

Sufficient Criteria for Biholomorphic Convex Mappings on D_p in C^n

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Abstract In this paper, two sufficient conditions for biholomorphic convex mappings on bounded convex balanced domain D_p in C^n are provided. From these, criteria for a biholomorphic convex mappings with particular form become direct. Moreover, some concrete biholomorphic convex mappings on D_p are also provided.

Keywords: Locally biholomorphic mapping; biholomorphic convex mapping; bounded convex balanced domain.

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1 Introduction and preliminaries

Suppose that n is a fixed positive integer and $p > 1$. Let C^n be the space of n complex variables $z = (z_1, z_2, \dots, z_n)$ with the usual inner product $\langle z, w \rangle = \sum_{j=1}^n z_j \bar{w}_j$, where $w = (w_1, w_2, \dots, w_n) \in C^n$. Let Ω be a domain in C^n . A mapping $f : \Omega \rightarrow C^n$ is said to be locally biholomorphic in Ω if f has a locally inverse at each point $z \in \Omega$ or, equivalently, if the first Fréchet derivative

$$Df(z) = \left(\frac{\partial f_j(z)}{\partial z_k} \right)_{1 \leq j, k \leq n}$$

is nonsingular at each point in Ω .

The second Fréchet derivative of a mapping $f : \Omega \rightarrow C^n$ is a symmetric bilinear operator $D^2f(z)(\cdot, \cdot)$ on $C^n \times C^n$, and $D^2f(z)(z, \cdot)$ is the linear operator obtained by restricting $D^2f(z)$ to $\{z\} \times C^n$. The matrix representation of $D^2f(z)(b, \cdot)$ is

$$D^2f(z)(b, \cdot) = \left(\sum_{l=1}^n \frac{\partial^2 f_j(z)}{\partial z_k \partial z_l} b_l \right)_{1 \leq j, k \leq n},$$

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where $f(z) = (f_1(z), \dots, f_n(z))$, $b = (b_1, \dots, b_n) \in C^n$. If $f \in H(\Omega)$, then for every $k = 0, 1, \dots$, there is a bounded symmetric k -linear operator $D^k f(0) : C^n \times C^n \times \dots \times C^n \rightarrow C^n$ such that

$$f(z) = \sum_{k=0}^{\infty} \frac{D^k f(0)}{k!}(z^k)$$

for $z \in \Omega$, where $D^0 f(0)(z^0) = f(0)$, $D^k f(0)(z^k) = D^k f(0)(z, z, \dots, z)$.

For a domain Ω in C^n . If for each $z \in \Omega$, we have $\lambda z \in \Omega$ for all $\lambda \in C$ with $|\lambda| \leq 1$, then we call Ω a balanced domain. The *Minkowski* functional of a balanced domain Ω is defined by

$$\rho(z) = \inf \left\{ t > 0, \frac{z}{t} \in \Omega \right\}, \quad z \in C^n.$$

Suppose that Ω is a bounded convex balanced domain in C^n , let $\rho(z)$ be the *Minkowski* functional of Ω , then $\rho(\bullet)$ is a norm of C^n , and

$$\Omega = \{z \in C^n : \rho(z) < 1\}, \quad \rho(\lambda z) = |\lambda| \rho(z),$$

where $\lambda \in C$, $z \in C^n$, and $\rho(z) = 0$ if and only if $z = 0$ (see [14]).

Assume $p_j > 1$ ($j = 1, 2, \dots, n$). Let $D_p = \{(z_1, z_2, \dots, z_n) \in C^n : \sum_{j=1}^n |z_j|^{p_j} < 1\}$, then D_p is a bounded convex balanced domain in C^n , and its *Minkowski* functional $\rho(z)$ satisfies the equality

$$\sum_{j=1}^n \left| \frac{z_j}{\rho(z)} \right|^{p_j} = 1. \quad (1.1)$$

We denote D_p by B_p^n for the special case of $p_1 = \dots = p_n = p$, at this time, we have $\rho(z) = \sqrt[p]{|z_1|^p + \dots + |z_n|^p}$. In particular, let $\Delta = B_p^1$ denote the unit disk in the complex plane C .

Let $H(D_p)$ be the class of holomorphic mappings $f(z) = (f_1(z), \dots, f_n(z))$ on bounded convex balanced domain D_p , where $z = (z_1, \dots, z_n) \in C^n$ and let $N(D_p)$ denote the class of all locally biholomorphic mappings $f : D_p \rightarrow C^n$ such that $f(0) = 0$, $Df(0) = I$, where I is the unit matrix of $n \times n$. If $f \in N(D_p)$ is a biholomorphic mapping on D_p and $f(D_p)$ is a convex domain in C^n , then we say that f is a biholomorphic convex mapping on D_p . The class of all biholomorphic convex mappings on D_p with $f(0) = 0$, $Df(0) = I$ is denoted by $K(D_p)$.

It is difficult to construct concrete biholomorphic convex mappings on some domains in C^n , even on the unit ball B_2^n . In 1995, Roper and Suffridge[13] proved that:

Theorem 1.1[13] If $f \in K$ and $F(z) = (f(z_1), \sqrt{f'(z_1)}z_0)$, where $z = (z_1, z_0) \in B^n$, $z_1 \in \Delta$, $z_0 = (z_2, \dots, z_n) \in C^{n-1}$, then $F \in K(B_2^n)$.

Which is popularly referred to as the Roper-Suffridge operator. Using this operator, we may construct a lot of concrete biholomorphic convex mappings on B_n^2 . After that, many authors had generalized Roper-Suffridge operator to Reinhardt domain D_p or Banach spaces[2, 3, 4, 5, 6, 7, 8, 12, 16]. However, according to the result in [1, 11], none of these concrete examples belongs to $K(D_p)$ ($p_j > 2, j = 1, 2, \dots, n$). Liu and Zhu[10] obtained some sufficient conditions for starlike mappings. Hamada and Kohr[4], Zhu[15] gave a

necessary and sufficient condition for biholomorphic convex mappings on bounded convex balanced domain D_p . From these, Liu and Zhu[9] had given some sufficient conditions of biholomorphic convex mappings on D_p as follows.

Theorem 1.2[9] Suppose that $n \geq 2, p_j \geq 2(j = 1, 2, \dots, n)$ and $g_j : \Delta \rightarrow C$ are analytic on Δ with $g_j(0) = g'_j(0) = 0(j = 1, 2, \dots, n - 1)$, $f_j(\zeta) \in N(\Delta)$ satisfy the conditions $|\zeta f''_j(\zeta)| \leq |f'_j(\zeta)|, (\zeta \in \Delta, j = 1, 2, \dots, n)$. Let

$$f(z) = (f_1(z_1) + g_1(z_n), f_2(z_2) + g_2(z_n), \dots, f_{n-1}(z_{n-1}) + g_{n-1}(z_n), f_n(z_n)).$$

If for any $z = (z_1, \dots, z_n) \in D_p \setminus \{0\}$, the inequality

$$\sum_{j=1}^{n-1} p_j \left[\left| \frac{g''_j(z_n)}{f'_j(z_j)} \right| + \left| \frac{g'_j(z_n)}{f'_j(z_j)} \right| \left| \frac{f''_n(z_n)}{f'_n(z_n)} \right| \right] \leq p_n \left(1 - \left| \frac{z_n f''_n(z_n)}{f'_n(z_n)} \right| \right) |z_n|^{p_n-2}$$

holds, then $f \in K(D_p)$.

Theorem 1.3[9] Suppose that $n \geq 2, p_j \geq 2(j = 1, 2, \dots, n)$, and $f_j \in N(\Delta)(j = 2, \dots, n)$, $f_1(z_1, \dots, z_n) : D_p \rightarrow C$ is holomorphic with $f_1(0, \dots, 0) = 0, (\partial f_1 / \partial z_1)(0, \dots, 0) = 1$. Let

$$f(z) = (f_1(z_1, z_2, \dots, z_n), f_2(z_2), \dots, f_n(z_n)),$$

where $z = (z_1, z_2, \dots, z_n)$. If for any $z = (z_1, \dots, z_n) \in D_p \setminus \{0\}$, we have

$$\begin{aligned} \frac{\partial f_1}{\partial z_1} \neq 0, \quad \sum_{j=1}^n |z_1 \frac{\partial^2 f_1}{\partial z_1 \partial z_j}| \leq \left| \frac{\partial f_1}{\partial z_1} \right|, \\ \left| \frac{\frac{\partial f_1}{\partial z_j} \cdot \frac{f''_j(z_j)}{f'_j(z_j)}}{\frac{\partial f_1}{\partial z_1}} \right| + \sum_{l=1}^n \left| \frac{\frac{\partial^2 f_1}{\partial z_j \partial z_l}}{\frac{\partial f_1}{\partial z_1}} \right| \leq \frac{p_j}{p_1} \left(1 - \left| \frac{z_j f''_j(z_j)}{f'_j(z_j)} \right| \right) \cdot |z_j|^{p_j-2}, j = 2, \dots, n, \end{aligned}$$

then $f \in K(D_p)$.

A problem is naturally posed: can we get some sufficient conditions such that the mapping of the form:

$$f(z) = (f_1(z_1, z_2, \dots, z_n), f_2(z_2) + g_2(z_n), \dots, f_{n-1}(z_{n-1}) + g_{n-1}(z_n), f_n(z_n))$$

is a biholomorphic convex mapping on D_p ?

The aim of this paper is to give an answer to the above problem, which extends Theorems 1.2 and 1.3. From these, we may construct some concrete biholomorphic convex mappings on D_p . In order to derive our main results, we need the following lemma.

Lemma 1.1 [4, 15] Suppose that $p_j \geq 2(j = 1, 2, \dots, n)$, $\rho(z)$ is the Minkowski functional of D_p , and $f \in N(D_p)$. Then $f \in K(D_p)$ if and only if for any $z = (z_1, z_2, \dots, z_n) \in D_p \setminus \{0\}$ and $b = (b_1, b_2, \dots, b_n) \in C^n \setminus \{0\}$ such that

$$\operatorname{Re} \left\{ \sum_{j=1}^n p_j \left| \frac{z_j}{\rho(z)} \right|^{p_j} \frac{b_j}{z_j} \right\} = 0,$$

we have

$$J_f(z, b) = \operatorname{Re} \left\{ \sum_{j=1}^n \frac{p_j^2}{2} \bullet \frac{|z_j|^{p_j-2}}{\rho(z)^{p_j}} |b_j|^2 + \sum_{j=1}^n p_j \left(\frac{p_j}{2} - 1 \right) \left| \frac{z_j}{\rho(z)} \right|^{p_j} \left(\frac{b_j}{z_j} \right)^2 - 2 \sum_{j=1}^n \frac{p_j}{\rho(z)} \left| \frac{z_j}{\rho(z)} \right|^{p_j} \left\langle Df(z)^{-1} D^2 f(z)(b, b), \frac{\partial \rho}{\partial \bar{z}} \right\rangle \right\} \geq 0.$$

2 Main results

We first establish a general sufficient condition for biholomorphic convex mappings on D_p , which is the main result of this paper.

Theorem 2.1 *Suppose that $n \geq 2, p_j \geq 2 (j = 1, 2, \dots, n)$ and $g_j : \Delta \rightarrow C$ are analytic on Δ with $g_j(0) = g'_j(0) = 0 (j = 2, \dots, n-1)$, $f_1(z_1, \dots, z_n) : D_p \rightarrow C$ is holomorphic with $f_1(0, \dots, 0) = 0, (\partial f_1 / \partial z_1)(0, \dots, 0) = 1, f_j(\zeta) \in N(\Delta)$ satisfy the conditions $|\zeta f''_j(\zeta)| \leq |f'_j(\zeta)|, \zeta \in \Delta, j = 2, \dots, n$. Let*

$$f(z) = (f_1(z_1, \dots, z_n), f_2(z_2) + g_2(z_n), \dots, f_{n-1}(z_{n-1}) + g_{n-1}(z_n), f_n(z_n)).$$

If f satisfies the following conditions

- (1) $\frac{\partial f_1}{\partial z_1} \neq 0, \sum_{l=1}^n \left| z_1 \frac{\partial^2 f_1}{\partial z_1 \partial z_l} \right| \leq \left| \frac{\partial f_1}{\partial z_1} \right|;$
- (2) $\frac{p_1}{\left| \frac{\partial f_1}{\partial z_1} \right|} \left(\sum_{l=1}^n \left| \frac{\partial^2 f_1}{\partial z_j \partial z_l} \right| + \left| \frac{\partial f_1}{\partial z_j} \frac{f''_j(z_j)}{f'_j(z_j)} \right| \right) \leq p_j \left(1 - \left| \frac{z_j f''_j(z_j)}{f'_j(z_j)} \right| \right) |z_j|^{p_j-2}, j = 2, \dots, n-1;$
- (3) $\frac{p_1}{\left| \frac{\partial f_1}{\partial z_1} \right|} \left(\sum_{l=1}^n \left| \frac{\partial^2 f_1}{\partial z_l \partial z_n} \right| + \sum_{j=2}^{n-1} \left| \frac{\partial f_1}{\partial z_j} \frac{g'_j(z_n)}{f'_j(z_j)} \right| + \left| \frac{\partial f_1}{\partial z_n} \frac{f''_n(z_n)}{f'_n(z_n)} \right| + \sum_{j=2}^{n-1} \left| \frac{g'_j(z_n)}{f'_j(z_j)} \frac{f''_n(z_n)}{f'_n(z_n)} \frac{\partial f_1}{\partial z_j} \right| \right) + \sum_{j=2}^{n-1} \frac{p_j}{|f'_j(z_j)|} \left(|g''_j(z_n)| + \left| \frac{g'_j(z_n) f''_n(z_n)}{f'_n(z_n)} \right| \right) \leq p_n \left(1 - \left| \frac{z_n f''_n(z_n)}{f'_n(z_n)} \right| \right) |z_n|^{p_n-2},$

for all $z = (z_1, \dots, z_n) \in D_p \setminus \{0\}$, then $f \in K(D_p)$.

Proof. By directly calculating the Fréchet derivatives of $f(z)$, we obtain

$$Df(z) = \begin{pmatrix} \frac{\partial f_1}{\partial z_1} & \frac{\partial f_1}{\partial z_2} & \dots & \frac{\partial f_1}{\partial z_{n-1}} & \frac{\partial f_1}{\partial z_n} \\ 0 & f'_2(z_2) & \dots & 0 & g'_2(z_n) \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & f'_{n-1}(z_{n-1}) & g'_{n-1}(z_n) \\ 0 & 0 & \dots & 0 & f'_n(z_n) \end{pmatrix},$$

$$Df(z)^{-1} = \begin{pmatrix} \frac{1}{\frac{\partial f_1}{\partial z_1}} & -\frac{\frac{\partial f_1}{\partial z_2}}{\frac{\partial f_1}{\partial z_1} f'_2(z_2)} & \dots & -\frac{\frac{\partial f_1}{\partial z_{n-1}}}{\frac{\partial f_1}{\partial z_1} f'_{n-1}(z_{n-1})} & -\frac{\frac{\partial f_1}{\partial z_n}}{\frac{\partial f_1}{\partial z_1} f'_n(z_n)} + \sum_{j=2}^{n-1} \frac{g'_j(z_n)}{f'_j(z_j)} \frac{\frac{\partial f_1}{\partial z_j}}{\frac{\partial f_1}{\partial z_1} f'_n(z_n)} \\ 0 & \frac{1}{f'_2(z_2)} & \dots & 0 & -\frac{g'_2(z_n)}{f'_2(z_2) f'_n(z_n)} \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & \frac{1}{f'_{n-1}(z_{n-1})} & -\frac{g'_{n-1}(z_n)}{f'_{n-1}(z_{n-1}) f'_n(z_n)} \\ 0 & 0 & \dots & 0 & \frac{1}{f'_n(z_n)} \end{pmatrix},$$

$$\begin{aligned}
D^2 f(z)(b, b) &= \begin{pmatrix} \sum_{l=1}^n \frac{\partial^2 f_1}{\partial z_1 \partial z_l} b_l & \sum_{l=1}^n \frac{\partial^2 f_1}{\partial z_2 \partial z_l} b_l & \cdots & \sum_{l=1}^n \frac{\partial^2 f_1}{\partial z_{n-1} \partial z_l} b_l & \sum_{l=1}^n \frac{\partial^2 f_1}{\partial z_n \partial z_l} b_l \\ 0 & f_2''(z_2) b_2 & \cdots & 0 & g_2''(z_n) b_n \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & f_{n-1}''(z_{n-1}) b_{n-1} & g_{n-1}''(z_n) b_n \\ 0 & 0 & \cdots & 0 & f_n''(z_n) b_n \end{pmatrix} \begin{pmatrix} b_1 \\ b_2 \\ \cdots \\ b_{n-1} \\ b_n \end{pmatrix} \\
&= \begin{pmatrix} \sum_{j=1}^n \sum_{l=1}^n \frac{\partial^2 f_1}{\partial z_j \partial z_l} b_l b_j \\ f_2''(z_2) b_2^2 + g_2''(z_n) b_n^2 \\ \cdots \\ f_{n-1}''(z_{n-1}) b_{n-1}^2 + g_{n-1}''(z_n) b_n^2 \\ f_n''(z_n) b_n^2 \end{pmatrix}.
\end{aligned}$$

According to (1.1) and $|z_j|^{p_j} = z_j^{p_j/2} \bar{z}_j^{p_j/2}$, direct computation yields

$$\frac{\partial \rho}{\partial \bar{z}_l} = \frac{p_l |z_l|^{p_l}}{2 \bar{z}_l \rho(z)^{p_l-1} \sum_{j=1}^n p_j \left| \frac{z_j}{\rho(z)} \right|^{p_j}}. \quad (2.1)$$

Fix $z = (z_1, z_2, \dots, z_n) \in D_p \setminus \{0\}$ and $b = (b_1, b_2, \dots, b_n) \in C^n \setminus \{0\}$ such that

$$\operatorname{Re} \left\{ \sum_{j=1}^n p_j \left| \frac{z_j}{\rho(z)} \right|^{p_j} \frac{b_j}{z_j} \right\} = 0.$$

From (1.1), we have $|z_j/\rho(z)| \leq 1$ ($j = 1, 2, \dots, n$) for all $z = (z_1, z_2, \dots, z_n) \in D_p$. By the hypotheses of Theorem 1, we obtain

$$\begin{aligned}
J_f(z, b) &\geq \sum_{j=1}^n \frac{p_j |z_j|^{p_j-2}}{\rho(z)^{p_j}} |b_j|^2 - 2 \sum_{j=1}^n \frac{p_j}{\rho(z)} \left| \frac{z_j}{\rho(z)} \right|^{p_j} \operatorname{Re} \left\langle Df(z)^{-1} D^2 f(z)(b, b), \frac{\partial \rho}{\partial \bar{z}} \right\rangle \\
&= \sum_{j=1}^n \frac{p_j |z_j|^{p_j-2}}{\rho(z)^{p_j}} |b_j|^2 - \operatorname{Re} \left\{ \left[\frac{1}{\partial z_1} \sum_{j=1}^n \sum_{l=1}^n \frac{\partial^2 f_1}{\partial z_j \partial z_l} b_j b_l - \sum_{j=2}^{n-1} \frac{\partial f_1}{\partial z_j} \frac{f_j''(z_j)}{f_j'(z_j)} b_j^2 \right. \right. \\
&\quad \left. \left. - \sum_{j=2}^{n-1} \frac{\partial f_1}{\partial z_j} \frac{g_j''(z_n)}{f_j'(z_j)} b_n^2 - \frac{\partial f_1}{\partial z_n} \frac{f_n''(z_n)}{f_n'(z_n)} b_n^2 + \sum_{j=2}^{n-1} \frac{g_j'(z_n) f_n''(z_n)}{f_j'(z_j) f_n'(z_n)} \frac{\partial f_1}{\partial z_1} b_n^2 \right] p_1 |z_1|^{p_1} \right. \\
&\quad \left. + \sum_{j=2}^{n-1} \left[\frac{f_j''(z_j)}{f_j'(z_j)} b_j^2 + \frac{g_j''(z_n)}{f_j'(z_j)} b_n^2 - \frac{g_j'(z_n) f_n''(z_n)}{f_j'(z_j) f_n'(z_n)} b_n^2 \right] \frac{p_j |z_j|^{p_j}}{z_j \rho(z)^{p_j}} + \frac{f_n''(z_n)}{f_n'(z_n)} b_n^2 \frac{p_n |z_n|^{p_n}}{z_n \rho(z)^{p_n}} \right\} \\
&\geq \sum_{j=1}^n \frac{p_j |z_j|^{p_j-2}}{\rho(z)^{p_j}} |b_j|^2 - \frac{1}{|\partial z_1|} \left[\sum_{j=1}^n \sum_{l=1}^n \left| \frac{\partial^2 f_1}{\partial z_j \partial z_l} \right| |b_j| |b_l| + \sum_{j=2}^{n-1} \left| \frac{\partial f_1}{\partial z_j} \frac{f_j''(z_j)}{f_j'(z_j)} \right| |b_j|^2 \right. \\
&\quad \left. + \sum_{j=2}^{n-1} \left| \frac{\partial f_1}{\partial z_j} \frac{g_j''(z_n)}{f_j'(z_j)} \right| |b_n|^2 + \left| \frac{\partial f_1}{\partial z_n} \frac{f_n''(z_n)}{f_n'(z_n)} \right| |b_n|^2 + \sum_{j=2}^{n-1} \left| \frac{g_j'(z_n) f_n''(z_n)}{f_j'(z_j) f_n'(z_n)} \frac{\partial f_1}{\partial z_j} \right| |b_n|^2 \right] \frac{p_1 |z_1|^{p_1-1}}{\rho(z)^{p_1}} \\
&\quad \left. - \sum_{j=2}^{n-1} \left[\left| \frac{f_j''(z_j)}{f_j'(z_j)} \right| |b_j|^2 + \left| \frac{g_j''(z_n)}{f_j'(z_j)} \right| |b_n|^2 + \left| \frac{g_j'(z_n) f_n''(z_n)}{f_j'(z_j) f_n'(z_n)} \right| |b_n|^2 \right] \frac{p_j |z_j|^{p_j-1}}{\rho(z)^{p_j}} \right.
\end{aligned}$$

$$\begin{aligned}
& - \left| \frac{f_n''(z_n)}{f_n'(z_n)} \right| |b_n|^2 \frac{p_n |z_n|^{p_n-1}}{\rho(z)^{p_n}} \\
\geq & |b_1|^2 \frac{p_1 |z_1|^{p_1-2}}{\rho(z)^{p_1}} \left[1 - \frac{1}{\left| \frac{\partial f_1}{\partial z_1} \right|} \sum_{l=1}^n \left| z_1 \frac{\partial^2 f_1}{\partial z_1 \partial z_l} \right| \right] + \sum_{j=2}^{n-1} |b_j|^2 \left[\frac{p_j |z_j|^{p_j-2}}{\rho(z)^{p_j}} \left(1 - \left| \frac{z_j f_j''(z_j)}{f_j'(z_j)} \right| \right) \right. \\
& - \frac{p_1}{\rho(z) \left| \frac{\partial f_1}{\partial z_1} \right|} \left| \frac{z_1}{\rho(z)} \right|^{p_1-1} \left(\sum_{l=1}^n \left| \frac{\partial^2 f_1}{\partial z_j \partial z_l} \right| + \left| \frac{\partial f_1}{\partial z_j} \frac{f_j''(z_j)}{f_j'(z_j)} \right| \right) \left. + |b_n|^2 \left[\frac{p_n |z_n|^{p_n-2}}{\rho(z)^{p_n}} \left(1 - \left| \frac{z_n f_n''(z_n)}{f_n'(z_n)} \right| \right) \right] \right. \\
& - \frac{p_1}{\rho(z) \left| \frac{\partial f_1}{\partial z_1} \right|} \left| \frac{z_1}{\rho(z)} \right|^{p_1-1} \left(\sum_{l=1}^n \left| \frac{\partial^2 f_1}{\partial z_l \partial z_n} \right| + \sum_{j=2}^{n-1} \left| \frac{\partial f_1}{\partial z_j} \frac{g_j''(z_n)}{f_j'(z_j)} \right| + \left| \frac{\partial f_1}{\partial z_n} \frac{f_n''(z_n)}{f_n'(z_n)} \right| \right. \\
& \left. + \sum_{j=2}^{n-1} \left| \frac{g_j'(z_n)}{f_j'(z_j)} \frac{f_n''(z_n)}{f_n'(z_n)} \frac{\partial f_1}{\partial z_j} \right| \right) - \sum_{j=2}^{n-1} \frac{p_j}{\rho(z)} \left| \frac{z_j}{\rho(z)} \right|^{p_j-1} \left(\left| \frac{g_j''(z_n)}{f_j'(z_j)} \right| + \left| \frac{g_j'(z_n)}{f_j'(z_j)} \frac{f_n''(z_n)}{f_n'(z_n)} \right| \right) \left. \right] \\
\geq & \sum_{j=2}^{n-1} \frac{|b_j|^2}{\rho(z)^{p_j}} \left[p_j |z_j|^{p_j-2} \left(1 - \left| \frac{z_j f_j''(z_j)}{f_j'(z_j)} \right| \right) - \frac{p_1 \rho(z)^{p_j-1}}{\left| \frac{\partial f_1}{\partial z_1} \right|} \left(\sum_{l=1}^n \left| \frac{\partial^2 f_1}{\partial z_j \partial z_l} \right| + \left| \frac{\partial f_1}{\partial z_j} \frac{f_j''(z_j)}{f_j'(z_j)} \right| \right) \right. \\
& + \frac{|b_n|^2}{\rho(z)^{p_n}} \left[p_n |z_n|^{p_n-2} \left(1 - \left| \frac{z_n f_n''(z_n)}{f_n'(z_n)} \right| \right) - \frac{p_1 \rho(z)^{p_n-1}}{\left| \frac{\partial f_1}{\partial z_1} \right|} \left(\sum_{l=1}^n \left| \frac{\partial^2 f_1}{\partial z_l \partial z_n} \right| + \sum_{j=2}^{n-1} \left| \frac{\partial f_1}{\partial z_j} \frac{g_j''(z_n)}{f_j'(z_j)} \right| \right. \right. \\
& \left. \left. + \left| \frac{\partial f_1}{\partial z_n} \frac{f_n''(z_n)}{f_n'(z_n)} \right| + \sum_{j=2}^{n-1} \left| \frac{g_j'(z_n)}{f_j'(z_j)} \frac{f_n''(z_n)}{f_n'(z_n)} \frac{\partial f_1}{\partial z_j} \right| \right) - \sum_{j=2}^{n-1} p_j \rho(z)^{p_n-1} \left(\left| \frac{g_j''(z_n)}{f_j'(z_j)} \right| + \left| \frac{g_j'(z_n)}{f_j'(z_j)} \frac{f_n''(z_n)}{f_n'(z_n)} \right| \right) \right].
\end{aligned}$$

Since $p_j \geq 2$ ($j = 1, \dots, n$) and $0 < \rho(z) < 1$ for every $z \in D_p \setminus \{0\}$, then we get that

$$\begin{aligned}
J_f(z, b) & \geq \sum_{j=2}^{n-1} \frac{|b_j|^2}{\rho(z)^{p_j}} \left[p_j |z_j|^{p_j-2} \left(1 - \left| \frac{z_j f_j''(z_j)}{f_j'(z_j)} \right| \right) - \frac{p_1}{\left| \frac{\partial f_1}{\partial z_1} \right|} \left(\sum_{l=1}^n \left| \frac{\partial^2 f_1}{\partial z_j \partial z_l} \right| + \left| \frac{\partial f_1}{\partial z_j} \frac{f_j''(z_j)}{f_j'(z_j)} \right| \right) \right] \\
& + \frac{|b_n|^2}{\rho(z)^{p_n}} \left[p_n |z_n|^{p_n-2} \left(1 - \left| \frac{z_n f_n''(z_n)}{f_n'(z_n)} \right| \right) - \frac{p_1}{\left| \frac{\partial f_1}{\partial z_1} \right|} \left(\sum_{l=1}^n \left| \frac{\partial^2 f_1}{\partial z_l \partial z_n} \right| + \sum_{j=2}^{n-1} \left| \frac{\partial f_1}{\partial z_j} \frac{g_j''(z_n)}{f_j'(z_j)} \right| \right. \right. \\
& \left. \left. + \left| \frac{\partial f_1}{\partial z_n} \frac{f_n''(z_n)}{f_n'(z_n)} \right| + \sum_{j=2}^{n-1} \left| \frac{g_j'(z_n)}{f_j'(z_j)} \frac{f_n''(z_n)}{f_n'(z_n)} \frac{\partial f_1}{\partial z_j} \right| \right) - \sum_{j=2}^{n-1} \frac{p_j}{\left| f_j'(z_j) \right|} \left(\left| \frac{g_j''(z_n)}{f_j'(z_j)} \right| + \left| \frac{g_j'(z_n)}{f_j'(z_j)} \frac{f_n''(z_n)}{f_n'(z_n)} \right| \right) \right] \\
& \geq 0.
\end{aligned}$$

Hence by Lemma 1.1, we obtain that $f \in K(D_p)$. This completes the proof of Theorem 1.

Remark 2.1 Setting $f_1(z) = f_1(z_1) + g_1(z_n)$ in Theorem 2.1, we get Theorem A or Theorem 1 in [9]. Setting $g_j(z_n) = 0$ ($j = 2, \dots, n-1$) in Theorem 2.1, we get Theorem B or Theorem 2 in [9].

If we replace $f_1(z)$ by $f_1(z_1) + g_1(z_2)$ in Theorem 2.1, we have the following corollary.

Corollary 2.1 *Suppose that $n \geq 3, p_j \geq 2, f_j(\zeta) \in N(\Delta)$ satisfy the conditions $|\zeta f_j''(\zeta)| \leq |f_j'(\zeta)|$ ($\zeta \in \Delta, j = 1, \dots, n$), and $g_j : \Delta \rightarrow C$ are analytic on Δ with $g_j(0) = g_j'(0) = 0$ ($j = 1, \dots, n-1$). Let*

$$f(z) = (f_1(z_1) + g_1(z_2), f_2(z_2) + g_2(z_n), \dots, f_{n-1}(z_{n-1}) + g_{n-1}(z_n), f_n(z_n)).$$

If f satisfies the following conditions

$$(1) \quad \frac{p_1}{|f'_1(z_1)|} \left(|g''_1(z_2)| + \left| \frac{g'_1(z_2)f''_2(z_2)}{f'_2(z_2)} \right| \right) \leq p_2 \left(1 - \left| \frac{z_2 f''_2(z_2)}{f'_2(z_2)} \right| \right) |z_2|^{p_2-2};$$

$$(2) \quad \frac{p_1}{|f'_1(z_1)|} \left(\left| \frac{g'_1(z_2)g''_2(z_2)}{f'_2(z_2)} \right| + \left| \frac{g'_2(z_2)}{f'_2(z_2)} \frac{f''_n(z_2)}{f'_n(z_2)} g'_1(z_2) \right| \right) \\ + \sum_{j=2}^{n-1} \frac{p_j}{|f'_j(z_j)|} \left(|g''_j(z_n)| + \left| \frac{g'_j(z_n)f''_n(z_n)}{f'_n(z_n)} \right| \right) \leq p_n \left(1 - \left| \frac{z_n f''_n(z_n)}{f'_n(z_n)} \right| \right) |z_n|^{p_n-2},$$

for all $z = (z_1, \dots, z_n) \in D_p \setminus \{0\}$, then $f \in K(D_p)$.

Next we give some concrete examples for biholomorphic mappings on D_p by applying these sufficient conditions.

Example 2.1 Suppose that $p_j \geq p_1 \geq 2(j = 2, \dots, n)$, and k_1, k_j are positive integers such that $k_1 \geq 1, k_j < p_j \leq k_j + 1(j = 2, 3, \dots, n), k = \max_{1 \leq j \leq n-1} \{k_j\}$. If

$$|a| \leq \left[(k+1) \sum_{j=1}^{n-1} (k_j+1) \right]^{-1}$$

and

$$|b| \leq \frac{p_n \sum_{j=1}^{n-1} (k_j+1) - p_n}{[p_2 \sum_{j=1}^{n-1} (k_j+1) + p_1 - p_2] k_n (k_n+1)},$$

then

$$f(z) = \left(z_1 + a z_1^{k_1+1} z_2^{k_2+1} \dots z_{n-1}^{k_{n-1}+1}, z_2 + b z_n^{k_n+1}, z_3, \dots, z_n \right) \in K(D_p).$$

Proof. Set $f_1(z) = z_1 + a z_1^{k_1+1} z_2^{k_2+1} \dots z_{n-1}^{k_{n-1}+1}, f_j(z_j) = z_j(j = 2, \dots, n), g_2(z_n) = b z_n^{k_n+1}$. Then

$$\sum_{l=1}^n \left| \frac{\partial^2 f_1}{\partial z_1 \partial z_l} \right| \leq (k_1+1) |a| \left[\sum_{l=2}^{n-1} (k_l+1) |z_1|^{k_1} + k_1 |z_1|^{k_1-1} \right] \\ \leq (k+1) |a| \left[\sum_{l=1}^{n-1} (k_l+1) - 1 \right] |z_1|^{k_1-1} \\ \leq 1 - (k+1) |a|, \\ \left| \frac{\partial f_1}{\partial z_1} \right| = |1 + a(k_1+1) z_1^{k_1} z_2^{k_2+1} \dots z_{n-1}^{k_{n-1}+1}| \\ \geq 1 - |a|(k_1+1) \geq 1 - |a|(k+1); \\ \frac{p_1}{\left| \frac{\partial f_1}{\partial z_1} \right|} \sum_{l=1}^n \left| \frac{\partial^2 f_1}{\partial z_j \partial z_l} \right| \leq \frac{p_1}{1 - |a|(k+1)} (k_j+1) |a| \left[\sum_{l \neq j} (k_l+1) |z_j|^{k_j} + k_j |z_j|^{k_j-1} \right] \\ \leq \frac{p_1}{1 - |a|(k+1)} (k+1) |a| \left[\sum_{l=1}^{n-1} (k_l+1) - 1 \right] |z_j|^{k_j-1}$$

$$\begin{aligned}
&\leq \frac{p_1}{1 - |a|(k+1)} [1 - (k+1)|a|] |z_j|^{p_j-2} |z_j|^{k_j+1-p_j} \\
&\leq p_1 |z_j|^{p_j-2} \\
&\leq p_j |z_j|^{p_j-2}, \quad j = 2, 3, \dots, n-1; \\
\frac{p_1}{\left| \frac{\partial f_1}{\partial z_1} \right|} \left| \frac{\partial f_1}{\partial z_2} \frac{g_2''(z_n)}{f_2'(z_2)} \right| &+ \frac{p_2}{|f_2'(z_2)|} |g_2''(z_n)| \\
&\leq \left[\frac{p_1}{1 - |a|(k+1)} |a|(k_2+1) |b|(k_n+1) k_n + p_2 |b|(k_n+1) k_n \right] |z_n|^{k_n-1} \\
&\leq |b|(k_n+1) k_n \left[\frac{p_1 |a|(k_2+1)}{1 - |a|(k+1)} + p_2 \right] |z_n|^{p_n-2} |z_n|^{k_n-p_n+1} \\
&\leq p_n |z_n|^{p_n-2}.
\end{aligned}$$

By Theorem 2.1, we obtain that $f(z) = (z_1 + a z_1^{k_1+1} z_2^{k_2+1} \dots z_{n-1}^{k_{n-1}+1}, z_2 + b z_n^{k_n+1}, z_3, \dots, z_n) \in K(D_p)$. This completes the proof.

Example 2.2 Suppose that $n \geq 3$, $p_j \geq 2$, $0 < |a_j| \leq \sqrt{3}/3$ ($j = 1, 2, \dots, n$), and k_1, k_2, k are positive integers such that $k_1 \geq 1$, $k_2 < p_2 \leq k_2 + 1$, and $k < p_n \leq k + 1$. Let

$$\begin{aligned}
f(z) = &(z_1 + a_1 z_1^{k_1+1} z_2^{k_2+1}, \frac{1}{2a_2} \log \frac{1 + a_2 z_2}{1 - a_2 z_2} + b_2 z_n^{k+1}, \dots, \\
&\frac{1}{2a_{n-1}} \log \frac{1 + a_{n-1} z_{n-1}}{1 - a_{n-1} z_{n-1}} + b_{n-1} z_n^{k+1}, \frac{1}{2a_n} \log \frac{1 + a_n z_n}{1 - a_n z_n}),
\end{aligned}$$

with $\log 1 = 0$. If

$$|a_1| \leq \frac{1}{(k_1+1)(k_1+k_2+2)},$$

and

$$\frac{p_1(k_2+1)|a_1||b_2|(1+|a_2|^2)}{1 - (k_1+1)|a_1|} + \sum_{j=2}^{n-1} p_j(1+|a_j|^2)|b_j| \leq \frac{p_n(1-3|a_n|^2)}{(k+1)[k+(2-k)|a_n|^2]}.$$

Then $f(z) \in K(D_p)$.

Example 2.3 Suppose that $p_j \geq 2$ ($j = 1, 2, \dots, n$), and k is a positive integer such that $k < p_n \leq k + 1$. If

$$\sum_{j=2}^{n-1} |b_j| \leq \frac{p_n}{k(k+1) \max_{2 \leq j \leq n-1} \{p_j\}}$$

for all $z = (z_1, \dots, z_n) \in D_p \setminus \{0\}$, then

$$f(z) = (z_1, z_2 + b_2 z_n^{k+1}, z_3 + b_3 z_n^{k+1}, \dots, z_{n-1} + b_{n-1} z_n^{k+1}, z_n) \in K(D_p).$$

Example 2.4 Suppose that $p_j \geq 2$ ($j = 1, 2, \dots, n$), and m, k are positive integers such that $m < p_2 \leq m + 1$ and $k < p_n \leq k + 1$. If

$$|a| \leq \frac{p_2}{p_1 m(m+1)}, |b| \leq \frac{p_n m}{k(k+1)(p_1 m + p_2 + p_2 m)},$$

then

$$f(z) = (z_1 + z_2 + az_2^{m+1}, z_2 + bz_n^{k+1}, z_3, \dots, z_n) \in K(D_p).$$

Finally we finish this section with an analogous theorem with Theorem 1.1 and a concrete example.

Theorem 2.2 *Suppose that $n \geq 3, p_j \geq 2$ and $g_j : \Delta \rightarrow C$ are analytic on Δ with $g_j(0) = g'_j(0) = 0 (j = 1, 2, \dots, n-1)$, $f_j(\zeta) \in H(\Delta)$ satisfy the conditions $f'_j(\zeta) \neq 0$, and $|\zeta f''_j(\zeta)| \leq |f'_j(\zeta)|, (\zeta \in \Delta, j = 1, 2, \dots, n)$. Let*

$$f(z) = (f_1(z_1) + g_1(z_{n-1}), \dots, f_{n-2}(z_{n-2}) + g_{n-2}(z_{n-1}), f_{n-1}(z_{n-1}) + g_{n-1}(z_n), f_n(z_n)).$$

If for any $z = (z_1, \dots, z_n) \in D_p^n$, we have

$$(1) \quad \sum_{j=1}^{n-2} p_j \left(\left| \frac{g''_j(z_{n-1})}{f'_j(z_j)} \right| + \left| \frac{g'_j(z_{n-1})f''_{n-1}(z_{n-1})}{f'_j(z_j)f'_{n-1}(z_{n-1})} \right| \right) \leq p_{n-1} |z_{n-1}|^{p_{n-1}-2} \left(1 - \left| \frac{z_{n-1}f''_{n-1}(z_{n-1})}{f'_{n-1}(z_{n-1})} \right| \right);$$

$$(2) \quad p_{n-1} \left(\left| \frac{g''_{n-1}(z_n)}{f'_{n-1}(z_{n-1})} \right| + \left| \frac{g'_{n-1}(z_n)f''_n(z_n)}{f'_{n-1}(z_{n-1})f'_n(z_n)} \right| \right) + \sum_{j=1}^{n-2} p_j \left(\left| \frac{g'_j(z_{n-1})g''_{n-1}(z_n)}{f'_j(z_j)f'_{n-1}(z_{n-1})} \right| \right. \\ \left. + \left| \frac{g'_j(z_{n-1})g'_{n-1}(z_n)f''_n(z_n)}{f'_j(z_j)f'_{n-1}(z_{n-1})f'_n(z_n)} \right| \right) \leq p_n |z_n|^{p_n-2} \left(1 - \left| \frac{z_n f''_n(z_n)}{f'_n(z_n)} \right| \right).$$

Then $f \in K(D_p^n)$.

Proof. By directly calculating the Fréchet derivatives of $f(z)$, we obtain

$$Df(z) = \begin{pmatrix} f'_1(z_1) & 0 & \cdots & 0 & g'_1(z_{n-1}) & 0 \\ 0 & f'_2(z_2) & \cdots & 0 & g'_2(z_{n-1}) & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & f'_{n-2}(z_{n-2}) & g'_{n-2}(z_{n-1}) & 0 \\ 0 & 0 & \cdots & 0 & f'_{n-1}(z_{n-1}) & g'_{n-1}(z_n) \\ 0 & 0 & \cdots & 0 & 0 & f'_n(z_n) \end{pmatrix},$$

$$Df(z)^{-1} = \begin{pmatrix} \frac{1}{f'_1(z_1)} & 0 & \cdots & 0 & -\frac{g'_1(z_{n-1})}{f'_1(z_1)f'_{n-1}(z_{n-1})} & \frac{g'_1(z_{n-1})g'_{n-1}(z_n)}{f'_1(z_1)f'_{n-1}(z_{n-1})f'_n(z_n)} \\ 0 & \frac{1}{f'_2(z_2)} & \cdots & 0 & -\frac{g'_2(z_{n-1})}{f'_2(z_2)f'_{n-1}(z_{n-1})} & \frac{g'_2(z_{n-1})g'_{n-1}(z_n)}{f'_2(z_2)f'_{n-1}(z_{n-1})f'_n(z_n)} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & \frac{1}{f'_{n-2}(z_{n-2})} & -\frac{g'_{n-2}(z_{n-1})}{f'_{n-2}(z_{n-2})f'_{n-1}(z_{n-1})} & \frac{g'_{n-2}(z_{n-1})g'_{n-1}(z_n)}{f'_{n-2}(z_{n-2})f'_{n-1}(z_{n-1})f'_n(z_n)} \\ 0 & 0 & \cdots & 0 & \frac{1}{f'_{n-1}(z_{n-1})} & -\frac{g'_{n-1}(z_n)}{f'_{n-1}(z_{n-1})f'_n(z_n)} \\ 0 & 0 & \cdots & 0 & 0 & \frac{1}{f'_n(z_n)} \end{pmatrix},$$

$$\begin{aligned}
D^2 f(z)(b, b) &= \begin{pmatrix} f_1''(z_1)b_1 & 0 & \cdots & 0 & g_1''(z_{n-1})b_{n-1} & 0 \\ 0 & f_2''(z_2)b_2 & \cdots & 0 & g_2''(z_{n-1})b_{n-1} & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & f_{n-2}''(z_{n-2})b_{n-2} & g_{n-2}''(z_{n-1})b_{n-1} & 0 \\ 0 & 0 & \cdots & 0 & f_{n-1}''(z_{n-1})b_{n-1} & g_{n-1}''(z_n)b_n \\ 0 & 0 & \cdots & 0 & 0 & f_n''(z_n)b_n \end{pmatrix} \begin{pmatrix} b_1 \\ b_2 \\ \cdots \\ b_{n-2} \\ b_{n-1} \\ b_n \end{pmatrix} \\
&= \begin{pmatrix} f_1''(z_1)b_1^2 + g_1''(z_{n-1})b_{n-1}^2 \\ f_2''(z_2)b_2^2 + g_2''(z_{n-1})b_{n-1}^2 \\ \cdots \\ f_{n-2}''(z_{n-2})b_{n-2}^2 + g_{n-2}''(z_{n-1})b_{n-1}^2 \\ f_{n-1}''(z_{n-1})b_{n-1}^2 + g_{n-1}''(z_n)b_n^2 \\ f_n''(z_n)b_n^2 \end{pmatrix}.
\end{aligned}$$

Taking $z = (z_1, z_2, \dots, z_n) \in D_p \setminus \{0\}$ and $b = (b_1, b_2, \dots, b_n) \in C^n \setminus \{0\}$ such that

$$\operatorname{Re} \left\{ \sum_{j=1}^n p_j \left| \frac{z_j}{\rho(z)} \right|^{p_j} \frac{b_j}{z_j} \right\} = 0.$$

From (1.1), we have $|z_j/\rho(z)| \leq 1 (j = 1, 2, \dots, n)$ for all $z = (z_1, z_2, \dots, z_n) \in D_p$. Notice that $p_j \geq 2 (j = 1, 2, \dots, n)$, by the hypotheses of Theorem 1, we obtain

$$\begin{aligned}
J_f(z, b) &\geq \sum_{j=1}^n \frac{p_j |z_j|^{p_j-2}}{\rho(z)^{p_j}} |b_j|^2 - 2 \sum_{j=1}^n \frac{p_j}{\rho(z)} \left| \frac{z_j}{\rho(z)} \right|^{p_j} \operatorname{Re} \left\langle Df(z)^{-1} D^2 f(z)(b, b), \frac{\partial \rho}{\partial \bar{z}} \right\rangle \\
&= \sum_{j=1}^n \frac{p_j |z_j|^{p_j-2}}{\rho(z)^{p_j}} |b_j|^2 - \operatorname{Re} \left[\sum_{j=1}^{n-2} \left(\frac{f_j''(z_j)}{f_j'(z_j)} b_j^2 + \frac{g_j''(z_{n-1})}{f_j'(z_j)} b_{n-1}^2 - \frac{g_j'(z_{n-1}) f_{n-1}''(z_{n-1})}{f_j'(z_j) f_{n-1}'(z_{n-1})} b_{n-1}^2 \right. \right. \\
&\quad \left. \left. - \frac{g_j'(z_{n-1}) g_{n-1}''(z_n)}{f_j'(z_j) f_{n-1}'(z_{n-1})} b_n^2 + \frac{g_j'(z_{n-1}) g_{n-1}'(z_n) f_n''(z_n)}{f_j'(z_j) f_{n-1}'(z_{n-1}) f_n'(z_n)} b_n^2 \right) \frac{p_j |z_j|^{p_j}}{z_j \rho(z)^{p_j}} \right. \\
&\quad \left. + \left(\frac{f_{n-1}''(z_{n-1})}{f_{n-1}'(z_{n-1})} b_{n-1}^2 + \frac{g_{n-1}''(z_n)}{f_{n-1}'(z_{n-1})} b_n^2 - \frac{g_{n-1}'(z_n) f_n''(z_n)}{f_{n-1}'(z_{n-1}) f_n'(z_n)} b_n^2 \right) \right. \\
&\quad \left. \frac{p_{n-1} |z_{n-1}|^{p_{n-1}}}{z_{n-1} \rho(z)^{p_{n-1}}} + \left(\frac{f_n''(z_n)}{f_n'(z_n)} b_n^2 \right) \frac{p_n |z_n|^{p_n}}{z_n \rho(z)^{p_n}} \right] \\
&\geq \sum_{j=1}^{n-2} \frac{p_j |z_j|^{p_j-2}}{\rho(z)^{p_j}} |b_j|^2 + \frac{p_{n-1} |z_{n-1}|^{p_{n-1}-2}}{\rho(z)^{p_{n-1}}} |b_{n-1}|^2 + \frac{p_n |z_n|^{p_n-2}}{\rho(z)^{p_n}} |b_n|^2 \\
&\quad - \sum_{j=1}^{n-2} \left[\left| \frac{f_j''(z_j)}{f_j'(z_j)} \right| |b_j|^2 + \left| \frac{g_j''(z_{n-1})}{f_j'(z_j)} \right| |b_{n-1}|^2 + \left| \frac{g_j'(z_{n-1}) f_{n-1}''(z_{n-1})}{f_j'(z_j) f_{n-1}'(z_{n-1})} \right| |b_{n-1}|^2 \right. \\
&\quad \left. + \left| \frac{g_j'(z_{n-1}) g_{n-1}''(z_n)}{f_j'(z_j) f_{n-1}'(z_{n-1})} \right| |b_n|^2 + \left| \frac{g_j'(z_{n-1}) g_{n-1}'(z_n) f_n''(z_n)}{f_j'(z_j) f_{n-1}'(z_{n-1}) f_n'(z_n)} \right| |b_n|^2 \right] \frac{p_j |z_j|^{p_j-1}}{\rho(z)^{p_j}} \\
&\quad - \left[\left| \frac{f_{n-1}''(z_{n-1})}{f_{n-1}'(z_{n-1})} \right| |b_{n-1}|^2 + \left| \frac{g_{n-1}''(z_n)}{f_{n-1}'(z_{n-1})} \right| |b_n|^2 + \left| \frac{g_{n-1}'(z_n) f_n''(z_n)}{f_{n-1}'(z_{n-1}) f_n'(z_n)} \right| |b_n|^2 \right] \\
&\quad \cdot \frac{p_{n-1} |z_{n-1}|^{p_{n-1}-1}}{\rho(z)^{p_{n-1}}} - \left| \frac{f_n''(z_n)}{f_n'(z_n)} \right| |b_n|^2 \frac{p_n |z_n|^{p_n-1}}{\rho(z)^{p_n}}
\end{aligned}$$

$$\begin{aligned}
&= \sum_{j=1}^{n-2} \frac{p_j |z_j|^{p_j-2}}{\rho(z)^{p_j}} |b_j|^2 \left(1 - \left| \frac{z_j f_j''(z_j)}{f_j'(z_j)} \right| \right) \\
&\quad + |b_{n-1}|^2 \left[\frac{p_{n-1} |z_{n-1}|^{p_{n-1}-2}}{\rho(z)^{p_{n-1}}} \left(1 - \left| \frac{z_{n-1} f_{n-1}''(z_{n-1})}{f_{n-1}'(z_{n-1})} \right| \right) \right. \\
&\quad \left. - \sum_{j=1}^{n-2} \frac{p_j |z_j|^{p_j-1}}{\rho(z)^{p_j}} \left(\left| \frac{g_j''(z_{n-1})}{f_j'(z_j)} \right| + \left| \frac{g_j'(z_{n-1}) f_{n-1}''(z_{n-1})}{f_j'(z_j) f_{n-1}'(z_{n-1})} \right| \right) \right] \\
&\quad + |b_n|^2 \left[\frac{p_n |z_n|^{p_n-2}}{\rho(z)^{p_n}} \left(1 - \left| \frac{z_n f_n''(z_n)}{f_n'(z_n)} \right| \right) \right. \\
&\quad \left. - \frac{p_{n-1} |z_{n-1}|^{p_{n-1}-1}}{\rho(z)^{p_{n-1}}} \left(\left| \frac{g_{n-1}''(z_n)}{f_{n-1}'(z_{n-1})} \right| + \left| \frac{g_{n-1}'(z_n) f_n''(z_n)}{f_{n-1}'(z_{n-1}) f_n'(z_n)} \right| \right) \right. \\
&\quad \left. - \sum_{j=1}^{n-2} \frac{p_j |z_j|^{p_j-1}}{\rho(z)^{p_j}} \left(\left| \frac{g_j'(z_{n-1}) g_{n-1}''(z_n)}{f_j'(z_j) f_{n-1}'(z_{n-1})} \right| + \left| \frac{g_j'(z_{n-1}) g_{n-1}'(z_n) f_n''(z_n)}{f_j'(z_j) f_{n-1}'(z_{n-1}) f_n'(z_n)} \right| \right) \right] \\
&\geq \frac{|b_{n-1}|^2}{\rho(z)^{p_{n-1}}} \left[p_{n-1} |z_{n-1}|^{p_{n-1}-2} \left(1 - \left| \frac{z_{n-1} f_{n-1}''(z_{n-1})}{f_{n-1}'(z_{n-1})} \right| \right) - \sum_{j=1}^{n-2} p_j \left(\left| \frac{g_j''(z_{n-1})}{f_j'(z_j)} \right| \right. \right. \\
&\quad \left. \left. + \left| \frac{g_j'(z_{n-1}) f_{n-1}''(z_{n-1})}{f_j'(z_j) f_{n-1}'(z_{n-1})} \right| \right) \right] + \frac{|b_n|^2}{\rho(z)^{p_n}} \left[p_n |z_n|^{p_n-2} \left(1 - \left| \frac{z_n f_n''(z_n)}{f_n'(z_n)} \right| \right) \right. \\
&\quad \left. - p_{n-1} \left(\left| \frac{g_{n-1}''(z_n)}{f_{n-1}'(z_{n-1})} \right| + \left| \frac{g_{n-1}'(z_n) f_n''(z_n)}{f_{n-1}'(z_{n-1}) f_n'(z_n)} \right| \right) \right. \\
&\quad \left. - \sum_{j=1}^{n-2} p_j \left(\left| \frac{g_j'(z_{n-1}) g_{n-1}''(z_n)}{f_j'(z_j) f_{n-1}'(z_{n-1})} \right| + \left| \frac{g_j'(z_{n-1}) g_{n-1}'(z_n) f_n''(z_n)}{f_j'(z_j) f_{n-1}'(z_{n-1}) f_n'(z_n)} \right| \right) \right] \\
&\geq 0.
\end{aligned}$$

Hence by Lemma 1.1, we obtain $f \in K(D_p)$, and the proof is complete.

Example 2.5 Suppose that $p_j \geq 2$, and $k_j (j = 1, 2, \dots, n-2)$ are positive integer such that $k_j \geq p_{n-1} - 1$, let

$$f(z) = (z_1 + a_1' z_1^2 + a_1 z_{n-1}^{k_1+1} + b_1 z_{n-1}^{k_1+2}, \dots, z_{n-2} + a_{n-2}' z_{n-2}^2 + a_{n-2} z_{n-1}^{k_{n-2}+1} + b_{n-2} z_{n-1}^{k_{n-2}+2}, z_{n-1}, z_n + a_n' z_n^2),$$

If $c = \max\{|a_j'| : j = 1, 2, \dots, n\} \leq \frac{1}{4}$ and $\sum_{j=1}^{n-2} p_j (k_j + 1) [k_j |a_j| + (k_j + 2) |b_j|] \leq p_{n-1} (1 - 2c)$, then $f(z) \in K(D_p)$.

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