Almost Quaternionic Structures on Quaternionic Kaehler Manifolds

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Abstract

In this work, we consider a Riemannian manifold M with an almost quaternionic structure V defined by a three-dimensional subbundle of $(1,1)$ tensors F, G and H such that $\{F, G, H\}$ is chosen to be a local basis for V. For such a manifold there exits a subbundle $\mathcal{H}(M)$ of the bundle of orthonormal frames $\mathcal{O}(M)$. If M admits a torsion-free connection reducible to a connection in $\mathcal{H}(M)$, then we give a condition such that the torsion tensor of the bundle vanishes. We also prove that if M admits a torsion-free connection reducible to a connection in $\mathcal{H}(M)$, then the tensors $\widetilde{F}^2$, $\widetilde{G}^2$, $\widetilde{H}^2$ are torsion-free, that is, they are integrable. Here $\widetilde{F}$, $\widetilde{G}$, $\widetilde{H}$ are the extended tensors of F, G and H defined on M. Finally, we show that if the torsions of $\widetilde{F}^2$, $\widetilde{G}^2$ and $\widetilde{H}^2$ vanish, then M admits a connection with torsion which is reducible to $\mathcal{H}(M)$, and this means that $\widetilde{F}^2$, $\widetilde{G}^2$ and $\widetilde{H}^2$ are integrable.

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1 Introduction

By imposing special structures on the tangent bundle of a manifold, it is possible to have different type of geometries. (Almost) complex, (almost) quaternionic and polynomial structures are examples of such type of structures [1-9].

In literature almost complex and almost quaternionic structures have been investigated widely, and a detailed review can be found in Kirichenko and Arseneva [5]. We begin by recalling basic results and definitions from literature.

An almost complex structure on a manifold M is a tensor field J: $TM \to TM$ satisfying the identity $J^2 = -id$. *An almost hypercomplex structure* on a $4m$-dimensional manifold M is a triple $S = (F, G, H)$ of almost complex structures

F, G and H satisfying the conditions

$$F^2 = G^2 = H^2 = -I, \quad H = FG, \quad FG + GF = FH + HF = GH + HG = 0, \tag{1.1}$$

where I denotes the identity transformation of $T_x(M)$. If each of the tensor fields F, G and H is a complex structure, then S is called *hypercomplex structure* on M.
Let M be a $4m$-dimensional Riemannian manifold admitting a three dimensional subbundle V of (1,1) tensors such that on a neighborhood U of each $x \in M$, V has a local base $\{F, G, H\}$. If on each such neigborhood, the tensors F, G and H satisfy the conditions (1.1), then the bundle V is called an *almost quaternionic structure* on M [2].

In [10], the Nijenhuis bracket of two tensor fields A and B of type $(1,1)$ is defined as the following tensor field of type $(1,2)$

$$\begin{aligned} [A,B](X,Y) &= [AX,BY] - A[BX,Y] - B[X,AY] \\ &+ [BX,AY] - B[AX,Y] - A[X,BY] + (AB+BA)[X,Y]\,. \end{aligned} \tag{1.2}$$

In particular, if $A = B$ we have

$$[A,A](X,Y) = 2\big([AX,AY] + A^2[X,Y] - A[AX,Y] - A[X,AY]\big). \tag{1.3}$$

In [6], using (1.2), the torsion tensor of V is defined by
$[V,V] = [F,F] + [G,G] + [H,H]$, where $[\ ,\ ]$ denotes the Nijenhuis bracket.

The Newlander-Nirenberg Theorem states that an almost complex structure is a complex structure if and only if it is integrable i.e., it has no torsion. Thus, if the tensor fields F, G and H are integrable, then the bracket of any two of them vanishes, that is, [1]

$$[F,F] = [G,G] = [H,H] = 0 \quad \text{and} \quad [F,G] = [F,H] = [G,H] = 0. \tag{1.4}$$

Also, it is shown that there exits a torsion-free connection that F, G and H are covariantly constant which means that V is a trivial bundle [1].

In literature, moreover, it is proved that if either M is a quaternionic Kaehler manifold, or if M is a complex manifold with almost complex structures, then the vanishing of torsion tensor of V is equivalent to the vanishing of all the Nijenhuis brackets of $\{F, G, H\}$. For a quaternionic Kaehler manifold, with a torsion-free connection ∇, it is given a condition implying that if $[V,V] = 0$

then $\nabla F = \nabla G = \nabla H = 0$. It follows that the bundle V admits a flat connection, hence it is trivial see [6].

In [1], a reducibility condition is given for a torsion-free connection on a quaternionic manifold M to be reducible to a connection in $\mathcal{H}(M)$, where $\mathcal{H}(M)$ is subbundle of the bundle of orthonormal frame $\mathcal{O}(M)$.

For a Riemannian manifold M of dimension $4m + n$ with an almost quaternionic structure V of rank $4m$, it is shown that the metric g satisfies $\mathrm{g}(X, \phi Y) + \mathrm{g}(\phi X, Y) = 0$ for any vector fields X, $Y \in \mathcal{X}(M)$, where $\mathcal{X}(M)$ is the algebra of vector fields on M, and any section ϕ of V. In [3] Doğanaksoy studied orthogonal plane fields P and Q defined by V admitting local basis $\{F, G, H\}$ and defined projection tensors $p : P \to T_xM$ and $q : Q \to T_xM$.

In this paper, by using the projection tensors p and q above, we define extended tensors $\widetilde{F}$, $\widetilde{G}$ and $\widetilde{H}$ on M, and we obtain reducibility conditions for a torsion-free connection on M to be a connection (either torsion-free or with torsion) in $\mathcal{H}(M)$. The reducibility condition corresponds to the integrability of extended tensors.

2 Almost Quaternionic Structures

We consider a Riemannian manifold M of dimension $4m + n$ admitting an almost quaternionic structure V. Let $\{F, G, H\}$ be a local basis for V on a neighborhood U of M. Since torsion tensors $[F, F]$, $[G, G]$, and $[H, H]$ are locally defined objects, to obtain a global condition for the triviality of V, a tensor $[V, V]$ of type $(1, 2)$ is defined globally on U by [6]

$$[V, V] = [F, F] + [G, G] + [H, H]. \tag{2.1}$$

Let $\{F, G, H\}$ and $\{F', G', H'\}$ be local bases for V defined on neighborhoods U and U', respectively, and assume that $U \cap U' \neq 0$.
Since U and U' are not disjoint, then in $U \cap U'$ we have

$$F' = a_{11}F + a_{12}G + a_{13}H\,, \tag{2.2}$$
$$G' = a_{21}F + a_{22}G + a_{23}H\,, \tag{2.3}$$
$$H' = a_{31}F + a_{32}G + a_{33}H \tag{2.4}$$

where a_{ij} are functions defined on $U \cap U'$. With the notation above, at any point $x \in U \cap U'$, $a_{ij} \in \mathrm{SO}(3)$ [1, 2].

We calculate the torsion tensor of $[V,V]$ defined in (2.1) by taking into account local bases $\{F,G,H\}$ and $\{F',G',H'\}$. Using equations (2.2)-(2.4) , we also, compute torsions $[G,F]$,$[F,H]$,$[H,F]$,$[G,H]$ and $[H,G]$ and show that the torsion tensor $[V,V]$ of the bundle V is independent of the choice of bases, that is, $[V,V]$ is well defined.
If a tensor f of type (1,1) on M satisfies the structure equation

$$f^3 + f = 0, \tag{2.5}$$

then f is called an f-structure. Since the tensors F, G and H satisfy the conditions (1.1), they are f-structures on M. In [3] it is shown that any cross-section ϕ of V of length 1 is f-structure on M, and F, G and H are of length 1.
Let $\{F,G,H\}$ be a basis for V in some neighborhood U of M. In [11], it is proved that each of $F, G,$ and H has a constant rank on U and from the conditions $F = GH$, $G = HF$, $H = FG$, it is seen that their ranks are all equal. Also, in [11], the rank of V is defined to be the rank of a basis element on some neighborhood U. By choosing $q = 1 + F^2 = 1 - p$, where 1 denotes the identity operator, it is obtained in [3] that

$$p + q = 1, \quad p^2 = p, \quad q^2 = q \tag{2.6}$$

and that

$$\phi p = p\,\phi = \phi, \quad \phi\, q = q\phi = 0\,, \tag{2.7}$$

for any cross-section ϕ of V. This shows that p and q are complementary projection operators. Then, there exist two distributions P and Q corresponding to p and q, respectively. If the rank of V is $4m$, then P is $4m$-dimensional and Q is n-dimensional.
Let g' be a Riemannian metric of M. Define g to be the tensor field of degree 2 on M as

$$\mathrm{g}(X,Y) = \begin{cases} 0\,, & X \in P(x)\,,\ Y \in Q(x) \\ \mathrm{g}'(X,Y)\,, & X, Y \in Q(x) \\ \mathrm{g}'(X,Y) + \mathrm{g}'(FX,FY) & \\ +\mathrm{g}'(GX,GY) + \mathrm{g}'(HX,HY)\,, & X, Y \in P(x) \end{cases} \tag{2.8}$$

where $\{F,G,H\}$ is a canonical local base for V, and $X, Y \in T_x(M)$. Since g' is a Riemannian, g satisfies all the conditions for a Riemannian metric [3].
For any cross-section ϕ of V, from (2.8) we have

$$\mathrm{g}(X,\phi Y) + \mathrm{g}(\phi X, Y) = 0\,, \quad X, Y \in \mathcal{X}(M)\,. \tag{2.9}$$

If M admits an almost quaternionic structure, then at each point x in M there is an orthonormal basis of TM , of the form

$$\{X_1,\dots,X_m,FX_1,\dots,FX_m,GX_1,\dots,GX_m,HX_1,\dots,HX_m\}, \tag{2.10}$$

and the set of all such frames at all points $x \in M$ constitutes a subbundle, denoted by $\mathcal{H}(M)$, of the bundle of orthonormal frames $\mathcal{O}(M)$.

We consider P and Q to be the orthogonal plane fields defined by almost substructure V, and let p and q be projections $P \to T_xM$ and $Q \to T_xM$, respectively [3].

Let us denote by f the tensor Jp defined on P. Since $pJ = J$, f satisfies the structure equation $f^3 + f = 0$. For any tensor J of type $(1,1)$ on M we define its extension as the tensor $\widetilde{J}$ of type $(1,1)$ on M by setting

$$\widetilde{J} = q + Jp = q + f\,. \tag{2.11}$$

It can be seen that the extended tensor $\widetilde{J}$ has the following properties

$$\widetilde{J}^2 = q - p\,, \tag{2.12}$$

$$\widetilde{J}^3 = q - Jp = 1 - \widetilde{J} + \widetilde{J}^2\,, \tag{2.13}$$

$$\widetilde{J}^4 = 1. \tag{2.14}$$

We recall that Γ is an affine (linear) connection on M. A tensor field, say J, is parallel with respect to Γ if and only if $\nabla J = 0$, where ∇ is the covariant differentiation with respect to Γ [10].

We give the following proposition from [4].

Proposition. *The followings are equivalent*
(a) $[\widetilde{J}^2, \widetilde{J}^2]$ *vanishes,*
(b) M *admits a torsion-free affine connection according to which* $\widetilde{J}^2$ *is parallel,*
(c) the plane field P *and its orthogonal complement* Q *are both integrable.*

3 Torsion-free connections reducible to a connection in $\mathcal{H}(M)$

Compatibility of a torsion-free connection Γ with the metric g' is equivalent to the reducibility of Γ to $\mathcal{O}(M)$. Furthermore, if Γ is reducible to $\mathcal{H}(M)$, then the manifold is called *quaternionic Kaehler manifold* [2].

We also quote the following theorem from [2].

Theorem. *Let M be an almost quaternionic manifold. A connection Γ in orthonormal frame $\mathcal{O}(M)$ is reducible to a connection in $\mathcal{H}(M)$ if and only if the covariant derivatives of the tensor fields of F, G and H satisfy the following conditions:*

$$\nabla F = aG - bH, \quad \nabla G = -aF + cH, \quad \nabla H = bF - cG \tag{3.1}$$

where a, b and c are 1-forms on M.

We now begin to state our main theorems related to the reducibility of a torsion free-connection.

Theorem 3.1. *If M admits a torsion-free connection Γ reducible to $\mathcal{H}(M)$ such that*

$$(\nabla\phi)(\phi X, Y) + (\nabla\phi)(X, \phi Y) = 0 \tag{3.2}$$

for any section $\phi \in \{F, G, H\}$, then the torsion tensor $[V, V]$ of the bundle V vanishes.

Proof. For any section $\phi \in \{F, G, H\}$, from (3.2) we have

$$(\nabla F)(FX, Y) + (\nabla F)(X, FY) = 0\,, \tag{3.3}$$
$$(\nabla G)(GX, Y) + (\nabla G)(X, GY) = 0\,, \tag{3.4}$$
$$(\nabla H)(HX, Y) + (\nabla H)(X, HY) = 0\,. \tag{3.5}$$

By using (3.1) in (3.3)-(3.5) , we obtain following relations

$$\begin{aligned}(\nabla F)(FX, Y) &= (\nabla_Y F)(FX) = a(Y)G(FX) - b(Y)H(FX)\\ &= -a(Y)H(X) - b(Y)G(X)\,, &(3.6)\\ (\nabla F)(X, FY) &= (\nabla_{FY} F)(X) = a(FY)G(X) - b(FY)H(X)\,. &(3.7)\end{aligned}$$

Similarly, we find

$$\begin{aligned}(\nabla G)(GX, Y) &= (\nabla_Y G)(GX) = -a(Y)F(GX) + c(Y)H(GX)\\ &= -a(Y)H(X) - c(Y)F(X)\,, &(3.8)\\ (\nabla G)(X, GY) &= (\nabla_{GY} G)(X) = -a(GY)F(X) + c(GY)H(X) &(3.9)\end{aligned}$$

and

$$\begin{aligned}(\nabla H)(HX, Y) &= (\nabla_Y H)(HX) = b(Y)F(HX) - c(Y)G(HX)\\ &= -b(Y)G(X) - c(Y)F(X)\,, &(3.10)\\ (\nabla H)(X, HY) &= (\nabla_{HY} H)(X) = b(HY)F(X) + b(Y)G(X)\,. &(3.11)\end{aligned}$$

For a torsion-free connection Γ reducible to $\mathcal{H}(M)$, if the 1-forms a, b and c defined in (3.1) are used in the equations (3.6)-(3.11), then we observe that they satisfy the following relations

$$\begin{aligned} b(X) &= a(FX), \\ c(X) &= -a(GX), \\ c(X) &= b(HX). \end{aligned} \tag{3.12}$$

In order to find the torsion tensor of V, we compute the torsions of tensors F, G and H :

$$\begin{aligned} \frac{1}{2}[F,F](X,Y) &= ([FX,FY] + F^2[X,Y] - F[FX,Y] - F[X,FY] \\ &= (\nabla_{FX}FY - \nabla_{FY}FX) + F^2(\nabla_X Y - \nabla_{FY}X) \\ &\quad - F(\nabla_{FX}Y - \nabla_Y FX) - F(\nabla_X FY - \nabla_{FY}X) \end{aligned}$$

$$\frac{1}{2}[F,F](X,Y) = (\nabla_{FX}F)Y - (\nabla_{FY}F)X + F((\nabla_Y F)X) - F((\nabla_X F)Y)\,. \tag{3.13}$$

Similarly, the tensors $[G,G]$ and $[H,H]$ are obtained as follows:

$$\frac{1}{2}[G,G](X,Y) = (\nabla_{GX}G)Y - (\nabla_{GY}G)X + G((\nabla_Y G)X) - G((\nabla_X G)Y) \tag{3.14}$$

and

$$\frac{1}{2}[H,H](X,Y) = (\nabla_{HX}H)Y - (\nabla_{HY}H)X + H((\nabla_Y H)X) - H((\nabla_X H)Y)\,. \tag{3.15}$$

Using the relations of (3.12) , we get the following equalities

$$\begin{aligned} \frac{1}{2}[F,F](X,Y) &= [a(FX)G(Y) - b(FX)H(Y)] - [a(FY)G(X) - b(FY)H(X)] \\ &\quad + F[a(Y)G(X) - b(Y)H(X)] - F[a(X)G(Y) - b(X)H(Y)] \\ \frac{1}{2}[F,F](X,Y) &= (b(Y) - a(FY))G(X) + (a(FX) - b(X))G(Y) \\ &\quad + (b(FY) + a(Y))H(X) + (-a(X) - b(FX))H(Y). \end{aligned} \tag{3.16}$$

By using (3.14) and (3.15) for G and H, we obtain

$$\begin{aligned} \frac{1}{2}[G,G](X,Y) &= (-a(GX) - c(X))F(Y) + (a(GY) + c(Y))F(X) \\ &\quad + (a(Y) - c(GY))H(X) + (c(GX) - a(X))H(Y) \end{aligned} \tag{3.17}$$

and

$$\frac{1}{2}[H,H](X,Y) = (b(HX) - c(X))F(Y) + (-b(HY) + c(Y))F(X) + (b(Y) + c(HY))G(X) + (-b(X) - c(HX))G(Y). \tag{3.18}$$

Substituting the conditions (3.12) in (3.16)-(3.18), we find $[F,F] = 0$, $[G,G] = 0$, $[H,H] = 0$, and therefore $[V,V] = 0$. This completes the proof.

□

Given tensor fields F, G and H, let us define their extended tensors $\widetilde{F}$, $\widetilde{G}$ and $\widetilde{H}$ as in (2.11). It is clear that the relations (2.12)-(2.14) hold for the tensors $\widetilde{F}$, $\widetilde{G}$ and $\widetilde{H}$.

To simplify our calculations we use the following notations

$$\widetilde{F} = q + Fp = q + f\,, \tag{3.19}$$

$$\widetilde{G} = q + Gp = q + g\,, \tag{3.20}$$

$$\widetilde{H} = q + Hp = q + h\,, \tag{3.21}$$

where $f = Fp$, $g = Gp$, and $h = Hp$. Since $p + q = 1$, where 1 denotes the identity operator, and tensors F, G, H are defined on distribution P of TM, using the projection tensor p of type (1,1) defined in Section 2, we write $q = 1 + f^2 = 1 - p$. Then, we have

$$p^2 = p\,, \quad q^2 = q\,. \tag{3.22}$$

Also using (2.12), (2.13), and (2.14), we show that the following relations hold:

$$\widetilde{F}^2 = q - p\,, \tag{3.23}$$

$$\widetilde{F}^3 = \widetilde{F}\widetilde{F}^2 = (q + f)(q - p) = q - f\,, \tag{3.24}$$

$$\widetilde{F}^3 + \widetilde{F} = 2q\,. \tag{3.25}$$

Similar calculations can be done for $\widetilde{G}$ and $\widetilde{H}$ by letting $q = 1 + g^2 = 1 - p$ and $q = 1 + h^2 = 1 - p$, respectively.

Theorem 3.2. *If M admits a torsion-free connection Γ reducible to $\mathcal{H}(M)$, then the tensors $\widetilde{F}^2 = \widetilde{G}^2 = \widetilde{H}^2 = q - p$ are torsion-free. That is, $[\widetilde{F}^2, \widetilde{F}^2] = 0$, $[\widetilde{G}^2, \widetilde{G}^2] = 0$ and $[\widetilde{H}^2, \widetilde{H}^2] = 0$, i.e., P and Q are integrable.*

Proof. Since Γ is reducible to $\mathcal{H}(M)$, equations in (3.1) hold.
We get $\nabla p = -\nabla q$ from $p+q=1$. Since $q = 1 + f^2 = 1 - p$, using (3.22) we obtain

$$\nabla p = -\nabla(f^2) = -\nabla(Fp \circ Fp) = -Fp\nabla(Fp) - \nabla(Fp)Fp\,. \tag{3.26}$$

By using (3.1) we get

$$\nabla p = -[(aG - bH)Fp + F(\nabla p)Fp + Fp(aG - bH)p - (\nabla p)], \tag{3.27}$$

which implies

$$\nabla p = 0\,. \tag{3.28}$$

So, ∇p and ∇q are both 0. It means that p and q are parallel. Furthermore, $\widetilde{F}^{\,2} = q - p$ implies that $\nabla \widetilde{F}^{\,2} = 0$. Then, $[\widetilde{F}^{\,2}, \widetilde{F}^{\,2}] = 0$. Similarly, we see that $[\widetilde{G}^{\,2}, \widetilde{G}^{\,2}] = 0$ and $[\widetilde{H}^{\,2}, \widetilde{H}^{\,2}] = 0$. Hence, P and Q are integrable by Proposition. □

Theorem 3.3. *If the torsions of $\widetilde{F}^{\,2}$, $\widetilde{G}^{\,2}$, and $\widetilde{H}^{\,2}$ vanish with respect to a torsion-free connection Γ, then M admits a connection $\overset{\circ}{\Gamma}$ reducible to $\mathcal{H}(M)$ whose torsion tensor is given by*

$$24\,T(X,Y) = 2\left([f,f] + [g,g] + [h,h]\right)(X,Y) - \left([f,f] + [g,g] + [h,h]\right)(pX, pY) \tag{3.29}$$

for any vector fields X and Y on M.

Proof. Since Γ is an arbitrary torsion-free affine connection on M with covariant derivative ∇, by using Proposition, we have $\nabla \widetilde{F}^2 = 0$, $\nabla \widetilde{G}^2 = 0$ and $\nabla \widetilde{H}^2 = 0$.
Below we give calculations only for the tensor $\widetilde{F}$. The calculations for the other tensors $\widetilde{G}$ and $\widetilde{H}$ can be obtained similarly.

To simplify the calculations, we introduce the tensors W_I and W_{II} of type $(1,2)$ as follows:

$$W_I(X,Y) = (\nabla_{\widetilde{F}Y}\widetilde{F})X + \widetilde{F}((\nabla_Y \widetilde{F})X) + \widetilde{F}^2((\nabla_{\widetilde{F}^3 Y}\widetilde{F})X) + \widetilde{F}^3((\nabla_{\widetilde{F}^2 Y}\widetilde{F})X)\,, \tag{3.30}$$

$$W_{II}(X,Y) = \widetilde{F}((\nabla_X \widetilde{F})Y) \tag{3.31}$$

for any vector fields X and Y on M. Using (3.30) we obtain

$$W_I(X, \widetilde{F}Y) = (\nabla_{\widetilde{F}^2 Y}\widetilde{F})X + \widetilde{F}((\nabla_{\widetilde{F}Y}\widetilde{F})X) + \widetilde{F}^2((\nabla_{\widetilde{Y}}\widetilde{F})X) + \widetilde{F}^3((\nabla_{\widetilde{F}^3 Y}\widetilde{F})X) \tag{3.32}$$

and

$$\widetilde{F}W_I(X,Y) = \widetilde{F}((\nabla_{\widetilde{F}Y}\widetilde{F})X) + \widetilde{F}^2((\nabla_Y\widetilde{F})X) + \widetilde{F}^3((\nabla_{\widetilde{F}^3Y}\widetilde{F})X) + ((\nabla_{\widetilde{F}^2Y}\widetilde{F})X)\,, \tag{3.33}$$

which gives

$$W_I(X,\widetilde{F}Y) - \widetilde{F}(W_I(X,Y)) = 0\,. \tag{3.34}$$

From (3.31) we also obtain

$$W_{II}(X,\widetilde{F}Y) - \widetilde{F}(W_{II}(X,Y)) = 2(\nabla_X\widetilde{F})Y\,. \tag{3.35}$$

As the tensor $\widetilde{F}^2$ is parallel, i.e, $\nabla\widetilde{F}^2 = 0$, we have $(\nabla\widetilde{F})\widetilde{F} = -\widetilde{F}(\nabla\widetilde{F})$. Using (2.6), (2.7), and (2.8), we see that the following relations hold:

$$\nabla\widetilde{F} = \nabla q + \nabla F p = \nabla F p = \nabla f\,. \tag{3.36}$$

Since $\nabla\widetilde{F}^2 = 0\,,\ \nabla p = \nabla q = 0$. Thus, $\nabla\widetilde{F}^3 = -\nabla\widetilde{F}\,.$

For any tensor field α, if α is parallel, then $\nabla_X\alpha Y = \alpha(\nabla_X Y)$. Therefore,

$$\nabla\widetilde{F}^3 = \nabla(\widetilde{F}^2\circ\widetilde{F}) = \nabla\widetilde{F}^2(\nabla\widetilde{F})\,. \tag{3.37}$$

In addition to the equations (3.34) and (3.35), the following relations hold:

$$W_{II}(X,\widetilde{F}Y) = \widetilde{F}((\nabla_X\widetilde{F})\widetilde{F}Y) = -\widetilde{F}^2((\nabla_X\widetilde{F})Y)\,, \tag{3.38}$$

$$\widetilde{F}W_{II}(X,Y) = \widetilde{F}^2((\nabla_X\widetilde{F})Y) = (\nabla_X\widetilde{F}^3)Y = (\nabla_X\widetilde{F})Y\,. \tag{3.39}$$

From (3.38) and (3.39) we get

$$W_{II}(X,\widetilde{F}Y) - \widetilde{F}(W_{II}(X,Y)) = -2(\nabla_X\widetilde{F}^3)Y = 2(\nabla_X\widetilde{F})Y. \tag{3.40}$$

Rearranging (3.30) we have

$$\begin{aligned} W_I(X,Y) &= (\nabla_{qY+fY}\widetilde{F})X + \widetilde{F}((\nabla_{pY+qY}\widetilde{F})X) + \widetilde{F}^2((\nabla_{qY-FY}\widetilde{F})X) \\ &\quad +\widetilde{F}^3((\nabla_{qY-pY}\widetilde{F})X) \\ &= ((\nabla_{qY}\widetilde{F})X)(1+\widetilde{F}+\widetilde{F}^2+\widetilde{F}^3) + ((\nabla_{FY}\widetilde{F})X)(1-\widetilde{F}^2) \\ &\quad +((\nabla_{pY}\widetilde{F})X)(\widetilde{F}-\widetilde{F}^3) \\ &= 4q((\nabla_{qY}\widetilde{F})X) + 2p((\nabla_{fY})\widetilde{F})X) + 2f((\nabla_{pY}\widetilde{F})X). \end{aligned} \tag{3.41}$$

Because $\nabla p = \nabla q = 0$, we obtain $\nabla\widetilde{F} = \nabla F p$. So, by the definition of the projection tensor q, we obtain $q(\nabla\widetilde{F}) = 0$. Moreover,

$$q(\nabla_X F)Y = 0 \quad \text{for any vector fields } X\,,\ \text{and } Y \in T_x(M)\,, \tag{3.42}$$

and

$$\begin{aligned} q(\nabla_X F)Y &= q((\nabla_X(FY) - F(\nabla_X Y)) \\ &= q(\nabla_X(FY)) - qF(\nabla_X Y) = q(\nabla_X(FY)) = 0\,. \end{aligned} \tag{3.43}$$

Since q is parallel, it follows that $q(\nabla_X(FY)) = \nabla_X(qFY) = 0$, which reduces (3.41) to

$$\begin{aligned} W_I(X,Y) = 4q[(\nabla_{qY}q)(X) + (\nabla_{qY}f)X] + 2p[(\nabla_{fY}q)X + (\nabla_{fY}f)X] \\ +2f[(\nabla_{pY}q)X + (\nabla_{pY}f)X]. \end{aligned} \tag{3.44}$$

Furthermore, we have $q((\nabla_{qY}(FX)) = p((\nabla_{FY}q)X) = F((\nabla_{pY}q)X) = 0$, which together with (3.43), implies

$$W_I(X,Y) = 2p[(\nabla_{fY}f)X] + 2f[(\nabla_{pY}f)X]. \tag{3.45}$$

(3.31) and (3.19) give

$$\begin{aligned} W_{II}(X,Y) &= \widetilde{F}((\nabla_X\widetilde{F})Y) \\ &= (q+f)(\nabla_X(q+f)Y) \\ &= q((\nabla_X q)Y) + q((\nabla_X f)Y) + f((\nabla_X q)Y) + f((\nabla_X f)Y)\,, \\ W_{II}(X,Y) &= f((\nabla_X f)Y). \end{aligned} \tag{3.46}$$

Let us now consider an affine connection $\overset{\circ}{\Gamma}$ whose covariant derivative $\overset{\circ}{\nabla}$ is given by

$$\overset{\circ}{\nabla}_X Y = \nabla_X Y - \frac{1}{8}W_I(X,Y) - \frac{1}{2}W_{II}(X,Y), \tag{3.47}$$

where X and Y are vector fields on M. Here ∇ is covariant differentiation of an affine connection, and W_I and W_{II} are bilinear mappings $T_x(M)\times T_x(M) \to T_x(M)$ at each $x \in M$, and $\overset{\circ}{\nabla}$ defines an affine connection on M.

We can see that by using (3.47), $\widetilde{F}$, $\widetilde{G}$, and $\widetilde{H}$ are parallel with respect to $\overset{\circ}{\Gamma}$ on M.

Let X and Y be arbitrary vector fields on M. Then,

$$\begin{aligned}
(\overset{\circ}{\nabla}_X \widetilde{F})Y &= \overset{\circ}{\nabla}_X(\widetilde{F}Y) - \widetilde{F}(\overset{\circ}{\nabla}_X Y) \\
&= \nabla_X(\widetilde{F}Y) - \frac{1}{8}W_I(X,\widetilde{F}Y) - \frac{1}{2}W_{II}(X,\widetilde{F}Y) \\
&\quad -\widetilde{F}[\nabla_X Y - \frac{1}{8}W_I(X,Y) - \frac{1}{2}W_{II}(X,Y)] \\
&= \nabla_X(\widetilde{F}Y) - \widetilde{F}(\nabla_X Y) - \frac{1}{8}(W_I(X,\widetilde{F}Y) - F(W_{II}(X,\widetilde{F}Y)) \\
&\quad -\frac{1}{2}[W_{II}(X,\widetilde{F}Y) - \widetilde{F}(W_{II}(X,Y))] \\
&= (\nabla_X\widetilde{F})Y - \frac{1}{2}(2(\nabla_X\widetilde{F})Y) = 0,
\end{aligned} \tag{3.48}$$

hence we conclude that $\overset{\circ}{\nabla}_X \widetilde{F} = 0$.
By (3.47) , the torsion of $\overset{\circ}{\Gamma}$ is obtained as

$$\begin{aligned}
T(X,Y) &= \overset{\circ}{\nabla}_X Y - \overset{\circ}{\nabla}_Y X - [X,Y] \\
&= \nabla_X Y - \nabla_Y X - [X,Y] - \frac{1}{8}(W_I(X,Y) - W_I(Y,X)) \\
&\quad -\frac{1}{2}(W_{II}(X,Y) - W_{II}(Y,X)).
\end{aligned} \tag{3.49}$$

Since Γ is torsion-free , we get

$$T(X,Y) = -\frac{1}{8}(W_I(X,Y) - W_I(Y,X)) - \frac{1}{2}(W_{II}(X,Y) - W_{II}(Y,X)). \tag{3.50}$$

Using (3.45) and (3.46) we find

$$\begin{aligned}
-8T(X,Y) &= W_I(X,Y) - W_I(Y,X) + 4W_{II}(X,Y) - 4W_{II}(Y,X) \\
&= 2p((\nabla_{fY}f)X) + 2f((\nabla_{pY}f)X) - 2p((\nabla_{fX}f)Y) \\
&\quad -2f((\nabla_{pX}f)Y) + 4f((\nabla_X f)Y) - 4f((\nabla_Y f)X).
\end{aligned} \tag{3.51}$$

Because $\nabla\widetilde{F}^2 = 0$ and $\nabla p = 0$, we see that $p(\nabla Fp) = p(\nabla f) = \nabla f$. Then, (3.51) is reduced to

$$\begin{aligned}
-8T(X,Y) &= 2(\nabla_{fY}f)X + 2f((\nabla_{pY}f)X) - 2(\nabla_{fX}f)Y - 2f((\nabla_{pX}f)Y) \\
&\quad +2f((\nabla_X)Y) + 2f((\nabla_{pX}f)Y) + 2(\nabla_{qX}f)Y - 2f((\nabla_Y f)X) \\
&\quad -2f((\nabla_{pY}f)X) - 2f((\nabla_{pY}f)X) - 2f((\nabla_{qY}f)X)\,.
\end{aligned} \tag{3.52}$$

Equation (3.52) can be simplified to

$$\begin{aligned}
-8T(X,Y) &= -2[(\nabla_{fX}f)Y - (\nabla_{fY}f)X - f((\nabla_X f)Y) + f((\nabla_Y f)X)] \\
&\quad +2f((\nabla_{qX}f)Y) - 2f((\nabla_{qY}f)X)\,.
\end{aligned} \tag{3.53}$$

By means of (1.3), we get

$$\frac{1}{2}[f,f](X,Y) = (\nabla_{fX}f)Y - (\nabla_{fY}f)X - f((\nabla_X f)Y) + f((\nabla_Y f)X)\,. \quad (3.54)$$

Substituting (3.54) in (3.53) yields

$$8T(X,Y) = [f,f](X,Y) - 2f((\nabla_{qX}f)Y) + 2F((\nabla_{qY}f)X)\,. \quad (3.55)$$

If torsion tensor of $[f,f]$ is calculated for vector fields pX and pY, by using (3.54), we get

$$\begin{aligned}[f,f](pX,pY) - [f,f](X,Y) = 2([fX,fY] - p[pX,pY] - f[pX,fY] - f[fX,pY] \\ - ([fX,fY] + f^2[X,Y] - f[fX,Y] - f[X,fY])).\end{aligned} \quad (3.56)$$

Moreover, we have

$$\begin{aligned} f[fX,Y] &= f[fX,pY+qY] = f[fX,pY] + f[fX,qY]\,, \\ f[X,fY] &= f[pX,fY] + f[qX,fY]\,. \end{aligned} \quad (3.57)$$

Since $(Fp)\circ(Fp) = -p$ and $f^2 = -p$, we obtain

$$f^2[X,Y] = -p[pX+qX,pY+qY] = -p[pX,pY] - p[pX,qY] - p[qX,pY]\,. \quad (3.58)$$

Substituting (3.57) and (3.58) in (3.56), we get

$$\begin{aligned}[f,f](pX,pY) - [f,f](X,Y) = 2(p[qX,pY] + p[pX,qY] + p[qX,qY] \\ +f[fX,qY] + f[qX,fY])\,.\end{aligned} \quad (3.59)$$

As $\nabla\widetilde{F}^2 = 0$ and Γ is torsion-free, by Proposition we have $p[qX,qY] = 0$. So, (3.59) becomes

$$\begin{aligned}[f,f](pX,pY) - [f,f](X,Y) = 2(p(\nabla_{pX}qY - \nabla_{qY}pX) + p(\nabla_{qX}pY - \nabla_{pY}qX) \\ + f(\nabla_{fX}qY - \nabla_{qY}fX) + f(\nabla_{qX}fY) - \nabla_{fY}qX)\,.\end{aligned} \quad (3.60)$$

By means of parallel properties of p and q, we get

$$p(\nabla_{pY}(qX)) = p(\nabla_{pX}(qY)) = F(\nabla_{FY}(qX)) = 0 \quad (3.61)$$

and

$$p(\nabla_{qX}(pY)) = p(\nabla_{qX}Y)\,, \quad p(\nabla_{qY}(pX)) = p(\nabla_{qY}X)\,.$$

Therefore, (3.60) reduces to

$$[f,f](pX,pY)-[f,f](X,Y)=2f((\nabla_{qX}f)Y)-2f((\nabla_{qY}f)X)\,.$$

Finally, torsion of f becomes

$$8T(X,Y)=2[f,f](X,Y)-[f,f](pX,pY). \tag{3.62}$$

Similarly, torsion tensors for g and h are obtained as

$$8T(X,Y) = 2[g,g](X,Y)-[g,g](pX,pY), \tag{3.63}$$

$$8T(X,Y) = 2[h,h](X,Y)-[h,h](pX,pY)\,, \tag{3.64}$$

which completes the proof. □

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