Corollary 1.1. *Let C be right semiperfect and $M, N \in \mathcal{M}^C$. If M is quasi-finite, then*

$$\overline{ext}_C^i(M, N) \cong ext_C^i(M, N), \quad ext_C^i(M, N)^* = Ext_C^i(N, M) \quad \forall i \geq 0.$$

The following Lemma will be useful.

Lemma 1.2. [11, Proposition 2.3.7] *There exists an isomorphism of left D -comodules*

$$M \square_C N^* \cong Com_C(N, M)$$

for every finite dimensional right C -comodule N and any $(D - C)$ -bicomodule M . Analogously, there exists an isomorphism of right D -comodules

$$K^* \square_C P \cong Com_C(P, K)$$

for every finite dimensional left C -comodule K and any $(C - D)$ -bicomodule P .

2 Semidualizing adjoint pairs

Let \mathcal{C} and \mathcal{D} be two abelian categories with arbitrary coproducts and products. Suppose that \mathcal{C} has a projective generator and \mathcal{D} has an injective cogenerator.

Let (F, G) be an adjoint situation with $F : \mathcal{C} \rightarrow \mathcal{D}$, $G : \mathcal{D} \rightarrow \mathcal{C}$. Then F is right exact and preserves arbitrary colimits and G is left exact and preserves arbitrary limits. Their left and right derived functors will be denoted by $\mathbf{L}_i F$ and $\mathbf{R}^i G \forall i \geq 0$ respectively.

We define the Auslander class of \mathcal{C} relative to F , denoted by $\mathcal{A}(\mathcal{C})$, consisting of the objects X satisfying:

- i) $\mathbf{L}_i F(X) = 0 \forall i \geq 1$
- ii) $\mathbf{R}^i G(F(X)) = 0 \forall i \geq 1$
- iii) the unit $u_X : X \rightarrow GF(X)$ is an isomorphism.

Analogously, the Bass class of \mathcal{D} relative to G denoted by $\mathcal{B}(\mathcal{D})$, consisting of the objects Y satisfying:

- i) $\mathbf{R}^i G(Y) = 0; \forall i \geq 1$
- ii) $\mathbf{L}_i F(G(Y)) = 0; \forall i \geq 1$
- iii) the counit $c_Y : FG(Y) \rightarrow Y$ is an isomorphism.

Proposition 2.1. *The functors F and G define an equivalence between $\mathcal{A}(\mathcal{C})$ and $\mathcal{B}(\mathcal{D})$.*

Proof. We only need to show that if $X \in \mathcal{A}(\mathcal{C})$ then $F(X)$ is in $\mathcal{B}(\mathcal{D})$. Now, the first condition for $F(X)$ to be in the Bass class becomes the second condition for X to be in the Auslander class. Moreover, $\mathbf{L}_i F(G(F(X))) = \mathbf{L}_i F(X) = 0$ by the third condition. The equality $F(u_X) \circ c_{F(X)} = id_{FGF(X)}$ shows that $c_{F(X)}$ is an isomorphism. ■

We now introduce the following definition.

Definition 2.1. 1) *We say that the adjoint pair (F, G) is right semidualizing if the class of injective objects is contained in $\mathcal{B}(\mathcal{D})$.*

2) *We say that the adjoint pair (F, G) is left semidualizing if the class of projective objects is contained in $\mathcal{A}(\mathcal{C})$.*

Example 2.2. *Let R and S be two rings and M be a semidualizing $(R - S)$ -bimodule ([22] or [2]), then the adjoint pairs $(M \otimes_S -, Hom_R(M, -))$ and $(- \otimes_R M, Hom_S(M, -))$ are left semidualizing. Under some finiteness conditions both notions coincide.*

In particular, if R and S are right and left noetherian rings and V is a semidualizing $(R - S)$ -bimodule, then $V[[x]]$ is a semidualizing $(R[[x]] - S[[x]])$ -bimodule (see [15, Example 3.5] for V a dualizing bimodule).

Example 2.3. *Let \mathcal{F} be a Gabriel topology on the ring R , $\mathbf{Mod}-(R, \mathcal{F})$ the quotient category of $\mathbf{Mod}-R$ with respect to \mathcal{F} , i.e. the full subcategory of $\mathbf{Mod}-R$ consisting of \mathcal{F} -closed modules, and $a : \mathbf{Mod}-R \rightarrow \mathbf{Mod}-(R, \mathcal{F})$ the localization functor. It is well-known [37, Proposition IX.1.11] that a is a left adjoint of the inclusion functor $i : \mathbf{Mod}-(R, \mathcal{F}) \rightarrow \mathbf{Mod}-R$ and a is exact. Hence condition ii) of the Bass class is trivially satisfied. On the other hand, $ai(X) \rightarrow X$ is an isomorphism for any object X in the quotient category and trivially $\mathbf{R}^i i(E) = 0$ for any injective object in the quotient category. Hence (a, i) is right semidualizing.*

The Auslander class will consist of all right R -modules such that $M \rightarrow ai(M)$ is an isomorphism, i.e. M is \mathcal{F} -closed and $\mathbf{R}^i i(a(M)) = 0$ and the Bass class is \mathcal{F} -closed and such that $\mathbf{R}^i i(Y) = 0$.

Example 2.4. *Let \mathbf{T} be the torsion class associated to an hereditary torsion theory (\mathbf{T}, \mathbf{F}) . There is an adjoint pair of functors (i, t) , where i is the inclusion functor and t is the torsion radical. In this case, this adjoint is left semidualizing and the Auslander and Bass class coincide. They consist of all torsion modules X with $\mathbf{R}^i t(X) = 0$, $i \geq 1$. An interesting case is $\mathbf{T} = \mathcal{M}^C = \text{Rat}(C^* - \text{Mod})$.*

Example 2.5. *Let R and S be two G -graded rings, with G a group. If M is a graded $R - S$ -bimodule then we have the adjoint pairs*

$$(M \otimes_S -, HOM_R(M, -)) \text{ and } (- \otimes_R M, HOM_S(M, -))$$

between the corresponding categories of graded modules (see [34, Proposition I.2.14]). The bimodule M is called a graded semidualizing bimodule if the mentioned adjoint pairs are left semidualizing. The results of this paper could be applied to this setting.

Example 2.6. Let \mathcal{A} be a Grothendieck category and D an object in \mathcal{A} . By [9, Proposition 1.1], if we consider $R = \text{End}_{\mathcal{A}}(D)$, then there is an adjoint situation

$$(T_D, \text{Hom}_{\mathcal{A}}(D, -)), \quad T_D : \text{Mod} - R \rightarrow \mathcal{A}, \quad \text{Hom}_{\mathcal{A}}(D, -) : \mathcal{A} \rightarrow \text{Mod} - R$$

and $T_D(R) = D$. It can be seen that this adjoint pair is left semidualizing if and only if D is a semidualizing object (in the sense that $\text{Ext}_{\mathcal{A}}^i(D, D^{(I)}) = 0$ and $\text{Hom}_{\mathcal{A}}(D, D^{(I)}) \cong \text{Hom}_{\mathcal{A}}(D, D)^{(I)}$ for any index set I). Analogously, we get conditions on D that guaranteeing that the adjoint pair is right semidualizing.

Let $\mathcal{L} \subset \mathcal{C}$ be a full subcategory of \mathcal{C} . As usual, we denote by $\text{add}(\mathcal{L})$ (resp. $\text{Add}(\mathcal{L})$) the full subcategory consisting of direct summands of finite direct sums (resp. arbitrary direct sums) of objects in \mathcal{L} . Analogously, we will define $\text{prod}(\mathcal{L})$ and $\text{Prod}(\mathcal{L})$ for products.

Lemma 2.7. Let (F, G) be a right semidualizing adjoint pair. Then

$$\text{add}(G(Y)) \subseteq \mathcal{A}(\mathcal{C})$$

for every object $Y \in \mathcal{B}(\mathcal{D})$.

If, in addition, G commutes with arbitrary coproducts, then

$$\text{Add}(G(Y)) \subseteq \mathcal{A}(\mathcal{C})$$

for every object $Y \in \mathcal{B}(\mathcal{D})$.

The dual result with F and products also holds.

Proof. Easy. ■

The following result is easy to prove by the adjoint property and the condition (iii) of the definitions of the classes $\mathcal{A}(\mathcal{C})$ and $\mathcal{B}(\mathcal{D})$.

Lemma 2.8. If $X, Y \in \mathcal{B}(\mathcal{D})$, then $\text{Hom}_{\mathcal{C}}(G(X), G(Y)) \cong \text{Hom}_{\mathcal{D}}(X, Y)$. If $X', Y' \in \mathcal{A}(\mathcal{C})$, then $\text{Hom}_{\mathcal{D}}(F(X'), F(Y')) \cong \text{Hom}_{\mathcal{C}}(X', Y')$.

We can also have the following lemma.

Lemma 2.9. 1) Let W be an injective cogenerator in \mathcal{D} . Then (F, G) is right semidualizing if and only if $W \in \mathcal{B}(\mathcal{D})$.

2) Let Q be a projective generator Q in \mathcal{C} . Then (F, G) is left semidualizing if and only if $Q \in \mathcal{A}(\mathcal{C})$.

Our aim now is to characterize the classes $\mathcal{A}(\mathcal{C})$ and $\mathcal{B}(\mathcal{D})$. We will do so in terms of two other natural classes of objects. Now let (F, G) be a right semidualizing adjoint pair and let $\mathcal{U} \subseteq \mathcal{C}$ be the class of objects U such that $U \cong G(E)$ for E injective object in \mathcal{D} .

Lemma 2.10. Let (F, G) be a right semidualizing adjoint pair. Then, every $X \in \mathcal{C}$ has an \mathcal{U} -preenvelope. If \mathcal{D} is a Grothendieck category then every $X \in \mathcal{C}$ has an \mathcal{U} -envelope.

Proof. Let $X \in \mathcal{C}$ and consider $i : F(X) \rightarrow E$ a monomorphism with E injective (which exists since \mathcal{D} has enough injectives). Then the composition $X \xrightarrow{u_X} GF(X) \xrightarrow{G(i)} G(E)$ is the desired preenvelope: let $U' \in \mathcal{U}$ and morphism $X \rightarrow U'$. Then we have $F(X) \rightarrow F(U') \cong FG(E') \cong E'$ for some injective E' , and so there is a morphism $E \rightarrow E'$ such that the preceding morphism $F(X) \rightarrow E'$ factors $F(X) \rightarrow E \rightarrow E'$. But then, we get $G(E) \rightarrow G(E')$ such that $X \rightarrow G(E) \rightarrow G(E')$ is the morphism $X \rightarrow G(E') \cong U'$.

Now, if \mathcal{D} is Grothendieck, then we can take $i : F(X) \rightarrow E$ an injective envelope. Let $f : G(E) \rightarrow G(E)$ such that $f \circ \phi = \phi$, where $\phi = G(i) \circ u_X$. Then $F(f) \circ F(\phi) = F(f) \circ FG(i) \circ F(u_X) = F(\phi)$. But since $F(\phi) = FG(i)F(u_X) = i$ is an injective envelope, it follows that $F(f)$ is an automorphism, so $f \cong GF(f)$ (because $G(E) \in \mathcal{A}(\mathcal{C})$) is an automorphism. \blacksquare

Theorem 2.11. *Let (F, G) be a right semidualizing adjoint pair and $X \in \mathcal{C}$. The following assertions are equivalent:*

- i) $X \in \mathcal{A}(\mathcal{C})$.
- ii) *There is an exact sequence*

$$\cdots P_1 \rightarrow P_0 \rightarrow U^0 \rightarrow U^1 \rightarrow \cdots$$

in \mathcal{C} , where every P_i is projective and every $U^i \in \mathcal{U}$, such that $X = \text{Ker}(U^0 \rightarrow U^1)$ and which remains exact when F is applied.

- iii) *There is an exact sequence*

$$0 \rightarrow X \rightarrow U^0 \rightarrow U^1 \rightarrow \cdots$$

in \mathcal{C} , where every $U^i \in \mathcal{U}$ such that remains exact when $\text{Hom}_{\mathcal{C}}(-, U)$ is applied for every $U \in \mathcal{U}$ and F leaves exact every projective resolution of X .

Proof. i) \Rightarrow ii) Let

$$\cdots \rightarrow P_1 \rightarrow P_0 \rightarrow X \rightarrow 0$$

be a projective resolution of X in \mathcal{C} . Then $\mathbf{L}_i F(X) = 0 \forall i \geq 1$ gives that

$$\cdots \rightarrow F(P_1) \rightarrow F(P_0) \rightarrow F(X) \rightarrow 0$$

is exact. Now let

$$0 \rightarrow F(X) \rightarrow E^0 \rightarrow E^1 \rightarrow \cdots$$

be an injective resolution of $F(X)$. If we denote $U^i = G(E^i)$ we get that the complex

$$0 \rightarrow X \cong GF(X) \rightarrow U^0 \rightarrow U^1 \rightarrow \cdots$$

is exact since $\mathbf{R}^i GF(X) = 0 \forall i \geq 1$. Finally the sequence also remains exact when F is applied by its construction.

- ii) \Leftrightarrow iii)

$$0 \rightarrow X \rightarrow U^0 \rightarrow U^1 \rightarrow \cdots$$

remains exact when $\text{Hom}_{\mathcal{C}}(-, U)$ for every $U \in \mathcal{U}$ is applied if and only if it remains exact when $\text{Hom}_{\mathcal{C}}(-, G(E))$ is applied for every injective $E \in \mathcal{D}$, and this assertion is equivalent to $\text{Hom}_{\mathcal{D}}(F(-), E)$ leaves the sequence exact for every injective $E \in \mathcal{D}$, and so, this is equivalent to that F leaves the sequence exact (since \mathcal{D} has an injective cogenerator by hypothesis). Finally, F leaves

$$\cdots \rightarrow P_1 \rightarrow P_0 \rightarrow X \rightarrow 0$$

exact if and only if F leaves exact any projective resolution of X . Here we use the fact that $\mathbf{L}_i F(X) = 0 \forall i \geq 1$.

iii) \Rightarrow i) The fact that F leaves exact any projective resolution of X implies that $\mathbf{L}_i F(X) = 0 \forall i \geq 1$. On the other hand, $U^i \cong G(E^i)$ for some $E^i \in \mathcal{D}$ injective, and so $F(U^i) \cong FG(E^i) \cong E^i$. Therefore, the natural morphism $U^i \rightarrow GF(U^i)$ is an isomorphism and

$$0 \rightarrow F(X) \rightarrow F(U^0) \rightarrow F(U^1) \rightarrow \cdots$$

is an injective resolution of $F(X)$, since \mathcal{D} has an injective cogenerator by hypothesis. But then, the complex

$$0 \rightarrow GF(X) \rightarrow GF(U^0) \rightarrow GF(U^1) \rightarrow \cdots$$

is isomorphic to the exact sequence

$$0 \rightarrow X \rightarrow U^0 \rightarrow U^1 \rightarrow \cdots,$$

which shows that $X \cong GF(X)$ and $\mathbf{R}^i GF(X) = 0 \forall i \geq 1$. ■

Proposition 2.12. *Let (F, G) be a right semidualizing adjoint pair and let $0 \rightarrow X' \rightarrow X \rightarrow X'' \rightarrow 0$ be exact in \mathcal{C} . If any two of X', X and X'' are in $\mathcal{A}(\mathcal{C})$, then so is the third.*

Proof. If $X'' \in \mathcal{A}(\mathcal{C})$, then $\mathbf{L}_1 F(X'') = 0$ and so

$$0 \rightarrow F(X') \rightarrow F(X) \rightarrow F(X'') \rightarrow 0$$

is exact. If $X \in \mathcal{A}(\mathcal{C})$, then we have an exact sequence

$$0 \rightarrow \mathbf{L}_1 F(X'') \rightarrow F(X') \rightarrow F(X) \rightarrow F(X'') \rightarrow 0$$

and so

$$0 \rightarrow G\mathbf{L}_1 F(X'') \rightarrow GF(X') \rightarrow GF(X)$$

is exact. Now, if $X, X' \in \mathcal{A}(\mathcal{C})$, we get that

$$0 \rightarrow GL_1 F(X'') \rightarrow X' \rightarrow X$$

is exact and therefore, since G is left exact, $L_1 F(X'') = 0$. In this way, if two of X', X and X'' are in $\mathcal{A}(\mathcal{C})$, then

$$0 \rightarrow F(X') \rightarrow F(X) \rightarrow F(X'') \rightarrow 0$$

is exact, which is equivalent to

$$0 \rightarrow Hom_{\mathcal{D}}(F(X''), E) \rightarrow Hom_{\mathcal{D}}(F(X), E) \rightarrow Hom_{\mathcal{D}}(F(X'), E) \rightarrow 0$$

is exact for every $E \in \mathcal{D}$ injective and so

$$0 \rightarrow Hom_{\mathcal{C}}(X'', G(E)) \rightarrow Hom_{\mathcal{C}}(X, G(E)) \rightarrow Hom_{\mathcal{C}}(X', G(E)) \rightarrow 0$$

is exact for every $E \in \mathcal{D}$ injective. Therefore

$$0 \rightarrow Hom_{\mathcal{C}}(X'', U) \rightarrow Hom_{\mathcal{C}}(X, U) \rightarrow Hom_{\mathcal{C}}(X', U) \rightarrow 0$$

is exact for every $U \in \mathcal{U}$. Now, by [13, Lemma 8.2.1], we can get an \mathcal{U} -resolution $0 \rightarrow X \rightarrow U^0 \rightarrow U^1 \rightarrow \dots$ such that it becomes exact when $Hom_{\mathcal{C}}(-, U)$ is applied for every $U \in \mathcal{U}$ from such \mathcal{U} -resolutions for X' and X'' , as well as a projective resolution. If we paste the two resolutions over $0 \rightarrow X' \rightarrow X \rightarrow X'' \rightarrow 0$ we get an exact sequence of complexes such that remains exact when we apply F . Now if two of these complexes are exact, then so is the third and so the result follows using the preceding theorem. \blacksquare

To get a dual characterization for $\mathcal{B}(\mathcal{D})$, and following what was done for $\mathcal{A}(\mathcal{C})$, we consider now the natural class \mathcal{W} of objects W in \mathcal{D} such that $W \cong F(P)$, where P is a projective object in \mathcal{C} . Also, let $\overline{\mathcal{W}}$ be the class of objects \overline{W} in \mathcal{D} such that $\overline{W} \cong F(\lim_i P_i)$, where $\lim_i P_i$ is a direct colimit of projective objects in \mathcal{C} .

Lemma 2.13. *Let (F, G) be a left semidualizing adjoint pair. Then every $Y \in \mathcal{D}$ has a \mathcal{W} -precover.*

If \mathcal{C} is a Grothendieck category and $\mathbf{R}^i G$ preserves arbitrary direct colimits for all $i \geq 0$, then every $Y \in \mathcal{D}$ has a $\overline{\mathcal{W}}$ -cover.

Proof. The proof of the first assertion is dual to that of Lemma 2.10 where we consider $P \rightarrow G(Y) \rightarrow 0$ exact with $P \in \mathcal{C}$ projective.

The second assertion is also dual to Lemma 2.10 by considering $P \rightarrow G(Y) \rightarrow 0$ a flat cover, which exists by [4, Theorem 3.2] and taking into account that the class of flat objects (i.e. direct colimits of projectives objects) in \mathcal{C} is contained in $\mathcal{A}(\mathcal{C})$ (by the hypothesis). \blacksquare

A dual proof to that of Theorem 2.11 gives the following.

Theorem 2.14. *Let (F, G) be a left semidualizing adjoint pair. The following assertions are equivalent for $Y \in \mathcal{D}$:*

- i) $Y \in \mathcal{B}(\mathcal{D})$.
- ii) *There is an exact sequence*

$$\cdots \rightarrow W_1 \rightarrow W_0 \rightarrow E^0 \rightarrow E^1 \rightarrow \cdots$$

in \mathcal{D} , where every E^i is injective and $W \in \mathcal{W}$, such that G leaves it exact and $Y = \text{Ker}(E^0 \rightarrow E^1)$.

- iii) *There is an exact \mathcal{W} -resolution*

$$\cdots \rightarrow W_1 \rightarrow W_0 \rightarrow Y \rightarrow 0$$

such that remains exact when $\text{Hom}_{\mathcal{D}}(W, -)$ is applied for every $W \in \mathcal{W}$ and G leaves exact every injective resolution of Y .

Proposition 2.15. *In the conditions of the preceding theorem, let $0 \rightarrow Y' \rightarrow Y \rightarrow Y'' \rightarrow 0$ be exact in \mathcal{D} . If any of two Y', Y and Y'' are in $\mathcal{B}(\mathcal{D})$, then so is the third.*

Remark 1. *As an immediate consequence of the fact that projective objects are in $\mathcal{A}(\mathcal{C})$ when (F, G) is left semidualizing and the last result we get that objects of finite projective dimension in \mathcal{C} belong to $\mathcal{A}(\mathcal{C})$. A dual result holds for objects of finite injective dimension in \mathcal{D} .*

We also can give information about Gorenstein projective and injective objects. We denote by \mathcal{GP} (resp. \mathcal{GI}) the class of Gorenstein projective object in \mathcal{C} (resp. Gorenstein injective objects in \mathcal{D}). For these definitions see [13]. The proof of the following proposition is similar to [14, Proposition 1.3], but we give it for completeness.

Proposition 2.16. *Let (F, G) be an adjoint pair.*

i) *If (F, G) is left semidualizing and \mathcal{D} has an injective cogenerator W such that $G(W)$ has finite projective dimension, then $\mathcal{GP} \subseteq \mathcal{A}(\mathcal{C})$.*

ii) *If (F, G) is right semidualizing and \mathcal{C} has a projective generator Q such that $F(Q)$ has finite injective dimension, then $\mathcal{GI} \subseteq \mathcal{B}(\mathcal{D})$.*

Proof. i) Let X be a Gorenstein projective object in \mathcal{C} . Then there exists an exact sequence

$$\cdots P_1 \xrightarrow{\delta_1} P_0 \xrightarrow{\delta_0} P_{-1} \rightarrow \cdots$$

where $X = \text{Ker}(\delta_0)$ and the sequence is $\text{Hom}_{\mathcal{C}}(-, P)$ -exact for every projective object $P \in \mathcal{C}$. Then $\text{Hom}_{\mathcal{C}}(-, G(W))$ leaves exact the above exact sequence ($G(W)$ has finite projective dimension), so $\text{Hom}_{\mathcal{D}}(F(-), W)$ also verifies it. Hence $F(-)$ leaves the sequence exact and $\mathbf{L}_i F(X) = 0$. Then

$$0 \rightarrow F(X) \rightarrow F(P_0) \rightarrow F(P_{-1})$$

and thus

$$0 \rightarrow GF(X) \rightarrow GF(P_0) \rightarrow GF(P_{-1})$$

is also exact. But $P_i \in \mathcal{A}(\mathcal{C})$, then $X \cong GF(X)$ canonically. Now let $0 \rightarrow X \rightarrow P_0 \rightarrow Y \rightarrow 0$ be exact. Then Y is also Gorenstein projective, so $Y \cong GF(Y)$ canonically.

Thus

$$0 \rightarrow F(X) \rightarrow F(P_0) \rightarrow F(Y) \rightarrow 0$$

is exact, and applying G , we get

$$0 \rightarrow X \rightarrow P_0 \rightarrow Y \rightarrow \mathbf{R}^1G(F(X)) \rightarrow \mathbf{R}^1G(F(P_0)) = 0.$$

This implies that $\mathbf{R}^1GF(X) = 0$. But then since we also have $\mathbf{R}^1GF(Y) = 0$ and $\mathbf{R}^2G(F(P_0)) = 0$, we get $\mathbf{R}^2GF(X) = 0$. Now by induction we get $\mathbf{R}^iGF(X) = 0$ for all $i \geq 1$.

Similar argument give ii). ■

In the following results we will study the orthogonal classes with respect to the Ext^1 of the classes \mathcal{U} and $\overline{\mathcal{W}}$. Our aim is to characterize when the pairs $({}^\perp\mathcal{U}, \mathcal{U})$ and $(\overline{\mathcal{W}}, \overline{\mathcal{W}}^\perp)$ are cotorsion theories (see [13] for the definitions). The proofs are similar to [16, Section 2], so they will be omitted.

Theorem 2.17. *Let (F, G) be a right semidualizing adjoint pair. The following assertions are equivalent:*

- i) $({}^\perp\mathcal{U}, \mathcal{U})$ is a cotorsion theory.
 - ii) Every injective object is in \mathcal{U} .
 - iii) Every \mathcal{U} -envelope $X \rightarrow U$ is a monomorphism.
 - iv) $u_X : X \rightarrow GF(X)$ is a monomorphism for all $X \in \mathcal{C}$.
 - v) $u_E : E \rightarrow GF(E)$ is a monomorphism for all $E \in \mathcal{C}$ injective.
- If $({}^\perp\mathcal{U}, \mathcal{U})$ is a cotorsion theory, then it is a perfect cotorsion theory.

Theorem 2.18. *If (F, G) is a left semidualizing adjoint pair, \mathcal{C} is a Grothendieck category and \mathbf{R}^iG preserves arbitrary direct colimits for all $i \geq 0$, then the following assertions are equivalent:*

- i) $(\overline{\mathcal{W}}, \overline{\mathcal{W}}^\perp)$ is a cotorsion theory.
 - ii) Every projective object is in \mathcal{W} .
 - iii) Every $\overline{\mathcal{W}}$ -cover $P \rightarrow Y$ is an epimorphism.
 - iv) $c_Y : FG(Y) \rightarrow Y$ is an epimorphism for all $Y \in \mathcal{D}$.
 - v) $c_P : FG(P) \rightarrow P$ is an epimorphism for all $P \in \mathcal{D}$ projective.
- If $(\overline{\mathcal{W}}, \overline{\mathcal{W}}^\perp)$ is a cotorsion theory, then it is a perfect cotorsion theory.

3 Tilting adjoint pairs

The aim of this section is to define a tilting adjoint pair and connect this definition with the definition of a semidualizing adjoint pair.

Let (F, G) be an adjoint pair as in Section 2. For $m \geq 0$ we will denote

$$\text{Ker}F_m = \{X \in \mathcal{C} \mid \mathbf{L}_i F(X) = 0 \forall i \neq m, i \geq 0\},$$

$$\text{Ker}G_m = \{X \in \mathcal{D} \mid \mathbf{R}^i G(X) = 0 \forall i \neq m, i \geq 0\}.$$

The following definition is due to B. Keller, [23].

Definition 3.1. *Suppose that F and G have finite cohomological dimension $\leq k$ and let m be an integer such that $0 \leq m \leq k$.*

We will say that the adjoint pair (F, G) is a tilting adjoint pair if the categories $\text{Ker}F_m$ and $\text{Ker}G_m$ are equivalent under the functors $\mathbf{L}_m F$ and $\mathbf{R}^m G$, $\forall 0 \leq m \leq k$.

Lemma 3.1. *Let (F, G) be a right semidualizing adjoint pair. Suppose that F has finite cohomological dimension $\leq k$. If $X \in \text{Ker}G_0$, then $G(X) \in \text{Ker}F_0$ and $c_X : FG(X) \rightarrow X$ is an isomorphism.*

Proof. Let $0 \rightarrow X \xrightarrow{d^0} E^0 \xrightarrow{d^1} E^1 \xrightarrow{d^2} \dots \xrightarrow{d^{k-1}} E^{k-1} \xrightarrow{d^k} E^k \rightarrow \dots$ be an injective resolution and let $L^i = \text{Coker}(d^{i-1})$ $i \geq 1$. Then we have an exact sequence

$$0 \rightarrow G(X) \rightarrow G(E^0) \rightarrow \dots \rightarrow G(E^k) \rightarrow G(E^{k+1})$$

If we denote $Y^i = G(L^i)$, since by the hypothesis it follows that $G(E^i) \in \text{Ker}F_0$, we get that $G(X)$, Y^1 and Y^2 are in $\text{Ker}F_0$ (F having finite cohomological dimension). Since, by hypothesis, $E^i \in \mathcal{B}(\mathcal{D})$, it follows that $c_{E^i} : FG(E^i) \rightarrow E^i$ are isomorphisms $\forall i \geq 0$. So, from the commutative diagram with exact rows

$$\begin{array}{ccccccc} 0 & \longrightarrow & FG(X) & \longrightarrow & FG(E^0) & \longrightarrow & FG(E^1) \\ & & \downarrow & & \downarrow \cong & & \downarrow \cong \\ 0 & \longrightarrow & X & \longrightarrow & E^0 & \longrightarrow & E^1 \end{array}$$

we get that $c_X : FG(X) \rightarrow X$ is an isomorphism. ■

Dually we get:

Lemma 3.2. *Let (F, G) be a left semidualizing adjoint pair. Suppose that G has finite cohomological dimension $\leq k$. If $Y \in \text{Ker}F_0$ then $F(Y) \in \text{Ker}G_0$ and $u_Y : Y \rightarrow GF(Y)$ is an isomorphism.*

Theorem 3.3. *Let (F, G) be a right semidualizing adjoint pair. Suppose that F has finite cohomological dimension $\leq k$. Let $m \geq 0$ be an integer and let $X \in \text{Ker}G_m$. Then $\mathbf{R}^m G(X) \in \text{Ker}F_m$ and there is an isomorphism $\mathbf{L}_m F(\mathbf{R}^m G(X)) \cong X$.*

Proof. By Lemma 3.1 we may suppose that $m \geq 1$. Now let

$$0 \rightarrow X \xrightarrow{d^0} E^0 \xrightarrow{d^1} E^1 \xrightarrow{d^2} \dots \xrightarrow{d^{k-1}} E^{m-1} \xrightarrow{d^m} E^m \rightarrow \dots$$

be an injective resolution and let $L^i = \text{Coker}(d^{i-1})$. By general properties of right derived functors $\mathbf{R}^i G(L^m) \cong \mathbf{R}^{i+m} G(X) = 0 \forall i \geq 1$ and so, $L^m \in \text{Ker}G_0$. Then,

by Lemma 3.1 $G(L^m) \in \text{Ker}F_0$ and $c_{L^m} : FG(L^m) \rightarrow L^m$ is an isomorphism. If we apply G to the injective resolution we get

$$0 \rightarrow G(X) \rightarrow G(E^0) \rightarrow \cdots \rightarrow G(E^{m-1}) \rightarrow G(E^m) \rightarrow Y \rightarrow 0$$

where $\mathbf{R}^1G(L^{n-1}) \cong \cdots \cong \mathbf{R}^nG(X) = Y$. Now $G(E^i) \in \text{Ker}F_0$ by hypothesis and so, from general properties of derived functors, it follows that $\mathbf{L}_iF(Y) = 0 \forall i \geq m+1$ and therefore

$$0 \rightarrow FG(E^0) \rightarrow \cdots \rightarrow FG(E^m) \rightarrow FG(Y) \rightarrow 0 \quad (*)$$

is exact.

Now, for $i = 0, \dots, m-2$, let

$$G(E^i) \rightarrow Y^{i+1} \rightarrow G(E^{i+1})$$

be a factorization and consider $Y = Y^{m-1}$. From the exact sequence

$$0 \rightarrow Y^i \rightarrow G(E^i) \rightarrow Y^{i+1} \rightarrow 0$$

we get an exact sequence

$$0 = \mathbf{L}_1F(G(E^i)) \rightarrow \mathbf{L}_1F(Y^{i+1}) \rightarrow F(Y^i) \rightarrow FG(E^i) \rightarrow F(Y^{i+1}) \rightarrow 0$$

for $i = 0, \dots, m-2$. But the fact that $(*)$ is exact is equivalent to $F(Y_0) = 0$ and $\mathbf{L}_1F(Y_i) = 0$ for $i = 0, \dots, m-2$. Now, since $G(E^i) \in \text{Ker}F_0$ $i = 0, \dots, m-2$, we get

$$\mathbf{L}_iF(Y^{m-1}) \cong \mathbf{L}_{i-1}F(Y^{m-2}) \cong \cdots \cong \mathbf{L}_1F(Y^{m-i+1}) = 0$$

and so $\mathbf{L}_iF(Y) = 0$, $i \geq 0$, $i \neq m$, i.e., $Y = \mathbf{R}^mG(X) \in \text{Ker}F_m$.

Moreover,

$$\mathbf{L}_mF(\mathbf{R}^mG(X)) \cong \text{Ker}(FG(E^0) \rightarrow FG(E^1)) \cong \text{Ker}(E^0 \rightarrow E^1) = X.$$

■

Now, a dual proof using Lemma 3.2 gives the following result.

Theorem 3.4. *Let (F, G) be a left semidualizing adjoint pair. Suppose that G has finite cohomological dimension $\leq k$. Let $m \geq 0$ be an integer and let $X \in \text{Ker}F_m$. Then $\mathbf{L}_mF(X) \in \text{Ker}G_m$ and there is an isomorphism $\mathbf{R}^mG(\mathbf{L}_mF(X)) \cong X$.*

Then, using Theorems 3.3 and 3.4 we get the following result which may be viewed as a generalization of Tilting Theorem.

Theorem 3.5. *Let (F, G) be an adjoint pair. Suppose that F and G have finite cohomological dimension $\leq k$.*

The following assertions are equivalent:

- 1) (F, G) is a right and left semidualizing adjoint pair.
- 2) (F, G) is a tilting adjoint pair.

Proof. We only have to prove $2) \Rightarrow 1)$. If $E \in \mathcal{D}$ is injective, then $E \in \text{Ker}G_0$ and so $G(E) \in \text{Ker}F_0$, therefore $\mathbf{L}_iF(G(E)) = 0 \forall i \geq 1$. Also, by the equivalence for $m = 0$ we get that $FG(E) \cong E$. Therefore $E \in \mathcal{B}(\mathcal{D})$.

The proof for projectives is dual. ■

Remark 2. If $k = 0$, then $\text{Ker}G_0 = \mathcal{D}$, $\text{Ker}F_0 = \mathcal{C}$ and (F, G) are mutually inverse equivalences of categories.

If $k = 1$, then F and G give an equivalence between the categories of F -acyclic objects in \mathcal{C} and G -acyclic objects in \mathcal{D} . Also $\text{Ker}F$ and $\text{Ker}G$ are equivalent via \mathbf{L}_1F and \mathbf{R}^1G .

Another consequence of Theorems 3.3 and 3.4 is the following.

Corollary 3.6. In the conditions of Theorem 3.5:

- i) If $X \in \mathcal{D}$ is such that $\mathbf{R}^iG(X) = 0$ for $i = 0, \dots, k$, then $X = 0$.
- ii) Let $Y \in \mathcal{C}$. If $\mathbf{L}_iF(Y) = 0$ for $i = 0, \dots, k$, then $Y = 0$.

If \mathcal{C} is an abelian category, we recall that the Grothendieck group of \mathcal{C} , denoted, $\mathcal{K}_0(\mathcal{C})$, is defined as follows: for each object $X \in \mathcal{C}$ there is a generator $[X]$; for each exact sequence

$$0 \rightarrow X' \rightarrow X \rightarrow X'' \rightarrow 0$$

in \mathcal{C} there is a relation

$$[X] = [X'] + [X''].$$

The following result generalizes [31, Theorem 1.19]. Its proof follows by the same arguments.

Theorem 3.7. Let (F, G) be an adjoint pair as above. Suppose that F and G have finite cohomological dimension $\leq k$. If (F, G) is a tilting adjoint pair, then the map

$$\mathbf{LF} : \mathcal{K}_0(\mathcal{C}) \rightarrow \mathcal{K}_0(\mathcal{D}), \quad [X] \mapsto \sum_{i \geq 0} [\mathbf{L}_iF(X)],$$

is an isomorphism. The inverse is given by

$$\mathbf{RG} : \mathcal{K}_0(\mathcal{D}) \rightarrow \mathcal{K}_0(\mathcal{C}), \quad [X] \mapsto \sum_{i \geq 0} [\mathbf{R}^iG(X)].$$

4 Semidualizing comodules

We start this section with an useful result that gives a characterization of semidualizing comodules.

Proposition 4.1. Let C be a coalgebra, T a quasi-finite right C -comodule and $D = e_C(T)$.

i) Suppose that $D = e_C(T)$ is a left semiperfect coalgebra and ${}_D T$ is a quasi-finite left D -comodule. Then, $\text{ext}_C^i(T, T) = 0 \forall i \geq 1$ if and only if the adjoint pair $(h_D(T, -), T \square_C -)$ is left semidualizing.

ii) Suppose that ${}_D T$ is a quasi-finite left D -comodule and $C \cong e_D(T)$ canonically. Then

$\text{ext}_D^i(T, T) = 0 \forall i \geq 1$ if and only if the adjoint pair $(h_D(T, -), T \square_C -)$ is right semidualizing.

iii) $\text{ext}_C^i(T, T) = 0 \forall i \geq 1$ if and only if the adjoint pair $(h_C(T, -), -\square_D T)$ is right semidualizing.

iv) If C is a right semiperfect coalgebra, then $\text{ext}_C^i(T, T) = 0 \forall i \geq 1$ if and only if the adjoint pair $(h_C(T, -), -\square_D T)$ is left semidualizing.

Proof. i) By hypothesis ${}^D \mathcal{M}$ has a projective generator and any projective left D -comodule is a direct sum of finite dimensional left D -comodules of the form E^* , where E is a finite dimensional injective right D -comodule. In particular, E^* is a direct summand of a finite direct sum of copies of D^* . Let P be a projective left D -comodule. In order to show that $P \in \mathcal{A}({}^D \mathcal{M})$ it is enough to take $P = E^*$ a finite dimensional projective left D -comodule.

We have, by Lemma 1.2, that

$$T \square_C h_D(T, E^*) \cong T \square_C \text{Com}_D(E^*, T)^* \cong \text{Com}_C(\text{Com}_D(E^*, T), T).$$

In this way the unit of the adjunction $u_{E^*} : E^* \rightarrow T \square_C h_D(T, E^*)$ is nothing but the evaluation map $E^* \rightarrow \text{Com}_C(\text{Com}_D(E^*, T), T)$. Since this last map is an isomorphism for D^* (by the same argument as above) we conclude that the same is true for E^* .

On the other hand, let

$$0 \rightarrow T \rightarrow I^0 \rightarrow I^1 \rightarrow \dots$$

be an injective resolution of T in \mathcal{M}^C . The homology of the complex

$$0 \rightarrow I^0 \square_C h_D(T, E^*) \rightarrow I^1 \square_C h_D(T, E^*) \rightarrow \dots$$

is precisely $\text{Tor}_C^i(T, h_D(T, E^*))$. But by the above

$$I^i \square_C h_D(T, E^*) \cong \text{Com}_C(\text{Com}_D(E^*, T), I^i)$$

naturally. Therefore $\text{Tor}_C^i(T, h_D(T, E^*)) \cong \text{Ext}_C^i(\text{Com}_D(E^*, T), T)$ which is a direct summand of

$$\text{Ext}_C^i(\text{Hom}_{D^*}((D^*)^n, T), T) \cong \text{Ext}_C^i(T, T)^n \cong (\text{ext}_C^i(T, T)^*)^n = 0.$$

ii) Immediate from the facts that $C \in \mathcal{B}({}^D \mathcal{M})$, $T \square_C -$ preserves arbitrary direct sums, and any injective left C -comodule is a direct summand of a direct sum of copies of C .

The proofs of (iii) and (iv) are similar to (i) and (ii). ■

Let C and D be right and left semiperfect coalgebras respectively over the same field k . Most of the results of this paper are valid for k a commutative ring such that C and D are k -flat, however we do not make such a generalization at this time. Let T be a $(D - C)$ -bicomodule such that $T_C \in \mathcal{M}^C$ and ${}_D T \in {}^D \mathcal{M}$ are quasi-finite. In that case, we have two adjoint situations, $(h_D(T, -), T \square_C -)$, where $h_D(T, -) : {}^D \mathcal{M} \rightarrow {}^C \mathcal{M}$, $T \square_C - : {}^C \mathcal{M} \rightarrow {}^D \mathcal{M}$ and similarly $(h_C(T, -), -_D \square T)$ on the right.

Definition 4.1. We will say that T is a semidualizing $(D - C)$ -bicomodule if the adjoint pairs $(h_D(T, -), T \square_C -)$ and $(h_C(T, -), -_D \square T)$ are right semidualizing.

The following result is immediate from the definition.

Proposition 4.2. A $(D - C)$ -bicomodule ${}_D T_C$ is a semidualizing bicomodule if and only if it satisfies the following two conditions:

- i) $h_D(T, T) \cong C$ and $h_C(T, T) \cong D$. (T is balanced)
- ii) $ext_D^i(T, T) = 0$ and $ext_C^i(T, T) = 0 \forall i \geq 1$. (T is self-orthogonal)

From a right C -comodule T , we can find conditions for ${}_D T_C$ to be semidualizing for $D = e_C(T)$.

Proposition 4.3. Let T be a quasi-finite right C -comodule and let $D = e_C(T)$ its endomorphisms coalgebra. Suppose ${}_D T$ is quasi-finite left D -comodule, $ext_C^i(T, T) = 0 \forall i \geq 1$ and let $D = e_C(T)$ be left semiperfect. If there is an exact sequence

$$0 \rightarrow T_n \rightarrow \cdots \rightarrow T_1 \rightarrow T_0 \rightarrow C \rightarrow 0$$

for some $n \geq 0$ with $T_i \in add(T_C)$ for $i = 0, \dots, n$, then ${}_D T_C$ is semidualizing.

Proof. Since $T \cong h_D(h_C(T, T), T)$ we get that $T_i \cong h_D(h_C(T_i, T), T)$ for $i = 0, \dots, n$ and so we get a commutative diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & h_D(h_C(T_n, T), T) & \longrightarrow & \cdots & \longrightarrow & h_D(h_C(T_0, T), T) & \longrightarrow & h_D(T, T) & \longrightarrow & 0 \\ & & \downarrow & & & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & T_n & \longrightarrow & \cdots & \longrightarrow & T_0 & \longrightarrow & C & \longrightarrow & 0, \end{array}$$

where the vertical arrows are isomorphisms. By hypothesis

$$0 \rightarrow T \cong h_C(C, T) \rightarrow h_C(T_0, T) \rightarrow \cdots \rightarrow h_C(T_n, T) \rightarrow 0$$

is exact and so is an injective resolution of ${}_D T$. Therefore $e_D(T) \cong C$ and $ext_D^i(T, T) = 0, \forall i \geq 1$. ■

5 n-cotilting comodules

In this section we will consider a particular class of semidualizing comodules that give equivalence of categories and which generalize other classical results.

Following the definitions of [36, Section 3] we give a general definition of n -cotilting comodule and n - f -cotilting comodule.

Definition 5.1. *Let T be a right C -comodule. We will say that T is a n -cotilting comodule if the following four conditions are satisfied.*

- (i) T is quasi-finite.
- (ii) $\text{inj.dim}_C T \leq n$.
- (iii) $\text{Ext}_C^i(T^I, T) = 0$ for any index set I and all $i \geq 1$.
- (iv) There exists an exact sequence

$$0 \rightarrow T_n \rightarrow T_{n-1} \rightarrow \cdots \rightarrow T_1 \rightarrow T_0 \rightarrow C \rightarrow 0$$

in \mathcal{M}^C with $T_i \in \text{Prod}(T)$ for all $i \geq 0$.

We define T to be an n - f -cotilting comodule if T satisfies (i) and the following three conditions:

- ii') T admits a finite injective resolution

$$0 \rightarrow T \rightarrow E^0 \rightarrow E^1 \rightarrow \cdots \rightarrow E^n \rightarrow 0$$

where the comodules E^i are quasi-finite injective and lie in $\text{add}(C)$ for every $i \geq 0$.

- iii') $\text{Ext}_C^i(T, T) = 0 \forall i \geq 1$.
- iv') There exists an exact sequence

$$0 \rightarrow T_n \rightarrow T_{n-1} \rightarrow \cdots \rightarrow T_1 \rightarrow T_0 \rightarrow C \rightarrow 0$$

in \mathcal{M}^C with $T_i \in \text{add}(T)$ for all $i \geq 0$.

We note that classical 1-cotilting, [30], is different from 1-cotilting in this new sense.

Example 5.1. 1) *Let C be a Gorenstein coalgebra [3, Definition 2.1]. Then C is right and left semiperfect and there is minimal projective resolution of quasi-finite comodules in \mathcal{M}^C (see [28, Theorem 3.1])*

$$0 \rightarrow P_k \rightarrow P_{k-1} \rightarrow \cdots \rightarrow P_0 \rightarrow C \rightarrow 0.$$

By [3, Lemma 2.2], any projective right C -comodule has finite injective dimension. We easily can see that $T = \bigoplus_{i=0}^k P_i$ is a generalized f -cotilting right C -comodule.

2) *Examples of 1- f -cotilting comodules over quivers coalgebras can be found in [36].*

In the following result, we show that n -cotilting comodules T over a quasi-co-Frobenius coalgebra C are trivial in the sense that $e_C(T)$ is Morita-Takeuchi equivalent to C .

Theorem 5.2. *If C is a quasi-co-Frobenius coalgebra, every n -cotilting comodule is an injective cogenerator.*

Proof. Let C be a quasi-co-Frobenius coalgebra, which is equivalent to that C is projective in \mathcal{M}^C and let T be in \mathcal{M}^C such that T is a n -cotilting comodule. Then, there is an exact sequence

$$0 \rightarrow T_n \rightarrow T_{n-1} \rightarrow \cdots \rightarrow T_0 \rightarrow C \rightarrow 0$$

with $T_i \in \text{Prod}(T)$. This sequence splits at C , so $C \oplus M = T^J$, for some $M \in \mathcal{M}^C$ and some set J . Therefore T is a cogenerator. Also there is an exact sequence

$$0 \rightarrow T \rightarrow I_0 \rightarrow I_1 \rightarrow \cdots \rightarrow I_n \rightarrow 0.$$

Since every injective is also projective, this sequence splits everywhere, so $T \oplus N = C^{(I)}$ for some $N \in \mathcal{M}^C$ and some set I . This implies that T is injective. \blacksquare

Now we show that duals of f -cotilting comodules are tilting modules in the sense of Miyashita, [31].

Proposition 5.3. *Let C be a right semiperfect coalgebra and let $T \in \mathcal{M}^C$. If T_C is a k - f -cotilting comodule, then $T_{C^*}^* \in \mathcal{M}_{C^*}$ is a tilting module of projective dimension less or equal than k .*

Proof. It is clear that $T_{C^*}^*$ is finitely generated. Now if we consider

$$0 \rightarrow T \rightarrow E^0 \rightarrow \cdots \rightarrow E^k \rightarrow 0$$

exact with E^i injective finitely cogenerated, we get that E^i is a direct summand of $C^{(n)}$ for some n and so, E^{i*} is a direct summand of $C^{(n)*}$, which gives that $pd(T^*) \leq k$.

On the other hand, from the exact sequence

$$0 \rightarrow T_k \rightarrow \cdots \rightarrow T_1 \rightarrow T_0 \rightarrow C \rightarrow 0$$

with $T_i \in \text{add}(T_C)$ for $i = 0, \dots, k$ we get

$$0 \rightarrow C^* \rightarrow T_0^* \rightarrow \cdots \rightarrow T_k^* \rightarrow 0$$

with $T_i^* \in \text{add}(T^*)$ $i = 0, \dots, k$.

Finally, from the exact sequence

$$0 \rightarrow T \rightarrow E^0 \rightarrow \cdots \rightarrow E^k \rightarrow 0$$

with E^i injective finitely cogenerated, we get by [30, Lemma 5.1] that the complexes,

$$0 \rightarrow \text{Com}_C(T, T) \rightarrow \text{Com}_C(T, E^0) \rightarrow \cdots \rightarrow \text{Com}_C(T, E^k) \rightarrow 0$$

and

$$0 \rightarrow \text{Hom}_{C^*}(T^*, T^*) \rightarrow \text{Hom}_{C^*}(E^{0*}, T^*) \rightarrow \cdots \rightarrow \text{Hom}_{C^*}(E^{k*}, T^*) \rightarrow 0$$

are isomorphic and so, give isomorphic homology groups. Now since $\text{ext}_C^i(T, T) = 0$, $\forall i \geq 1$ we get that $\text{Ext}_C^i(T, T) = 0$, $\forall i \geq 1$ and so, $\text{Ext}_{C^*}^i(T^*, T^*) = 0$, $\forall i \geq 1$. \blacksquare

Proposition 5.4. *Let C be right semiperfect and let $D = e_C(T)$ be left semiperfect. If T is a k -f-cotilting comodule, then ${}_D T$ is k -f-cotilting and $e_D(T) \cong C$.*

Proof. By Proposition 4.3 we get that ${}_D T_C$ is semidualizing and $e_D(T) \cong C$. On the other hand, by the hypothesis, there is an exact sequence in \mathcal{M}^C

$$0 \rightarrow T \xrightarrow{d^0} E^0 \xrightarrow{d^1} \dots \xrightarrow{d^k} E^k \rightarrow 0$$

where every E^i is a direct summand of $C^{(n)}$. Now consider $L_i = \text{Coker}(d^{i-1})$, $i = 1, \dots, k-1$. Then for $j = 1, \dots, k-2$, $\text{ext}_C^i(L_j, T) \cong \text{ext}_C^{i-1}(L_{j+1}, T)$. In particular, for $j = 1, \dots, k-1$, we get that

$$\text{ext}_C^1(L_j, T) \cong \text{ext}_C^2(L_{j-1}, T) \cong \dots \cong \text{ext}_C^{k+1}(L_{j-k}, T) = 0$$

taking into account that $Y_{-1} = T$, and so we get an exact sequence in ${}^D \mathcal{M}$

$$0 \rightarrow h_C(E^k, T) \rightarrow h_C(E^{k-1}, T) \rightarrow h_C(E^0, T) \rightarrow h_C(T, T) = D \rightarrow 0.$$

Now, since $h_C(C, T) \cong {}_D T$ and E^i are direct summands of $C^{(n)}$, we get that $h_C(E^i, T) \in \text{add}({}_D T)$ for $i = 0, \dots, k$.

Finally, let

$$0 \rightarrow T_k \xrightarrow{d_k} T_{k-1} \xrightarrow{d_{k-1}} \dots \xrightarrow{d_1} T_0 \xrightarrow{d_0} C \rightarrow 0$$

be exact in \mathcal{M}^C and $T_i \in \text{add}(T_C)$. Since $\text{ext}_C^i(T, T) = 0$, $\forall i \geq 1$ it follows that $\text{ext}_C^i(T_j, T) = 0$, $\forall i \geq 1$ and $j = 0, \dots, k$. By the same reasoning as above, we get an exact sequence in ${}^D \mathcal{M}$

$$0 \rightarrow T \cong h_C(C, T) \rightarrow h_C(T_0, T) \rightarrow \dots \rightarrow h_C(T_k, T) \rightarrow 0$$

Now the fact that for $j = 0, \dots, k$, $h_C(T_j, T)$ is a direct summand of $h_C(T^{(n)}, T) \cong D^{(n)}$ gives that $\text{id}({}_D T) \leq k$. ■

For the next Theorem we need to introduce some notation.

Let k be a non-negative fixed integer. For $T \in \mathcal{M}^C$ and $m \geq 0$ we will denote

$$\begin{aligned} \text{Kerext}_m(T_C) &= \{M \in \mathcal{M}^C : \text{ext}_C^i(T, M) = 0 \text{ for } i = 0, \dots, k+m \text{ and } i \neq m\} \\ \text{KerTor}_m(T_C) &= \{M \in {}^C \mathcal{M} : \text{Tor}_C^i(T, M) = 0 \text{ for } i = 0, \dots, k+m \text{ and } i \neq m\} \end{aligned}$$

For ${}^C T' \in {}^C \mathcal{M}$ $\text{Kerext}_m({}^C T')$ and $\text{KerTor}_m({}^C T')$ are defined analogously.

Now, by Theorem 3.5 we have:

Theorem 5.5. *Let C and D be right and left semiperfect coalgebras respectively and T a $(D-C)$ -bicomodule such that T_C and ${}_D T$ are quasi-finite. Suppose:*

i) ${}_D T_C$ is a semidualizing bicomodule.

ii) $\text{id}(T_C) \leq k$ and $\text{id}({}_D T) \leq k$.

Let m be an integer such that $0 \leq m \leq k$.

Then the categories $\text{KerTor}_m(T_C)$ and $\text{Kerext}_m({}_D T)$ are equivalent under the functors $\text{Tor}_C^m(T, -)$ and $\text{ext}_D^m(T, -)$.

At the end of Section 3 we have defined the Grothendieck group of an abelian category. For a coalgebra C we denote by $\mathcal{K}_0(C)$ the Grothendieck group of ${}^C\mathcal{M}$ and by $K_0(C)$ the Grothendieck group of ${}^C\mathcal{M}_{f.d.}$, the full subcategory of finitely dimensional left C -comodules.

Theorem 5.6. *Under conditions of Theorem 5.5, the Grothendieck groups of C and D are isomorphic, that is, $K_0(C) \cong K_0(D)$ and $\mathcal{K}_0(C) \cong \mathcal{K}_0(D)$. In particular the number of isomorphism classes of simple left C -comodules is equal to the number of isomorphic classes of simple left D -comodules.*

Proof. Follows from Theorem 3.7 noting that derived functors of $h_D(T, -)$ and $T \square_C -$ preserve finite dimensional comodules. ■

Corollary 5.7. *Under conditions of Theorem 5.5, C is finite dimensional if and only if D is finite dimensional.*

Proof. Follows by Theorem 5.6 and [11, Exercise. 3.3.13]. ■

Acknowledgement. First and third authors are supported by the grants MTM2011-27090 from Ministerio de Ciencia e Innovación FEDER and P07-FQM-0312 from Junta de Andalucía. The second author is supported by DGES BFM2002-02717 grant and Junta de Andalucía FQM 0211.

We thank the referee for pointing out the article [35] and for his/her helpful suggestions for improving the paper.

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