Existence and properties of solutions of a boundary problem for a Love's equation

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Abstract

In this paper, we use the Faedo - Galerkin method, compactness method and monotone method in order to study a nonlinear Love's equation with mixed nonhomogeneous conditions. The results obtained here are existence of a weak solution, uniqueness, regularity and asymptotic behavior of solutions.

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1 Introduction

In this paper, we consider the following equation with initial conditions and mixed nonhomogeneous conditions

$$u_{tt} - u_{xx} - \varepsilon u_{xxtt} + \lambda \left| u_t \right|^{q-2} u_t + K \left| u \right|^{p-2} u = F(x,t),\ x \in \Omega = (0,1),\ 0 < t < T, \tag{1.1}$$

$$\varepsilon u_{xtt}(0,t) + u_x(0,t) = hu(0,t) + g(t), \tag{1.2}$$

$$u(1,t) = 0, \tag{1.3}$$

$$u(x,0) = \tilde{u}_0(x),\ u_t(x,0) = \tilde{u}_1(x), \tag{1.4}$$

where $p > 1$, $q > 1$, $\varepsilon > 0$, $\lambda > 0$, $K > 0$, $h \geq 0$ are constants and $\tilde{u}_0$, $\tilde{u}_1$, F, g are given functions satisfying conditions specified later.

When $F = 0$, $\lambda = K = 0$, $\Omega = (0, L)$, Equation (1.1) is related to the Love's equation

$$u_{tt} - \tfrac{E}{\rho} u_{xx} - 2\mu^2 k^2 u_{xxtt} = 0, \tag{1.5}$$

presented by V. Radochová in 1978 (see [8]). This equation describes the vertical oscillations of a rod, which was established from Euler's variational equation of an energy function

$$\int_0^T dt \int_0^L \left[\tfrac{1}{2}F\rho\left(u_t^2+\mu^2k^2u_{tx}^2\right)-\tfrac{1}{2}F\left(Eu_x^2+\rho\mu^2k^2u_xu_{xtt}\right)\right]dx, \tag{1.6}$$

the parameters in (1.6) have the following meanings: u is the displacement, L is the length of the rod, F is the area of cross-section, k is the cross-section radius, E is the Young modulus of the material and ρ is the mass density. By using the Fourier method, Radochová [8] obtained a classical solution of problem (1.5) associated with initial conditions (1.4) and boundary conditions

$$u(0,t)=u(L,t)=0, \tag{1.7a}$$

or

$$\begin{cases} u(0,t)=0, \\ \varepsilon u_{xtt}(L,t)+c^2u_x(L,t)=0, \end{cases} \tag{1.7b}$$

where $c^2=\frac{E}{\rho}$, $\varepsilon=2\mu^2k^2$. On the other hand, the asymptotic behaviour of the solution of problem (1.4), (1.5), (1.7a) or (1.7b) as $\varepsilon\to 0_+$ are also established by the method of small parameter.

Equations of Love waves or equations for waves of Love types have been studied by many authors, we refer to [3], [4], [7] and references therein.

In [1], Ang and Dinh established a uniqueness and global existence for the problem (1.1) – (1.4) with $\varepsilon=K=h=0$, $\lambda=1$, $1<q<2$, $F(x,t)=0$. In this latter case this problem governs the motion of a linear viscoelastic bar.

In this paper, we shall use the Faedo - Galerkin method, compactness method and monotone method in order to study problem (1.1) - (1.4). The results obtained are existence of a weak solution, uniqueness, regularity and asymptotic behavior of solutions.

The paper consists of four sections. Section 2 is devoted to the study of the existence a weak solution for problem (1.1) – (1.4) with $\tilde{u}_0$, $\tilde{u}_1\in V=\{v\in H^1: v(1)=0\}$, $p>1$, $q>1$. Here, a energy lemma (as given in Lemma 2.4) is also established in order to pass the limit of a approximate problem and prove the uniqueness in case $p\geq 2$. In section 3, we consider the regularity of solution for problem (1.1) - (1.4) with $\tilde{u}_0$, $\tilde{u}_1\in V\cap H^2$, $p\geq 2$, $q\geq 2$ and some other conditions. In case $p=q=2$, we show that the regularity of solutions depending on the regularity of data. Finally, the asymptotic behavior of solutions as $\varepsilon\to 0_+$ is discussed in Section 4. The results obtained here may be considered as the generalizations of those in [8].

2 Existence and uniqueness of a solution

First, we put $\Omega=(0,1)$; $Q_T=\Omega\times(0,T)$, $T>0$ and we denote the usual function spaces used in this paper by the notations $C^m\left(\overline{\Omega}\right)$, $W^{m,p}=W^{m,p}(\Omega)$, $L^p=W^{0,p}(\Omega)$, $H^m=W^{m,2}(\Omega)$, $1\leq p\leq\infty$, $m=0,1,...$ Let $\langle\cdot,\cdot\rangle$ be either the scalar product in L^2 or the dual pairing of a continuous linear functional and an element of a function space. The notation $\|\cdot\|$ stands for the norm in L^2 and we denote by $||\cdot||_X$ the norm in the Banach space X. We call X' the dual space of X. We denote by $L^p(0,T;X)$, $1\leq p\leq\infty$ for the Banach space of the real functions $u:(0,T)\to X$ measurable, such that

$$\|u\|_{L^p(0,T;X)}=\left(\int_0^T\|u(t)\|_X^p\,dt\right)^{1/p}<\infty \text{ for } 1\leq p<\infty,$$

and

$$\|u\|_{L^\infty(0,T;X)} = \underset{0<t<T}{ess\sup}\|u(t)\|_X \ \text{ for } p=\infty.$$

Let $u(t)$, $u'(t) = u_t(t)$, $u''(t) = u_{tt}(t)$, $u_x(t)$, $u_{xx}(t)$ denote $u(x,t)$, $\frac{\partial u}{\partial t}(x,t)$, $\frac{\partial^2 u}{\partial t^2}(x,t)$, $\frac{\partial u}{\partial x}(x,t)$, $\frac{\partial^2 u}{\partial x^2}(x,t)$, respectively.

On H^1 we shall use the following norm

$$\|v\|_{H^1} = \left(\|v\|^2 + \|v_x\|^2\right)^{1/2}.$$

We put

$$V = \{v \in H^1 : v(1) = 0\}. \tag{2.1}$$

Then V is a closed subspace of H^1 and on V, $v \longmapsto ||v||_{H^1}$ and $v \longmapsto ||v_x||$ are equivalent norms.

Then the following lemmas are known as a standard one.

Lemma 2.1. *The imbedding $H^1 \hookrightarrow C^0([0,1])$ is compact and*

$$||v||_{C^0(\overline{\Omega})} \le \sqrt{2}||v||_{H^1} \text{ for all } v \in H^1. \tag{2.2}$$

Lemma 2.2. *The imbedding $V \hookrightarrow C^0([0,1])$ is compact and*

$$||v||_{C^0(\overline{\Omega})} \le ||v_x|| \text{ for all } v \in V. \tag{2.3}$$

We remark that the weak formulation of the initial-boundary value problem (1.1) - (1.4) can be given in the following manner: Find $u \in L^\infty(0,T;V)$, with $u_t \in L^\infty(0,T;V)$, such that u satisfies the following variational equation

$$\begin{cases} \frac{d}{dt}\left[\langle u_t(t), w\rangle + \varepsilon\langle u_{xt}(t), w_x\rangle\right] + \langle u_x(t), w_x\rangle + (hu(0,t) + g(t))\, w(0) \\ \qquad +\lambda\langle |u_t(t)|^{q-2}\, u_t(t), w\rangle + K\langle |u|^{p-2}\, u, w\rangle = \langle F(t), w\rangle, \end{cases} \tag{2.4}$$

for all $w \in V$, a.e., $t \in (0,T)$, together with the initial conditions

$$u(0) = \tilde{u}_0,\ u_t(0) = \tilde{u}_1. \tag{2.5}$$

Next, we need the following assumptions:

- (H_1) $p > 1$, $q > 1$, $\lambda > 0$, $K > 0$, $\varepsilon > 0$, $h \ge 0$;
- (H_2) $\tilde{u}_0, \tilde{u}_1 \in V$;
- (H_3) $F \in L^1(0,T;L^2)$;
- (H_4) $g \in W^{1,1}(0,T)$.

Then, we have the following theorem.

Theorem 2.3. *Let $T > 0$. Suppose that $(H_1) - (H_4)$ hold. Then, there exists a weak solution u of problem (1.1) - (1.4) such that*

$$u \in L^\infty(0,T;V),\ u_t \in L^\infty(0,T;V). \tag{2.6}$$

Furthermore, if $p \ge 2$, the solution is unique.

Proof. The proof is a combination of Galerkin method and compactness arguments, and consits of four steps.

Step 1. *The Faedo-Galerkin approximation* (introduced by Lions [6]). Consider the basis in V

$$w_j(x) = \sqrt{\tfrac{2}{1+\lambda_j^2}}\cos(\lambda_j x),\ \lambda_j = (2j-1)\tfrac{\pi}{2},\ j\in\mathbb{N},$$

constructed by the eigenfunctions of the Laplace operator $-\Delta = -\frac{\partial^2}{\partial x^2}$. Put

$$u_m(t) = \textstyle\sum_{j=1}^m c_{mj}(t)w_j, \tag{2.7}$$

where the coefficients $c_{mj}^{(k)}$ satisfy the system of nonlinear ordinary differential equations

$$\begin{cases} \langle u_m''(t), w_j\rangle + \langle u_{mx}(t) + \varepsilon u_{mx}''(t), w_{jx}\rangle + \lambda\left\langle |u_m'(t)|^{q-2}\, u_m'(t), w_j\right\rangle \\ \qquad +K\left\langle |u_m(t)|^{p-2}\, u_m(t), w_j\right\rangle + (hu_m(0,t)+g(t))\, w_j(0) = \langle F(t), w_j\rangle,\ 1\le j\le m, \\ u_m(0) = \tilde{u}_{0m},\ u_m'(0) = \tilde{u}_{1m}, \end{cases} \tag{2.8}$$

where

$$\begin{cases} \tilde{u}_{0m} = \sum_{j=1}^m \alpha_{mj}w_j \to \tilde{u}_0 \text{ strongly in } V, \\ \tilde{u}_{1m} = \sum_{j=1}^m \beta_{mj}w_j \to \tilde{u}_1 \text{ strongly in } V. \end{cases} \tag{2.9}$$

From the assumptions of Theorem 2.3, system (2.8) has a solution u_m on an interval $[0, T_m]\subset[0,T]$. The following estimates allow one to take $T_m = T$ for all m (see [2]).

Step 2. Multiplying the j^{th} equation of (2.8) by $c_{mj}'(t)$ and summing up with respect to j, afterwards, integrating by parts with respect to the time variable from 0 to t, after some rearrangements, we get

$$\begin{aligned} S_m(t) &= S_m(0) + 2g(0)\tilde{u}_{0m}(0) + 2\textstyle\int_0^t \langle F(s), u_m'(s)\rangle\, ds \\ &\quad +2\textstyle\int_0^t g'(s)u_m(0,s)ds - 2g(t)u_m(0,t) \\ &= S_m(0) + 2g(0)\tilde{u}_{0m}(0) + \textstyle\sum_{j=1}^3 I_j, \end{aligned} \tag{2.10}$$

where

$$\begin{aligned} S_m(t) &= \|u_m'(t)\|^2 + \|u_{mx}(t)\|^2 + \varepsilon\,\|u_{mx}'(t)\|^2 + hu_m^2(0,t) \\ &\quad +\tfrac{2K}{p}\,\|u_m(t)\|_{L^p}^p + 2\lambda\textstyle\int_0^t \|u_m'(s)\|_{L^q}^q\, ds. \end{aligned} \tag{2.11}$$

By (2.9), (2.11) and the imbedding $H^1 \hookrightarrow C^0\left(\overline{\Omega}\right)$, there exists a positive constant $\bar{C}_0$ depending only on $\tilde{u}_0$, $\tilde{u}_1$, h, K, p, $g(0)$ and ε, such that

$$\begin{aligned} S_m(0) + 2g(0)\tilde{u}_{0m}(0) + \tfrac{2K}{p}\,\|\tilde{u}_{0m}\|_{L^p}^p &= \|\tilde{u}_{1m}\|^2 + \|\tilde{u}_{0mx}\|^2 + \varepsilon\,\|\tilde{u}_{1mx}\|^2 \\ &\quad +h\tilde{u}_{0m}^2(0) + 2g(0)\tilde{u}_{0m}(0) + \tfrac{2K}{p}\,\|\tilde{u}_{0m}\|_{L^p}^p \le \tfrac{1}{2}\bar{C}_0, \text{ for all } m. \end{aligned} \tag{2.12}$$

Using (2.3) and the following inequalities

$$2ab \le \beta a^2 + \tfrac{1}{\beta}b^2,\ \text{ for all } a,\ b\in\mathbb{R},\ \beta>0, \tag{2.13}$$

and

$$|u_m(0,t)| \le \|u_m(t)\|_{C^0(\overline{\Omega})} \le \|u_{mx}(t)\| \le \sqrt{S_m(t)}, \tag{2.14}$$

we can estimate all terms in the right − hand side of (2.10) as follows

$$\begin{aligned} I_1 = 2\textstyle\int_0^t \langle F(s), u_m'(s)\rangle\, ds &\le \textstyle\int_0^t \|F(s)\|\, ds + \int_0^t \|F(s)\|\,\|u_m'(s)\|^2\, ds \\ &\le C_T + \textstyle\int_0^t \|F(s)\|\, S_m(s)\, ds, \end{aligned} \tag{2.15}$$

where C_T indicates a constant depending on T;

$$\begin{array}{l} I_2 = 2\int_0^t g'(s)u_m(0,s)ds \le 2\int_0^t |g'(s)|\sqrt{S_m(s)}ds \\ \quad \le \int_0^t |g'(s)|\,ds + \int_0^t |g'(s)|\,S_m(s)ds \\ \quad \le C_T + \int_0^t |g'(s)|\,S_m(s)ds, \end{array} \tag{2.16}$$

with $C_T \ge \int_0^T |g'(s)|\,ds$;

$$I_3 = -2g(t)u_m(0,t) \le 2\,\|g\|_{L^\infty(0,T)}\sqrt{S_m(t)} \le C_T + \tfrac{1}{2}S_m(t), \tag{2.17}$$

for all $\beta > 0$, $C_T \ge 2\,\|g\|^2_{L^\infty(0,T)}$.

Combining (2.10), (2.12), (2.15) - (2.17) and choose $\beta = \frac{1}{2}$, the result is

$$S_m(t) \le C_T + \int_0^t d_T^{(1)}(s)S_m(s)\,ds,\ \ 0 \le t \le T_m, \tag{2.18}$$

where $d_T^{(1)}(s) = 2\left[\|F(s)\| + |g'(s)|\right]$, $d_T^{(1)} \in L^1(0,T)$.

By Gronwall's lemma, we deduce from (2.18) that

$$S_m(t) \le C_T \exp\left[\int_0^T d_T^{(1)}(s)ds\right] \le C_T, \text{ for all } t \in [0,T], \tag{2.19}$$

where C_T always indicates a bound depending on T. Thus, we can take constant $T_m = T$ for all m.

On the other hand, we deduce from (2.11) and (2.19) that

$$\left\{ \begin{array}{l} \left\| |u_m|^{p-2}\,u_m \right\|^{p'}_{L^\infty(0,T;L^{p'})} = \|u_m\|^p_{L^\infty(0,T;L^p)} \le \frac{p}{2K}C_T \le C_T, \\ \left\| |u'_m|^{q-2}\,u'_m \right\|^{q'}_{L^{q'}(Q_T)} = \int_0^T \|u'_m(s)\|^q_{L^q}\,ds \le \frac{1}{2\lambda}C_T \le C_T, \end{array} \right. \tag{2.20}$$

where C_T always indicates a bound depending on T as above.

Step 3. *Limiting process.* From (2.11), (2.19), (2.20) we deduce the existence of a subsequence of $\{u_m\}$, denoted by the same symbol such that

$$\left\{ \begin{array}{cccc} u_m \to u & \text{in} & L^\infty(0,T;V) & \text{weakly*}, \\ u'_m \to u' & \text{in} & L^\infty(0,T;V) & \text{weakly*}, \\ u_m \to u & \text{in} & L^\infty(0,T;L^p) & \text{weakly*}, \\ u'_m \to u' & \text{in} & L^q(Q_T) & \text{weakly}, \\ |u_m|^{p-2}\,u_m \to \chi_0 & \text{in} & L^\infty(0,T;L^{p'}) & \text{weakly*}, \\ |u'_m|^{q-2}\,u'_m \to \chi_1 & \text{in} & L^{q'}(Q_T) & \text{weakly}. \end{array} \right. \tag{2.21}$$

By the compactness lemma of Lions ([6], p. 57), from $(2.21)_{1,2}$, there exists a subsequence of $\{u_m\}$, still denoted by $\{u_m\}$, such that

$$u_m \to u \quad \text{strongly in} \quad L^2(Q_T) \quad \text{and a.e. in } Q_T. \tag{2.22}$$

By means of the continuity of function $x \longmapsto |x|^{p-2}x$, we have

$$|u_m|^{p-2}u_m \to |u|^{p-2}u \quad \text{a.e. in } Q_T. \tag{2.23}$$

Using Lions's Lemma ([6], Lemma 1.3, p.12), it follows from $(2.20)_1$ and (2.23) that

$$|u_m|^{p-2}u_m \to |u|^{p-2}u \quad \text{in } L^{p'}(Q_T) \quad \text{weakly.} \tag{2.24}$$

By $(2.21)_5$ and (2.24), we deduce that

$$\chi_0 = |u|^{p-2}u. \tag{2.25}$$

Passing to the limit in (2.8) by (2.9), (2.21), (2.24) and (2.25), we have u satisfying the problem

$$\begin{cases} \frac{d}{dt}\left[\langle u'(t), v\rangle + \varepsilon \langle u'_x(t), v_x\rangle\right] + \langle u_x(t), v_x\rangle + \lambda \langle \chi_1(t), v\rangle + K\left\langle |u(t)|^{p-2}u(t), v\right\rangle \\ \qquad\qquad + (hu(0,t) + g(t))\,v(0) = \langle F(t), v\rangle, \text{ for all } v \in V, \\ u(0) = \tilde{u}_0,\ u'(0) = \tilde{u}_1. \end{cases} \tag{2.26}$$

It remains to prove that $\chi_1 = |u'|^{q-2}u'$. We need the following lemmas.

Lemma 2.4. *Let u be the weak solution of the following problem*

$$\begin{cases} u'' - u_{xx} - \varepsilon u''_{xx} = \Phi,\ 0 < x < 1,\ 0 < t < T, \\ \varepsilon u''_x(0,t) + u_x(0,t) = G(t),\ u(1,t) = 0, \\ u(0) = \tilde{u}_0,\ u'(0) = \tilde{u}_1, \\ u \in L^\infty(0,T;V), \quad u' \in L^\infty(0,T;V), \\ \tilde{u}_0,\ \tilde{u}_1 \in V,\ G \in L^2(0,T),\ \Phi \in L^1(0,T;L^2). \end{cases} \tag{2.27}$$

Then we have

$$\begin{aligned} &\tfrac{1}{2}\|u'(t)\|^2 + \tfrac{1}{2}\|u_x(t)\|^2 + \tfrac{\varepsilon}{2}\|u'_x(t)\|^2 + \textstyle\int_0^t G(s)u'(0,s)ds \\ &\qquad \geq \tfrac{1}{2}\|\tilde{u}_1\|^2 + \tfrac{1}{2}\|\tilde{u}_{0x}\|^2 + \tfrac{\varepsilon}{2}\|\tilde{u}_{1x}\|^2 + \textstyle\int_0^t \langle \Phi(s), u'(s)\rangle\, ds, \ \ a.e.,\ t \in [0,T]. \end{aligned} \tag{2.28}$$

Furthermore, if $\tilde{u}_0 = \tilde{u}_1 = 0$, there is equality in (2.28).

Proof of Lemma 2.4.
The idea of the proof is the same as in ([5], Lemma 2.1, p. 79). Fix t_1, t_2, $0 < t_1 < t_2 < T$ and let $v(x,t)$ be the function defined as follows

$$v(x,t) = \theta_m(t)[(\theta_m(t)u'(x,t)) * \rho_k(t) * \rho_k(t)], \tag{2.29}$$

where

(*i*) θ_m is a continuous, piecewise linear function on $[0,T]$ defined as follows:

$$\theta_m(t) = \begin{cases} 0, & \text{if}, \quad t \in [0,T] \setminus [t_1 + 1/m,\ t_2 - 1/m], \\ 1, & \text{if}, \quad t \in [t_1 + 2/m, t_2 - 2/m], \\ m(t - t_1 - 1/m), & \text{if}, \quad t \in [t_1 + 1/m, t_1 + 2/m], \\ -m(t - t_2 + 1/m), & \text{if}, \quad t \in [t_2 - 2/m, t_2 - 1/m]. \end{cases} \tag{2.30}$$

(*ii*) $\{\rho_k\}$ is a regularizing sequence in $C_c^\infty(\mathbb{R})$, i.e.,

$$\rho_k \in C_c^\infty(\mathbb{R}), \quad \rho_k(t) = \rho_k(-t), \quad \textstyle\int_{-\infty}^{+\infty} \rho_k(t)dt = 1,\ \text{supp } \rho_k \subset [-1/k, 1/k]. \tag{2.31}$$

(iii) $(*)$ is the convolution product in the time variable, ie.,

$$(u * \rho_k)(x,t) = \int_{-\infty}^{+\infty} u(x,t-s)\rho_k(s)ds. \tag{2.32}$$

We take the scalar product of the function $v(x,t)$ in (2.29) with Eq. $(2.27)_1$, then integrate with respect to the time variable from 0 to T, and we have

$$X_{mk} + Y_{mk} = Z_{mk}, \tag{2.33}$$

where

$$\begin{cases} X_{mk} = \int_0^T \langle u''(t), v(t)\rangle dt, \\ Y_{mk} = -\int_0^T \langle \frac{\partial}{\partial x}\left(u_x(t) + \varepsilon u_{xtt}(t)\right), v(t)\rangle dt, \\ Z_{mk} = \int_0^T \langle \Phi(t), v(t)\rangle dt. \end{cases} \tag{2.34}$$

By using the properties of the functions $\theta_m(t)$ and $\rho_k(t)$ we can show after some lengthy calculation

$$\begin{cases} \lim\limits_{k\to+\infty} X_{mk} = -\int_0^T \theta_m\theta_m' \|u'(t)\|^2 dt, \\ \lim\limits_{k\to+\infty} Y_{mk} = -\int_0^T \theta_m\theta_m' \|u_x(t)\|^2 dt - \varepsilon\int_0^T \theta_m\theta_m' \|u_x'(t)\|^2 dt + \int_0^T \theta_m^2 G(t)u'(0,t)dt, \\ \lim\limits_{k\to+\infty} Z_{mk} = \int_0^T \theta_m^2 \langle \Phi(t), u'(t)\rangle dt. \end{cases} \tag{2.35}$$

Letting $m \to \infty$, (2.33) – (2.35) yield

$$\begin{aligned} &\tfrac{1}{2}\|u'(t_2)\|^2 - \tfrac{1}{2}\|u'(t_1)\|^2 + \tfrac{1}{2}\|u_x(t_2)\|^2 - \tfrac{1}{2}\|u_x(t_1)\|^2 + \tfrac{\varepsilon}{2}\|u_x'(t_2)\|^2 - \tfrac{\varepsilon}{2}\|u_x'(t_1)\|^2 \\ &\quad + \int_{t_1}^{t_2} G(t)u'(0,t)dt = \int_{t_1}^{t_2}\langle \Phi(t), u'(t)\rangle dt, \ \text{a.e., } t_1\ t_2 \in (0,T),\ t_1 < t_2. \end{aligned} \tag{2.36}$$

From (2.36), using the weak lower semicontinuity of the functional $v \longmapsto \|v\|^2$, we obtain (2.28) by taking $t_2 = t$ and passing to the limit as $t_1 \to 0_+$.

In the case of $\tilde{u}_0 = \tilde{u}_1 = 0$, we prolong u, Φ, G by 0 as $t < 0$ and we deduce equality (2.36) is true for almost $t_1 < t_2 < T$. Taking $t_1 < 0$ in (2.36), its right-hand side is 0, we take $t_1 \to 0_-$, we have equality (2.28).

The proof of Lemma 2.4 is completed.

Remark 2.1. Lemma 2.4 is a relative generalization of a lemma of Lions ([6], Lemma 6.1, p. 224).

We now prove that $\chi_1 = |u'|^{q-2} u'$.

From (2.10) and (2.11) we deduce

$$\begin{aligned} &2\lambda\int_0^t \left\langle |u_m'(s)|^{q-2} u_m'(s), u_m'(s)\right\rangle ds = 2\lambda\int_0^t \|u_m'(s)\|_{L^q}^q ds \\ &\qquad = \|\tilde{u}_{1m}\|^2 + \varepsilon\|\tilde{u}_{1mx}\|^2 + \|\tilde{u}_{0mx}\|^2 + h\tilde{u}_{0m}^2(0) + \tfrac{2K}{p}\|\tilde{u}_{0m}\|_{L^p}^p \\ &\qquad - \|u_m'(t)\|^2 - \varepsilon\|u_{mx}'(t)\|^2 - \|u_{mx}(t)\|^2 - hu_m^2(0,t) - \tfrac{2K}{p}\|u_m(t)\|_{L^p}^p \\ &\qquad + 2\int_0^t \langle F(s), u_m'(s)\rangle ds - 2\int_0^t g(s)u_m'(0,s)ds. \end{aligned} \tag{2.37}$$

Using Lemma 2.4, with $\Phi = F - K\,|u|^{p-2}\,u - \lambda\chi_1$, $G(t) = hu(0,t) + g(t)$, it follows from (2.8), (2.9), (2.21), (2.28), (2.37) that

$$
\begin{aligned}
&2\lambda \limsup_{m\to\infty} \int_0^t \left\langle |u_m'(s)|^{q-2}\,u_m'(s), u_m'(s)\right\rangle ds \\
&\quad \le \|\tilde u_1\|^2 + \varepsilon\,\|\tilde u_{1x}\|^2 + \|\tilde u_{0x}\|^2 + h\tilde u_0^2(0) + \tfrac{2K}{p}\,\|\tilde u_0\|_{L^p}^p \\
&\quad - \liminf_{m\to\infty}\,\|u_m'(t)\|^2 - \varepsilon \liminf_{m\to\infty} \|u_{mx}'(t)\|^2 - \liminf_{m\to\infty} \left(\|u_{mx}(t)\|^2 + hu_m^2(0,t)\right) \\
&\quad -\tfrac{2K}{p} \liminf_{m\to\infty}\,\|u_m(t)\|_{L^p}^p + 2\int_0^t \langle F(s), u'(s)\rangle\, ds - 2\int_0^t g(s)u'(0,s)ds \\
&\quad \le \|\tilde u_1\|^2 + \varepsilon\,\|\tilde u_{1x}\|^2 + \|\tilde u_{0x}\|^2 + h\tilde u_0^2(0) + \tfrac{2K}{p}\,\|\tilde u_0\|_{L^p}^p \\
&\quad - \|u'(t)\|^2 - \varepsilon\,\|u_x'(t)\|^2 - \|u_x(t)\|^2 - hu^2(0,t) \\
&\quad -\tfrac{2K}{p}\,\|u(t)\|_{L^p}^p + 2\int_0^t \langle F(s), u'(s)\rangle\, ds - 2\int_0^t g(s)u'(0,s)ds \\
&\quad \le \|\tilde u_1\|^2 + \|\tilde u_{0x}\|^2 + \varepsilon\,\|\tilde u_{1x}\|^2 - \|u'(t)\|^2 - \|u_x(t)\|^2 - \varepsilon\,\|u_x'(t)\|^2 \\
&\quad +2\int_0^t \left\langle F(s) - K|u(s)|^{p-2}u(s) - \lambda\chi_1(s), u'(s)\right\rangle ds - 2\int_0^t \left(hu(0,s) + g(s)\right) u'(0,s)ds \\
&\quad +2\lambda\int_0^t \langle \chi_1(s), u'(s)\rangle\, ds \le 2\lambda \int_0^t \langle \chi_1(s), u'(s)\rangle\, ds.
\end{aligned} \tag{2.38}
$$

Consider

$$
\phi_m(t) = \int_0^t \left\langle |u_m'(s)|^{q-2}\,u_m'(s) - |v(s)|^{q-2}\,v(s), u_m'(s) - v(s)\right\rangle ds \ge 0, \tag{2.39}
$$

for all $v \in L^q(Q_T)$.

Combining $(2.21)_{2-6}$, (2.38) and (2.39), we have

$$
0 \le \limsup_{m\to\infty}\, \phi_m(t) \le \int_0^t \left\langle \chi_1(s) - |v(s)|^{q-2}\,v(s), u'(s) - v(s)\right\rangle ds, \text{ for all } v \in L^q(Q_T). \tag{2.40}
$$

In (2.40), choose $v(s) = u'(s) - \delta w$, with $\delta > 0$ and $w \in L^q(Q_T)$. Apply the argument of Minty and Browder (see Lions [6], p. 172), we obtain $\chi_1 = |u'|^{q-2}\,u'$.

The proof of existence is completed.

Step 4. *Uniqueness of the solution.* Assume now that $p \ge 2$ holds.

Let u, v be two weak solutions of the problem (1.1) – (1.4), such that

$$
u, v \in L^\infty(0,T;V) \text{ and } u', v' \in L^\infty(0,T;V). \tag{2.41}
$$

Then $w = u - v$ is the weak solution of the following problem

$$
\begin{cases}
w_{tt} - \varepsilon w_{xxtt} - w_{xx} = -\lambda\left(|u'|^{q-2}\,u' - |v'|^{q-2}\,v'\right) - K\left(|u|^{p-2}u - |v|^{p-2}v\right) = 0, \\
\varepsilon w_{xtt}(0,t) + w_x(0,t) = hw(0,t),\ w(1,t) = 0, \\
w(x,0) = w_t(x,0) = 0, \\
w,\ w' \in L^\infty(0,T;V).
\end{cases} \tag{2.42}
$$

Using Lemma 2.4 with $\tilde u_0 = \tilde u_1 = 0$, $\Phi = -\lambda\left(|u'|^{q-2}\,u' - |v'|^{q-2}\,v'\right) - K\left(|u|^{p-2}u - |v|^{p-2}v\right)$, $G(t) = hw(0,t)$, we obtain

$$
\begin{aligned}
&\sigma(t) + 2\lambda\int_0^t \left\langle |u'(s)|^{q-2}\,u'(s) - |v'(s)|^{q-2}\,v'(s), u'(s) - v'(s)\right\rangle ds \\
&\qquad = -2K\int_0^t \left\langle |u(s)|^{p-2}u(s) - |v(s)|^{p-2}v(s), w'(s)\right\rangle ds, \text{ a.e. } t \in [0,T],
\end{aligned} \tag{2.43}
$$

where

$$\sigma(t) = \|w'(t)\|^2 + \|w_x(t)\|^2 + \varepsilon \|w'_x(t)\|^2 + hw^2(0,t). \tag{2.44}$$

Using the following inequality

$$\left| \, |x|^{p-2}x - |y|^{p-2}y \right| \leq (p-1)M^{p-2} |x-y| \,, \, \forall x, y \in [-M, M], \, \forall M > 0, \, \forall p \geq 2, \tag{2.45}$$

with $M = \|u\|_{L^\infty(0,T;V)} + \|v\|_{L^\infty(0,T;V)}$, and note that

$$\begin{aligned} &\int_0^t \left\langle |u'(s)|^{q-2} u'(s) - |v'(s)|^{q-2} v'(s), u'(s) - v'(s) \right\rangle ds \geq 0, \\ &\sigma(t) = \|w'(t)\|^2 + \|w_x(t)\|^2 + \varepsilon \|w'_x(t)\|^2 \geq 2 \|w'(t)\| \|w_x(t)\| \,, \end{aligned} \tag{2.46}$$

we deduce from (2.43), (2.46) that

$$\begin{aligned} \sigma(t) &\leq -2K \int_0^t \langle |u(s)|^{p-2}u(s) - |v(s)|^{p-2}v(s), w'(s) \rangle \, ds \\ &\leq 2K(p-1)M^{p-2} \int_0^t \|w(s)\| \|w'(s)\| \, ds \leq K(p-1)M^{p-2} \int_0^t \sigma(s) ds. \end{aligned} \tag{2.47}$$

By Gronwall's lemma, it follows from (2.47) that $\sigma \equiv 0$, i.e., $u \equiv v$. Theorem 2.3 is proved completely. □

3 The regularity of solutions

In this section, we study the regularity of solutions of problem (1.1) – (1.4) corresponding to $(\tilde{u}_0, \tilde{u}_1) \in \left(V \cap H^2\right) \times \left(V \cap H^2\right)$.

Henceforth, we strengthen the hypotheses and assume that:

- (H'_1) $p \geq 2$, $q \geq 2$, $\lambda > 0$, $K > 0$, $\varepsilon > 0$, $h \geq 0$;
- (H'_2) $\tilde{u}_0$, $\tilde{u}_1 \in V \cap H^2$;
- (H'_3) F, $F' \in L^1(0,T;L^2)$;
- (H'_4) $g \in W^{2,1}(0,T)$.

First, we have the following theorem.

Theorem 3.1. *Let $T > 0$. Suppose that $(H'_1) - (H'_4)$ hold. Then problem (1.1) - (1.4) has a unique weak solution*

$$u \in L^\infty \left(0,T; V \cap H^2\right), \text{ such that } u_t, \, u_{tt} \in L^\infty \left(0,T; V \cap H^2\right). \tag{3.1}$$

Remark 3.1.

The regularity obtained by (3.1) shows that problem (1.1) – (1.4) has a unique strong solution

$$u \in C^1 \left(0,T; V \cap H^2\right), \;\; u_{tt} \in L^\infty \left(0,T; V \cap H^2\right). \tag{3.2}$$

Proof. The proof consists of four Steps as follows.

Step 1. The Faedo-Galerkin approximation. By the same argument as in Theorem 2.3, we obtain the approximate solution $u_m(t)$ of problem (1.1) – (1.4) in the form (2.7), where the coefficient functions c_{mj} satisfy the system (2.8), with

$$\begin{cases} \tilde{u}_{0m} = \sum_{j=1}^m \alpha_{mj} w_j \to \tilde{u}_0 \text{ strongly in } V \cap H^2, \\ \tilde{u}_{1m} = \sum_{j=1}^m \beta_{mj} w_j \to \tilde{u}_1 \text{ strongly in } V \cap H^2. \end{cases} \tag{3.3}$$

Step 2. *A priori estimates* I. Using assumptions $(H_1') - (H_4')$, similarly, we get

$$\begin{array}{c} S_m(t) = \|u_m'(t)\|^2 + \|u_{mx}(t)\|^2 + \varepsilon \|u_{mx}'(t)\|^2 + hu_m^2(0,t) \\ + \frac{2K}{p} \|u_m(t)\|_{L^p}^p + 2\lambda \int_0^t \|u_m'(s)\|_{L^q}^q \, ds \leq C_T, \end{array} \tag{3.4}$$

for all $t \in [0,T]$ and for all m, and C_T always indicates a bound depending on T.

A priori estimates II. Now differentiating $(2.8)_1$ with respect to t, we have

$$\begin{array}{l} \langle u_m'''(t), w_j \rangle + \langle u_{mx}'(t) + \varepsilon u_{mx}'''(t), w_{jx} \rangle + K(p-1) \left\langle |u_m(t)|^{p-2} \, u_m'(t), w_j \right\rangle \\ \quad + \lambda(q-1) \left\langle |u_m'(t)|^{q-2} \, u_m''(t), w_j \right\rangle + (hu_m'(0,t) + g'(t)) \, w_j(0) = \langle F'(t), w_j \rangle , \end{array} \tag{3.5}$$

for all $1 \leq j \leq m$.

Multiplying the j th equation of (3.5) by $c_{mj}''(t)$, summing up with respect to j and then integrating with respect to the time variable from 0 to t, we obtain

$$\begin{array}{rl} X_m(t) = & X_m(0) + 2g'(0)\tilde{u}_{1m}(0) + 2 \int_0^t \langle F'(s), u_m''(s) \rangle \, ds \\ & -2K(p-1) \int_0^t \left\langle |u_m(s)|^{p-2} \, u_m'(s), u_m''(s) \right\rangle ds \\ & -2g'(t)u_m'(0,t) + 2 \int_0^t g''(s)u_m'(0,s)ds \\ \equiv & X_m(0) + 2g'(0)\tilde{u}_{1m}(0) + \sum_{j=1}^4 J_j, \end{array} \tag{3.6}$$

where

$$\begin{array}{c} X_m(t) = \|u_m''(t)\|^2 + \|u_{mx}'(t)\|^2 + \varepsilon \|u_{mx}''(t)\|^2 + h \, |u_m'(0,t)|^2 \\ +2\lambda(q-1) \int_0^t ds \int_0^1 |u_m'(x,s)|^{q-2} \, |u_m''(x,s)|^2 \, dx. \end{array} \tag{3.7}$$

First, we estimate $\eta_m = \|u_m''(0)\|^2 + \varepsilon \|u_{mx}''(0)\|^2$.

Letting $t \to 0_+$ in Eq. $(2.8)_1$, multiplying the result by $c_{mj}''(0)$, then

$$\begin{array}{l} \|u_m''(0)\|^2 + \varepsilon \|u_{mx}''(0)\|^2 + \langle \tilde{u}_{0mx}, u_{mx}''(0) \rangle + \lambda \left\langle |\tilde{u}_{1m}|^{q-2} \, \tilde{u}_{1m}, u_m''(0) \right\rangle \\ \qquad + (h\tilde{u}_{0m}(0) + g(0) \;) \, u_m''(0,0) \\ \qquad +K \left\langle |\tilde{u}_{0m}|^{p-2} \, \tilde{u}_{0m}, u_m''(0) \right\rangle = \langle F(0), u_m''(0) \rangle . \end{array} \tag{3.8}$$

Note that

$$|u_m''(0,0)| \leq \|u_m''(0)\|_{C^0([0,1])} \leq \|u_{mx}''(0)\| \leq \tfrac{1}{\sqrt{\varepsilon}} \sqrt{\eta_m}. \tag{3.9}$$

This implies that

$$\begin{array}{rl} \eta_m = \|u_m''(0)\|^2 + \varepsilon \|u_{mx}''(0)\|^2 \leq & \|\tilde{u}_{0mx}\| \, \|u_{mx}''(0)\| + |h\tilde{u}_{0m}(0) + g(0)| \; |u_m''(0,0)| \\ & + \left[\lambda \left\| |\tilde{u}_{1m}|^{q-1} \right\| + K \left\| |\tilde{u}_{0m}|^{p-1} \right\| + \|F(0)\| \; \right] \|u_m''(0)\| \\ \leq & \frac{1}{2\gamma} \|\tilde{u}_{0mx}\|^2 + \frac{\gamma}{2} \|u_{mx}''(0)\|^2 + \frac{1}{2\gamma} \left(|h\tilde{u}_{0m}(0) + g(0)| \right)^2 + \frac{1}{2\varepsilon}\gamma\eta_m \\ & + \frac{1}{2\gamma} \left[\lambda \left\| |\tilde{u}_{1m}|^{q-1} \right\| + K \left\| |\tilde{u}_{0m}|^{p-1} \right\| + \|F(0)\| \; \right]^2 + \frac{\gamma}{2} \|u_m''(0)\|^2 \\ \leq & \frac{1}{2\gamma} \|\tilde{u}_{0mx}\|^2 + \frac{\gamma}{2\varepsilon}\eta_m + \frac{1}{2\gamma} \left(|h\tilde{u}_{0m}(0) + g(0)| \right)^2 + \frac{1}{2\varepsilon}\gamma\eta_m \\ & + \frac{1}{2\gamma} \left[\lambda \left\| |\tilde{u}_{1m}|^{q-1} \right\| + K \left\| |\tilde{u}_{0m}|^{p-1} \right\| + \|F(0)\| \; \right]^2 + \frac{\gamma}{2}\eta_m \\ \leq & \frac{1}{2\gamma} \|\tilde{u}_{0mx}\|^2 + \frac{1}{2\gamma} \left(|h\tilde{u}_{0m}(0) + g(0)| \right)^2 \\ & + \frac{1}{2\gamma} \left[\lambda \left\| |\tilde{u}_{1m}|^{q-1} \right\| + K \left\| |\tilde{u}_{0m}|^{p-1} \right\| + \|F(0)\| \; \right]^2 \\ & + \frac{\gamma}{2} \left[1 + \frac{2}{\varepsilon} \right] \eta_m, \text{ for all } \gamma > 0. \end{array} \tag{3.10}$$

Choose $\gamma > 0$, such that $\frac{\gamma}{2}\left[1+\frac{2}{\varepsilon}\right] \le \frac{1}{2}$, we have

$$\begin{aligned} \eta_m = \|u''_m(0)\|^2 + \varepsilon \|u''_{mx}(0)\|^2 \le \tfrac{1}{\gamma}\|\tilde{u}_{0mx}\|^2 + \tfrac{1}{\gamma}\left(|h\tilde{u}_{0m}(0) + g(0)|\right)^2 \\ +\tfrac{1}{\gamma}\left[\lambda \left\| |\tilde{u}_{1m}|^{q-1} \right\| + K \left\| |\tilde{u}_{0m}|^{p-1} \right\| + \|F(0)\| \right]^2 \le \overline{X}_0 \text{ for all } m, \end{aligned} \tag{3.11}$$

where $\overline{X}_0$ is a constant depending only on p, q, K, λ, F, $\tilde{u}_0$, $\tilde{u}_1$, h, $g(0)$ and ε.

By (3.3), (3.7) and (3.11), we get

$$\begin{aligned} X_m(0) + 2g'(0)\tilde{u}_{1m}(0) &= \eta_m + \|\tilde{u}_{1mx}\|^2 + h\tilde{u}^2_{1mx}(0) + 2g'(0)\tilde{u}_{1m}(0) \\ &\le \overline{X}_0 + \|\tilde{u}_{1mx}\|^2 + h\tilde{u}^2_{1mx}(0) + 2g'(0)\tilde{u}_{1m}(0) \le \tfrac{1}{2}X_0, \text{ for all } m, \end{aligned} \tag{3.12}$$

where X_0 is a constant depending only on p, q, K, λ, F, $\tilde{u}_0$, $\tilde{u}_1$, h, $g(0)$ and ε.

A combination of (2.3), (2.14), (3.7) and the following inequalities

$$X_m(t) \ge \|u''_m(t)\|^2 + \|u'_{mx}(t)\|^2 + \varepsilon \|u''_{mx}(t)\|^2, \tag{3.13}$$

$$|u'_m(0,t)| \le \|u'_m(t)\|_{C^0(\overline{\Omega})} \le \|u'_{mx}(t)\| \le \sqrt{X_m(t)}, \tag{3.14}$$

all terms on the right-hand side of (3.6) are estimated as follows

$$\begin{aligned} J_1 = 2\int_0^t \langle F'(s), u''_m(s)\rangle\, ds \le \|F'\|_{L^1(0,T;L^2)} + \int_0^t \|F'(s)\|\, X_m(s)\, ds \\ \le C_T + \int_0^t \|F'(s)\|\, X_m(s)\, ds; \end{aligned} \tag{3.15}$$

$$\begin{aligned} J_2 &= -2K(p-1)\int_0^t \left\langle |u_m(s)|^{p-2}\, u'_m(s), u''_m(s)\right\rangle ds \\ &\le 2K(p-1)\int_0^t \|u_{mx}(s)\|^{p-2}\, \|u'_m(s)\|\, \|u''_m(s)\|\, ds \\ &\le 2K(p-1)\int_0^t \left(\sqrt{S_m(s)}\right)^{p-2} \sqrt{S_m(s)}\sqrt{X_m(s)} ds \\ &\le 2(p-1)\sqrt{C_T^{p-1}}\int_0^t \sqrt{X_m(s)} ds \le C_T + \int_0^t X_m(s) ds; \end{aligned} \tag{3.16}$$

$$\begin{aligned} J_3 = -2g'(t)u'_m(0,t) \le 2|g'(t)|\,|u'_m(0,t)| \le 2|g'(t)|\sqrt{X_m(t)} \\ \le \tfrac{1}{\beta}\|g'\|^2_{L^\infty(0,T)} + \beta X_m(t) \le \tfrac{1}{\beta}C_T + \beta X_m(t); \end{aligned} \tag{3.17}$$

$$\begin{aligned} J_4 = 2\int_0^t g''(s)u'_m(0,s) ds \le 2\int_0^t |g''(s)|\sqrt{X_m(s)} ds \\ \le \int_0^t |g''(s)|\left[1 + X_m(s)\right] ds \le C_T + \int_0^t |g''(s)|\, X_m(s)\, ds, \end{aligned} \tag{3.18}$$

where C_T also indicates a bound depending on T and $C_T \ge \int_0^T |g''(s)|\, ds$.

Combining (3.6), (3.12), (3.15) − (3.18) and choose $\beta = \frac{1}{2}$, the result is

$$X_m(t) \le C_T + 2\int_0^t \left(1 + |g''(s)| + \|F'(s)\|\right) X_m(s)\, ds,\ 0 \le t \le T, \tag{3.19}$$

where C_T indicates a bound depending on T as above.

By Gronwall's lemma, we deduce from (3.19) that

$$X_m(t) \le C_T \exp\left[2\int_0^T \left(1 + |g''(s)| + \|F'(s)\|\right) ds\right] \le C_T, \text{ for all } t \in [0,T], \tag{3.20}$$

where C_T always indicates a bound depending on T.

Step 3. *Limiting process.* From (3.4), (3.7), (3.20), we deduce the existence of a subsequence of $\{u_m\}$ still also so denoted, such that

$$\begin{cases} u_m \to u & \text{in} \quad L^\infty(0,T;V) \quad \text{weakly*}, \\ u'_m \to u' & \text{in} \quad L^\infty(0,T;V) \quad \text{weakly*}, \\ u''_m \to u'' & \text{in} \quad L^\infty(0,T;V) \quad \text{weakly*}. \end{cases} \tag{3.21}$$

By the compactness lemma of Lions ([6], p. 57), from (3.21), there exists a subsequence of $\{u_m\}$, denoted by the same symbol, such that

$$\begin{cases} u_m \to u & \text{strongly in} \quad L^2(Q_T) \quad \text{and a.e. in } Q_T, \\ u'_m \to u' & \text{strongly in} \quad L^2(Q_T) \quad \text{and a.e. in } Q_T. \end{cases} \tag{3.22}$$

Using again the inequality (2.45), with $M = C_T$, we deduce from (3.22) that

$$\begin{array}{ll} |u_m|^{p-2}u_m \to |u|^{p-2}u & \text{strongly in} \quad L^2(Q_T), \\ |u'_m|^{q-2}u'_m \to |u'|^{q-2}u' & \text{strongly in} \quad L^2(Q_T). \end{array} \tag{3.23}$$

Passing to the limit in (2.8), by (3.3), (3.21) – (3.23), we have u satisfying the problem

$$\begin{cases} \langle u''(t), v\rangle + \langle u_x(t) + \varepsilon u''_x(t), v_x\rangle + \lambda \left\langle |u'(t)|^{q-2}u'(t), v\right\rangle + K \left\langle |u(t)|^{p-2} u(t), v\right\rangle \\ \qquad\qquad + (hu(0,t) + g(t))\, v(0) = \langle F(t), v\rangle\,, \text{ for all } v \in V, \\ u(0) = \tilde{u}_0,\ u'(0) = \tilde{u}_1. \end{cases} \tag{3.24}$$

On the other hand, (3.21) and $(3.24)_1$ yield

$$\tfrac{\partial^2}{\partial x^2}\left(u + \varepsilon u_{tt}\right) = u_{tt} + \lambda |u_t|^{q-2}u_t + K\,|u|^{p-2}\,u - F(t) \in L^\infty(0,T;L^2). \tag{3.25}$$

Hence

$$u + \varepsilon u_{tt} \equiv \Psi \in L^\infty(0,T;V\cap H^2). \tag{3.26}$$

Furthermore, by $u_{tt} + \frac{1}{\varepsilon}u \equiv \frac{1}{\varepsilon}\Psi$, it follows that

$$\begin{array}{l} u(t) = \cos\left(\sqrt{\tfrac{1}{\varepsilon}}t\right)\tilde{u}_0 + \sqrt{\varepsilon}\sin\left(\sqrt{\tfrac{1}{\varepsilon}}t\right)\tilde{u}_1 \\ \qquad + \sqrt{\varepsilon}\int_0^t \sin\left(\sqrt{\tfrac{1}{\varepsilon}}(t-s)\right)\tfrac{1}{\varepsilon}\Psi(s)ds \in L^\infty(0,T;V\cap H^2). \end{array} \tag{3.27}$$

Then

$$u_{tt} = \tfrac{1}{\varepsilon}\left(\Psi - u\right) \in L^\infty(0,T;V\cap H^2), \text{ and } u_t = \tilde{u}_1 + \int_0^t u_{tt}(s)ds \in L^\infty(0,T;V\cap H^2). \tag{3.28}$$

Thus u, u_t, $u_{tt} \in L^\infty(0,T;V\cap H^2)$ and the existence of the solution is proved completely.

Step 4. *Uniqueness of the solution.* Let u, v be two weak solutions of problem (1.1) - (1.4), such that

$$u,\, v \in C^1\left(0,T;V\cap H^2\right), \text{ with } u',\, v',\, u'',\, v'' \in L^\infty\left(0,T;V\cap H^2\right). \tag{3.29}$$

Then $w = u - v$ verifies

$$\begin{cases} \langle w''(t), z\rangle + \langle w_x(t) + \varepsilon w''_x(t), z_x\rangle + \lambda \left\langle |u'(t)|^{q-2}u'(t) - |v'(t)|^{q-2}v(t), z\right\rangle \\ \qquad\qquad + hw(0,t)z(0) = -K\left\langle |u(t)|^{p-2}u(t) - |v(t)|^{p-2}v(t), z\right\rangle, \text{ for all } z \in V, \\ w(0) = w'(0) = 0. \end{cases} \tag{3.30}$$

We take $z = w = u - v$ in (3.30) and integrating with respect to t, we obtain

$$\sigma(t) = -2K \int_0^t \left\langle |u(s)|^{p-2}u(s) - |v(s)|^{p-2}v(s), w'(s)\right\rangle ds, \tag{3.31}$$

where

$$\begin{aligned} \sigma(t) &= \|w'(t)\|^2 + \varepsilon \|w'_x(t)\|^2 + \|w_x(t)\|^2 + hw^2(0,t) \\ &\quad +2\lambda \int_0^t \left\langle |u'(s)|^{q-2} u'(s) - |v'(s)|^{q-2} v'(s), u'(s) - v'(s)\right\rangle ds. \end{aligned} \tag{3.32}$$

Using again the inequality (2.45), with $M = \max\{\|u\|_{L^\infty(0,T;V)}, \|v\|_{L^\infty(0,T;V)}\}$, we get

$$\left| |u(x,s)|^{p-2}u(x,s) - |v(x,s)|^{p-2}v(x,s)\right| \leq (p-1)M^{p-2} |w(x,s)|, \text{ for all } (x,s) \in Q_T, \tag{3.33}$$

and the following inequalities

$$\begin{aligned} &\int_0^t \left\langle |u'(s)|^{q-2} u'(s) - |v'(s)|^{q-2} v'(s), u'(s) - v'(s)\right\rangle ds \geq 0, \\ &\sigma(t) \geq \|w'(t)\|^2 + \varepsilon \|w'_x(t)\|^2 + \|w_x(t)\|^2 \geq 2 \|w'(t)\| \|w_x(t)\|, \end{aligned} \tag{3.34}$$

so

$$\begin{aligned} \sigma(t) &\leq -2K \int_0^t \left\langle |u(s)|^{p-2}u(s) - |v(s)|^{p-2}v(s), w'(s)\right\rangle ds \\ &\leq 2K(p-1)M^{p-2} \int_0^t \|w(s)\| \|w'(s)\| ds \leq K(p-1)M^{p-2} \int_0^t \sigma(s) ds. \end{aligned} \tag{3.35}$$

By Gronwall's lemma, it follows from (3.35) that $\sigma \equiv 0$, i.e., $u \equiv v$.
Theorem 3.1 is proved completely. □

Next, we continue to consider the regularity of solution of problem (1.1) - (1.4), corresponding to $p = q = 2$.

$$\begin{cases} Lu \equiv u'' - u_{xx} - \varepsilon u''_{xx} + \lambda u' + Ku = F(x,t), \; 0 < x < 1, \; 0 < t < T, \\ L_0 u \equiv \varepsilon u''_x(0,t) + u_x(0,t) - hu(0,t) = g(t), \\ u(1,t) = 0, \\ u(0) = \tilde{u}_0, \; u'(0) = \tilde{u}_1. \end{cases} \tag{3.36}$$

For this purpose, we also assume that $\varepsilon > 0$, $K > 0$, $\lambda > 0$, $h \geq 0$. Furthermore, we will impose stronger assumptions. With $r \in N$, we assume that

$(H_2^{[r]})$ $\tilde{u}_0, \tilde{u}_1 \in V \cap H^{r+2}$.
$(H_3^{[r]})$ The function F satisfies

$$\begin{cases} \frac{\partial^j F}{\partial t^j} \in L^\infty(0,T; V \cap H^r), \quad 0 \leq j \leq r, \\ \frac{\partial^{r+1} F}{\partial t^{r+1}} \in L^1(0,T; V \cap H^r). \end{cases}$$

$(H_4^{[r]})$ $g \in W^{r+1,1}(0,T)$, $r \geq 1$.

First, we define the sequences $\{\tilde{u}_0^{[k]}\}$, $\{\tilde{u}_1^{[k]}\}$, $k = 0, 1, ..., r+2$ by the following recurrent formulas

$$\begin{cases} \tilde{u}_0^{[0]} = \tilde{u}_0, \; \tilde{u}_1^{[0]} = \tilde{u}_1, \\ \tilde{u}_0^{[k]} = \tilde{u}_1^{[k-1]}, \; k \in \{1, 2, ..., r+1\}, \; r \geq 1, \end{cases} \tag{3.37}$$

where $\tilde{u}_0^{[k]}$ is defined by the following problem

$$\begin{cases} -\varepsilon \Delta \tilde{u}_0^{[k]} + \tilde{u}_0^{[k]} = \frac{\partial^{k-2} F}{\partial t^{k-2}}(\cdot, 0) + \Delta \tilde{u}_0^{[k-2]} - K\tilde{u}_0^{[k-2]} - \lambda \tilde{u}_1^{[k-2]} \equiv \Phi^{[k]}, \; 0 < x < 1, \\ \varepsilon \tilde{u}_{0x}^{[k]}(0) = -\tilde{u}_{0x}^{[k-2]}(0) + h\tilde{u}_0^{[k-2]}(0) + \frac{d^{k-2}g}{dt^{k-2}}(0) \equiv \Phi_0^{[k]}, \quad \tilde{u}_0^{[k]}(1) = 0. \end{cases} \tag{3.38}$$

Then, we have the following Lemma.

Lemma 3.2. *Suppose that* $(H_2^{[r]}) - (H_4^{[r]})$ *hold. Then problem (3.38) has a unique weak solution* $\tilde{u}_0^{[k]} \in V$. *Furthermore, we have* $\tilde{u}_0^{[k]} \in V \cap H^{r+2}$, $k = 2, 3, ..., r+1$.

Proof of Lemma 3.2.
The weak solution of problem (3.38) is obtained from the following variational problem.

Find $U \in V$ such that

$$a(U, w) = \langle , w \rangle, \text{ for all } w \in V, \tag{3.39}$$

where

$$\begin{cases} a(U, w) = \langle \varepsilon U_x, w_x \rangle + \langle U, w \rangle, \\ \langle , w \rangle = \langle \Phi^{[k]}, w \rangle - \Phi_0^{[k]} w(0). \end{cases} \tag{3.40}$$

Using the Lax-Milgram's theorem, Problem (3.39) has a unique weak solution $\tilde{u}_0^{[k]} \in V$.

We shall prove that

$$\tilde{u}_0^{[k]} \in V \cap H^{r+2},\ k \in \{1, 2, ..., r+1\},\ r \geq 1. \tag{3.41}$$

(i) $k = 1 : \tilde{u}_0^{[1]} = \tilde{u}_1^{[0]} = \tilde{u}_1 \in V \cap H^{r+2}$. (by $(H_2^{[r]})$).

(ii) Suppose by induction that $\tilde{u}_0^{[1]}, ..., \tilde{u}_0^{[k-1]} \in V \cap H^{r+2}$ hold. We shall prove that $\tilde{u}_0^{[k]} \in V \cap H^{r+2}$ holds.

In fact, by $(H_3^{[r]})$, we have $\frac{\partial^{k-2} F}{\partial t^{k-2}}(\cdot, 0) \in V \cap H^r$, $2 \leq k \leq r+2$. Hence, by induction we obtain

$$\Phi^{[k]} = \frac{\partial^{k-2} F}{\partial t^{k-2}}(\cdot, 0) + \Delta \tilde{u}_0^{[k-2]} - K \tilde{u}_0^{[k-2]} - \lambda \tilde{u}_0^{[k-1]} \in V \cap H^r. \tag{3.42}$$

On the other hand, by $\tilde{u}_0^{[k]} \in V$ and (3.42), we obtain

$$\varepsilon \Delta \tilde{u}_0^{[k]} = \tilde{u}_0^{[k]} - \Phi^{[k]} \in V. \tag{3.43}$$

Then $\tilde{u}_0^{[k]} \in V \cap H^3$.

Similarly, we have also $\tilde{u}_0^{[k]} \in V \cap H^{2s+1}$, with $s \in \mathbb{N}$, $2s - 1 \leq r < 2s + 1$. Then

$$\varepsilon \Delta \tilde{u}_0^{[k]} = \tilde{u}_0^{[k]} - \Phi^{[k]} \in V \cap H^r. \tag{3.44}$$

Thus

$$\tilde{u}_0^{[k]} \in V \cap H^{r+2}. \tag{3.45}$$

Lemma 3.2 is proved completely.

Now, formally differentiating problem (3.36) with respect to time up to order r and letting $u^{[r]} = \frac{\partial^r u}{\partial t^r}$ we are led to consider the solution $u^{[r]}$ of problem $(Q^{[r]})$:

$$(Q^{[r]}) \begin{cases} L u^{[r]} = \frac{\partial^r F}{\partial t^r}(x, t), \quad (x, t) \in Q_T, \\ L_0 u^{[r]} = \frac{d^r g}{d t^r}(t), \quad u^{[r]}(1, t) = 0, \\ u^{[r]}(0) = \tilde{u}_0^{[r]}, \quad u_t^{[r]}(0) = \tilde{u}_1^{[r]}. \end{cases} \tag{3.46}$$

From the assumptions $(H_2^{[r]}) - (H_4^{[r]})$ we deduce that $\tilde{u}_0^{[r]}$, $\tilde{u}_1^{[r]}$, $\frac{\partial^r F}{\partial t^r}$ and $\frac{d^r g}{d t^r}$ satisfy the conditions of Theorem 3.1. So, the problem $(Q^{[r]})$ has a unique weak solution $u^{[r]}$ such that

$$u^{[r]} \in C^1\left(0, T; V \cap H^2\right), \quad u_{tt}^{[r]} \in L^\infty\left(0, T; V \cap H^2\right). \tag{3.47}$$

Moreover, from the uniqueness of a weak solution we have $u^{[r]} = \frac{\partial^r u}{\partial t^r}$. Hence we deduce from (3.47) that the solution u of problem (3.36) satisfy

$$u \in C^{r+1}\left(0, T; V \cap H^2\right), \ \frac{\partial^{r+2} u}{\partial t^{r+2}} \in L^\infty\left(0, T; V \cap H^2\right). \tag{3.48}$$

Next we shall prove by induction on r that

$$u \in C^{r+1}\left(0, T; V \cap H^{r+2}\right), \ \frac{\partial^{r+2} u}{\partial t^{r+2}} \in L^\infty(0, T; V \cap H^{r+2}), \ r \geq 1. \tag{3.49}$$

(i) In the case of $r = 1$, the proof of (3.49) is easy, hence we omit the details. We only prove with $r \geq 2$.

(ii) Suppose by induction that (3.49) holds for $r - 1$. i.e.,

$$u \in C^{r}\left(0, T; V \cap H^{r+1}\right), \ \frac{\partial^{r+1} u}{\partial t^{r+1}} \in L^\infty(0, T; V \cap H^{r+1}). \tag{3.50}$$

We need prove that (3.49) holds. To achieve this, we only have to prove that

$$\begin{cases} \frac{\partial^{r} u}{\partial t^{r}} \in L^\infty(0, T; V \cap H^{r+2}), \\ \frac{\partial^{r+1} u}{\partial t^{r+1}} \in L^\infty(0, T; V \cap H^{r+2}), \\ \frac{\partial^{r+2} u}{\partial t^{r+2}} \in L^\infty(0, T; V \cap H^{r+2}), \ r \geq 1. \end{cases} \tag{3.51}$$

By $(Q^{[r]})_1$, we have

$$\left(u^{[r]} - \varepsilon \Delta u^{[r]}\right)'' - \Delta u^{[r]} + K u^{[r]} + \lambda u_t^{[r]} = \frac{\partial^r F}{\partial t^r}. \tag{3.52}$$

Put

$$\begin{cases} W = u^{[r]} - \varepsilon \Delta u^{[r]}, \\ \tilde{w}_0 = \tilde{u}_0^{[r]} - \varepsilon \Delta \tilde{u}_0^{[r]}, \\ \tilde{w}_1 = \tilde{u}_1^{[r]} - \varepsilon \Delta \tilde{u}_1^{[r]} = \tilde{u}_0^{[r+1]} - \varepsilon \Delta \tilde{u}_0^{[r+1]}, \end{cases} \tag{3.53}$$

it follows that

$$\begin{cases} W'' + \frac{1}{\varepsilon} W = \frac{1}{\varepsilon} u^{[r]} - K u^{[r]} - \lambda u_t^{[r]} + \frac{\partial^r F}{\partial t^r} \equiv \Psi^{[r]} \in L^\infty(0, T; V \cap H^r), \\ W(0) = \tilde{w}_0 \in V \cap H^r, \\ W'(0) = \tilde{w}_1 \in V \cap H^r. \end{cases} \tag{3.54}$$

Thus

$$\begin{aligned} W(t) = & \cos\left(\sqrt{\tfrac{1}{\varepsilon}} t\right) \tilde{w}_0 + \sqrt{\varepsilon} \sin\left(\sqrt{\tfrac{1}{\varepsilon}} t\right) \tilde{w}_1 \\ & + \sqrt{\varepsilon} \int_0^t \sin\left(\sqrt{\tfrac{1}{\varepsilon}} (t - s)\right) \Psi^{[r]}(s) ds \in L^\infty(0, T; V \cap H^r). \end{aligned} \tag{3.55}$$

By (3.50) and (3.55), it follows that

$$\Delta u^{[r]} = \frac{1}{\varepsilon} u^{[r]} - \frac{1}{\varepsilon} W \in L^\infty(0, T; V \cap H^r). \tag{3.56}$$

Thus

$$u^{[r]} \in L^\infty(0, T; V \cap H^{r+2}). \tag{3.57}$$

On the other hand, by $(3.54)_1$, we obtain

$$W'' = -\frac{1}{\varepsilon} W + \Psi^{[r]} \in L^\infty(0, T; V \cap H^r). \tag{3.58}$$

It follows from (3.47), (3.58) and $r \geq 2$, that

$$\Delta u_{tt}^{[r]} = \tfrac{1}{\varepsilon} u_{tt}^{[r]} - \tfrac{1}{\varepsilon} W'' \in L^\infty(0,T; V \cap H^2). \tag{3.59}$$

Consequently

$$u_{tt}^{[r]} \in L^\infty(0,T; V \cap H^4). \tag{3.60}$$

Similarly, we have also $u_{tt}^{[r]} \in L^\infty(0,T; H^{2s})$, with $s \in \mathbb{N}$, $2s-2 \leq r < 2s$. Then

$$\Delta u_{tt}^{[r]} = \tfrac{1}{\varepsilon} u_{tt}^{[r]} - \tfrac{1}{\varepsilon} W'' \in L^\infty(0,T; V \cap H^r). \tag{3.61}$$

So

$$u_{tt}^{[r]} \in L^\infty(0,T; V \cap H^{r+2}). \tag{3.62}$$

On the other hand

$$u_t^{[r]} = \tilde{u}_1^{[r]} + \textstyle\int_0^t u_{tt}^{[r]}(s)ds \in L^\infty(0,T; V \cap H^{r+2}). \tag{3.63}$$

Combining (3.57), (3.62) and (3.63), by induction arguments on r, we conclude that (3.49) holds and the following theorem is proved.

Theorem 3.3. *Let $(H_2^{[r]})$–$(H_4^{[r]})$ hold. Then the unique solution $u(x,t)$ of problem (3.36)satisfies (3.49).*

□

4 Asymptotic behavior of solutions as $\varepsilon \to 0_+$

In this part, we assume that $p > 2$, $q > 1$, $\lambda > 0$, $K > 0$, $h \geq 0$ and $(\tilde{u}_0, \tilde{u}_1, F)$ satisfy the assumptions (H_2), (H_3). Let $\varepsilon > 0$. By theorem 2.3, the problem (1.1) – (1.4) has a unique weak solution $u = u_\varepsilon$ depending on ε.

We consider the following perturbed problem, where ε is a small parameter:

$$(P_\varepsilon) \begin{cases} u_{tt} - u_{xx} - \varepsilon u_{xxtt} + \lambda \left|u_t\right|^{q-2} u_t + K \left|u\right|^{p-2} u = F(x,t),\ 0<x<1,\ 0<t<T, \\ \varepsilon u_{xtt}(0,t) + u_x(0,t) = hu(0,t) + g(t),\ \ u(1,t) = 0, \\ u(0) = \tilde{u}_0,\ u'(0) = \tilde{u}_1. \end{cases} \tag{4.1}$$

We shall study the asymptotic behavior of the solution u_ε of problem (P_ε) as $\varepsilon \to 0_+$.

Theorem 4.1. *Let $T > 0$, $p > 2$, $q > 1$, $\lambda > 0$, $K > 0$. Let (H_2), (H_3) hold. Then*

(i) The problem (P_0) corresponding to $\varepsilon = 0$ has a unique weak solution $\bar{u}_0$ satisfying

$$\bar{u}_0 \in L^\infty(0,T;V),\ \bar{u}_0' \in L^\infty\left(0,T;L^2\right). \tag{4.2}$$

(ii) If $\bar{u}_0'' \in L^2(0,T;H^2)$, then solution u_ε converges strongly in W_T to $\bar{u}_0$, as $\varepsilon \to 0_+$, where

$$W_T = \{v \in L^\infty(0,T;V) : v' \in L^\infty(0,T;L^2)\}. \tag{4.3}$$

Furthermore, we have the estimation

$$\|u_\varepsilon' - \bar{u}_0'\|_{L^\infty(0,T;L^2)} + \|u_\varepsilon - \bar{u}_0\|_{L^\infty(0,T;V)} \leq C_T\sqrt{\varepsilon}, \tag{4.4}$$

where C_T is a posistive constant depending only on T.

Proof. First, we note that if the small parameter $\varepsilon > 0$ satisfy $0 < \varepsilon < 1$ then a priori estimates of the sequence $\{u_m\}$ in the proof of Theorem 2.3 for problem (P_ε) satisfy

$$\|u_m'(t)\|^2 + \|u_{mx}(t)\|^2 + \varepsilon \|u_{mx}'(t)\|^2 + \|u_m(t)\|_{L^p}^p + \int_0^t \|u_m'(s)\|_{L^q}^q \, ds \leq C_T, \tag{4.5}$$

for all $t \in [0, T]$ and for all m, and C_T is a constant depending only on T, p, q, λ, K, $\tilde{u}_0$, $\tilde{u}_1$, F (independent of ε). Hence, the limit $u = u_\varepsilon$ of the sequence $\{u_m\}$ as $m \to +\infty$, in suitable function spaces is a unique weak solution of problem (P_ε) satisfying

$$\|u_\varepsilon'(t)\|^2 + \|u_{\varepsilon x}(t)\|^2 + \varepsilon \|u_{\varepsilon x}'(t)\|^2 + \|u_\varepsilon(t)\|_{L^p}^p + \int_0^t \|u_\varepsilon'(s)\|_{L^q}^q \, ds \leq C_T, \tag{4.6}$$

for all $t \in [0, T]$ and for all $\varepsilon \in (0, 1)$.

Let $\{\varepsilon_m\}$ be a sequence such that $\varepsilon_m > 0$, $\varepsilon_m \to 0$ as $m \to +\infty$. We put $u_m = u_{\varepsilon_m}$, we deduce from (4.6) that, there exists a subsequence of the sequence $\{u_m\}$ still denoted by $\{u_m\}$, such that

$$\begin{cases} u_m \to \bar{u}_0 & \text{in} & L^\infty(0, T; V) & \text{weakly*}, \\ u_m' \to \bar{u}_0' & \text{in} & L^\infty(0, T; L^2) & \text{weakly*}, \\ \sqrt{\varepsilon_m} u_m' \to \zeta & \text{in} & L^\infty(0, T; V) & \text{weakly*}, \\ u_m \to \bar{u}_0 & \text{in} & L^\infty(0, T; L^p) & \text{weakly*}, \\ u_m' \to \bar{u}_0' & \text{in} & L^q(Q_T) & \text{weakly}, \\ |u_m|^{p-2} u_m \to \chi_0 & \text{in} & L^\infty(0, T; L^{p'}) & \text{weakly*}, \\ |u_m'|^{p-2} u_m' \to \chi_1 & \text{in} & L^{q'}(Q_T) & \text{weakly}. \end{cases} \tag{4.7}$$

By the compactness lemma of Lions ([6]: p. 57), $(4.7)_{1,2}$ lead to the existence of a subsequence still denoted by $\{u_m\}$, such that

$$u_m \to \bar{u}_0 \quad \text{strongly in } L^2(Q_T) \text{ and a.e. in } \quad Q_T. \tag{4.8}$$

It follows from $(4.7)_{2,3}$, that $\zeta = 0$. Hence, we obtain from $(4.7)_3$ that

$$\sqrt{\varepsilon_m} u_m' \to 0 \quad \text{in } L^\infty(0, T; V) \text{ weakly*}. \tag{4.9}$$

Similarly

$$|u_m|^{p-2} u_m \to |\bar{u}_0|^{p-2} \bar{u}_0 = \chi_0 \quad \text{strongly in} \quad L^2(Q_T), \tag{4.10}$$

and

$$\chi_1 = |\bar{u}_0'|^{q-2} \bar{u}_0'. \tag{4.11}$$

By passing to the limit, as in the proof of Theorem 2.3, we conclude that $\bar{u}_0$ is a unique weak solution of problem (P_0) corresponding to $\varepsilon = 0$ satisfying

$$\bar{u}_0 \in L^\infty(0, T; V), \ \bar{u}_0' \in L^\infty\left(0, T; L^2\right). \tag{4.12}$$

(ii) Put $u = u_\varepsilon - \bar{u}_0$, then u is the weak solution of the following problem

$$\begin{cases} u'' - \Delta u - \varepsilon \Delta u'' + \lambda \left(|u_\varepsilon'|^{q-2} u_\varepsilon' - |\bar{u}_0'|^{q-2} \bar{u}_0' \right) + K \left(|u_\varepsilon|^{p-2} u_\varepsilon - |\bar{u}_0|^{p-2} \bar{u}_0 \right) \\ \qquad\qquad = \varepsilon \Delta \bar{u}_0'', \ 0 < x < 1, \ 0 < t < T, \\ \varepsilon u_x''(0, t) + u_x(0, t) = h u(0, t) - \varepsilon \bar{u}_0''(0, t), \ \ u(1, t) = 0, \\ u(0) = u'(0) = 0. \end{cases} \tag{4.13}$$

Using again Lemma 2.4, in a manner similar to the above part, we obtain

$$\begin{aligned}\sigma(t) &= 2\varepsilon \int_0^t \langle \Delta \bar{u}_0'', u'(s)\rangle\, ds + 2\varepsilon \int_0^t \bar{u}_{0x}''(0,s)u'(0,s)ds \\ &\quad -2K \int_0^t \left\langle |u_\varepsilon|^{p-2} u_\varepsilon - |\bar{u}_0|^{p-2} \bar{u}_0, u'(s) \right\rangle ds,\end{aligned} \tag{4.14}$$

where

$$\begin{aligned}\sigma(t) &= \|u'(t)\|^2 + \varepsilon \|u_x'(t)\|^2 + \|u_x(t)\|^2 + hu^2(0,t) \\ &\quad +2\lambda \int_0^t \left\langle |u_\varepsilon'|^{q-2} u_\varepsilon' - |\bar{u}_0'|^{q-2} \bar{u}_0', u'(s) \right\rangle ds.\end{aligned} \tag{4.15}$$

Note that

$$\begin{cases} \int_0^t \left\langle |u_\varepsilon'|^{q-2} u_\varepsilon' - |\bar{u}_0'|^{q-2} \bar{u}_0', u'(s) \right\rangle \geq 0, \\ \sigma(t) \geq \varepsilon \|u_x'(t)\|^2, \\ \sigma(t) \geq \|u'(t)\|^2 + \|u_x(t)\|^2 \geq 2\|u_x(t)\|\,\|u'(t)\|. \end{cases} \tag{4.16}$$

By (2.45), (4.6), (4.16), we estimate all terms in the right – hand side of (4.14) as follows

$$\begin{aligned}2\varepsilon \int_0^t \langle \Delta \bar{u}_0''(s), u'(s)\rangle\, ds &\leq 2\varepsilon \int_0^t \|\Delta \bar{u}_0''(s)\|\,\|u'(s)\|\, ds \leq 2\varepsilon \int_0^t \|\bar{u}_0''(s)\|_{H^2} \|u'(s)\|\, ds \\ &\leq \varepsilon^2 \int_0^t \|\bar{u}_0''(s)\|_{H^2}^2 ds + \int_0^t \|u'(s)\|^2 ds \\ &\leq \varepsilon^2 \|\bar{u}_0''\|_{L^2(0,T;H^2)}^2 + \int_0^t \sigma(s)ds;\end{aligned} \tag{4.17}$$

$$\begin{aligned}2\varepsilon \int_0^t \bar{u}_{0x}''(0,s)u'(0,s)ds &\leq 2\sqrt{2}\varepsilon \int_0^t \|\bar{u}_{0x}''(s)\|_{H^1} \|u_x'(s)\|\, ds \\ &\leq 2\sqrt{2}\varepsilon \int_0^t \|\bar{u}_0''(s)\|_{H^2} \|u_x'(s)\|\, ds \\ &\leq 2\varepsilon \int_0^t \|\bar{u}_0''(s)\|_{H^2}^2 ds + \varepsilon \int_0^t \|u_x'(s)\|^2 ds \\ &\leq 2\varepsilon \|\bar{u}_0''\|_{L^2(0,T;H^2)}^2 + \int_0^t \sigma(s)ds;\end{aligned} \tag{4.18}$$

$$\begin{aligned}-2K \int_0^t \left\langle |u_\varepsilon|^{p-2} u_\varepsilon - |\bar{u}_0|^{p-2} \bar{u}_0, u'(s) \right\rangle ds &\leq 2K(p-1)C_T^{p-2} \int_0^t \|u(s)\|\,\|u'(s)\|\, ds \\ &\leq K(p-1)C_T^{p-2} \int_0^t \sigma(s)ds.\end{aligned} \tag{4.19}$$

Combining (4.14), (4.17) - (4.19), it implies that

$$\sigma(t) \leq 3\varepsilon \|\bar{u}_0''\|_{L^2(0,T;H^2)}^2 + \left[2 + K(p-1)C_T^{p-2}\right] \int_0^t \sigma(s)ds. \tag{4.20}$$

By Gronwall's lemma, (4.20) leads to

$$\sigma(t) \leq 3\varepsilon \|\bar{u}_0''\|_{L^2(0,T;H^2)}^2 \exp(T\left[2 + K(p-1)C_T^{p-2}\right]) \equiv \bar{C}_T\varepsilon, \ \ \forall t \in [0,T]. \tag{4.21}$$

Hence

$$\|u_\varepsilon' - \bar{u}_0'\|_{L^\infty(0,T;L^2)} + \|u_\varepsilon - \bar{u}_0\|_{L^\infty(0,T;H^1)} \leq C_T\sqrt{\varepsilon}, \tag{4.22}$$

where C_T is a constant depending only on T.

Theorem 4.1 is proved completely. □

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