ON CENTRALIZERS OF BANACH ALGEBRAS

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Abstract. Let $\mathcal{A}$ be a unital Banach algebra and $\mathcal{M}$ be a unital Banach $\mathcal{A}$-bimodule. The main results characterize a continuous linear map $\varphi : \mathcal{A} \to \mathcal{M}$ that satisfies $a\varphi(a^{-1}) = \varphi(1)$ or $a\varphi(a^{-1}) + \varphi(a^{-1})a = 2\varphi(1)$ for all a in principal component of invertible elements of $\mathcal{A}$. The proof is based on the consideration of a continuous bilinear map satisfying a related condition.

1. Introduction

Throughout this paper all algebras and vector spaces will be over the complex field $\mathbb{C}$. Let $\mathcal{A}$ be an algebra and $\mathcal{M}$ be an $\mathcal{A}$-bimodule. Recall that a linear (additive) map $\varphi : \mathcal{A} \to \mathcal{M}$ is said to be a *right (left) centralizer* if $\varphi(ab) = a\varphi(b)$ ($\varphi(ab) = \varphi(a)b$) for each $a, b \in \mathcal{A}$. It is called a *centralizer* if φ is both a left centralizer and a right centralizer. Also φ is called *right (left) Jordan centralizer* if $\varphi(a^2) = a\varphi(a)$ ($\varphi(a^2) = \varphi(a)a$) for all $a \in \mathcal{A}$. We say that φ is a *Jordan centralizer* if $\varphi(ab + ba) = a\varphi(b) + \varphi(b)a = b\varphi(a) + \varphi(a)b$ for all $a, b \in \mathcal{A}$. In case $\mathcal{A}$ has a unity 1 and $\mathcal{M}$ is aunital $\mathcal{A}$-bimodule, $\varphi : \mathcal{A} \to \mathcal{M}$ is a right (left) centralizer if and only if φ is of the form $\varphi(a) = a\varphi(1)$ ($\varphi(a) = \varphi(1)a$) for all $a \in \mathcal{A}$. Also φ is a centralizer if and only if $\varphi(a) = a\varphi(1) = \varphi(1)a$ for each $a \in \mathcal{A}$.

Clearly, each (right, left) centralizer is a (right, left) Jordan centralizer. The converse is, in general, not true (see Example 2.6). It is natural and interesting to find some conditions under which a (right, left) Jordan centralizer is a (right, left) centralizer. Zalar has proved in [20] that any right (left) Jordan centralizer on a 2-torsion free semiprime ring is a right (left) centralizer. Vukman [14] has showed that an additive map $\varphi : \mathcal{R} \to \mathcal{R}$, where $\mathcal{R}$ is a 2-torsion free semiprime ring, with the property that $2\varphi(a^2) = a\varphi(a) + \varphi(a)a$ for all $a \in \mathcal{A}$, is a centralizer. We refer the reader to [5, 8, 11, 15, 16] and references therein for results concerning centralizers on rings and algebras.

In recent years, several authors studied the linear (additive) maps that behave like homomorphisms, derivations or right (left) centalizers when acting on special products (for instance, see [6, 9, 10, 12, 19] and the references therein). The question of characterizing these linear (additive) maps can be sometimes effectively solved by considering bilinear maps that preserve certain product properties (for instance, see [1, 2, 3, 4, 7, 17, 18]).

In this article we study the (right, left) centralizers and Jordan centralizers on Banach algebras through identity products and identity Jordan products, respectively, by consideration of bilinear maps satisfying a related condition.

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Let $\mathcal{A}$ be a Banach algebra with unity 1 and $\mathcal{M}$ be a unital Banach $\mathcal{A}$-bimodule. Here and subsequently, $'\circ'$ denotes the Jordan product $a \circ b = ab + ba$ on $\mathcal{A}$ and $'\bullet'$ denotes the Jordan product on $\mathcal{M}$:

$$a \bullet m = m \bullet a = am + ma, \quad a \in \mathcal{A}, m \in \mathcal{M}.$$

Denote by $Inv(\mathcal{A})$ the set of invertible elements of $\mathcal{A}$. $Inv(\mathcal{A})$ is an open subset of $\mathcal{A}$ and hence it is a disjoint union of open connected subsets, the components of $Inv(\mathcal{A})$. The component containing 1 is called the *principal component* of $Inv(\mathcal{A})$ and it is denoted by $Inv_0(\mathcal{A})$. We denote by $exp(\mathcal{A})$ the range of the exponential function in $\mathcal{A}$, i.e.

$$exp(\mathcal{A}) = \{exp(a) \mid a \in \mathcal{A}\}$$

and we have $exp(\mathcal{A}) \subseteq Inv_0(\mathcal{A})$.

In this paper we characterize the continuous linear maps $\varphi : \mathcal{A} \to \mathcal{M}$ satisfying

$$a \in Inv_0(\mathcal{A}) \Rightarrow a\varphi(a^{-1}) = \varphi(1) \quad (I1)$$

or

$$a \in Inv_0(\mathcal{A}) \Rightarrow a \bullet \varphi(a^{-1}) = 2\varphi(1) \quad (I2).$$

Obviously, if $\varphi : \mathcal{A} \to \mathcal{M}$ satisfying

$$a, b \in \mathcal{A}, \quad ab = 1 \Rightarrow a\varphi(b) = \varphi(1)$$

or

$$a, b \in \mathcal{A}, \quad a \circ b = 1 \Rightarrow a \bullet \varphi(b) = \varphi(1),$$

then φ satisfies $(I1)$ or $(I2)$, respectively. For characterization continuous linear maps satisfying $(I1)$ or $(I2)$, we first study continuous bilinear maps $\phi : \mathcal{A} \times \mathcal{A} \to \mathcal{X}$ into some Banach space $\mathcal{X}$ with the property that

$$a \in Inv_0(\mathcal{A}) \Rightarrow \phi(a, a^{-1}) = \phi(1, 1).$$

Finally, we provide some classes of Banach algebras $\mathcal{A}$ and Banach $\mathcal{A}$-bimodules $\mathcal{M}$ such that continuous linear maps $\varphi : \mathcal{A} \to \mathcal{M}$ satisfying $(I2)$ are centralizers.

2. Bilinear maps

From this point up to the last section $\mathcal{A}$ is a Banach algebra with unity 1.

Theorem 2.1. *Let $\mathcal{X}$ be a Banach space and let $\phi : \mathcal{A} \times \mathcal{A} \to \mathcal{X}$ be a continuous bilinear map with the property that*

$$a \in Inv_0(\mathcal{A}) \Rightarrow \phi(a, a^{-1}) = \phi(1, 1).$$

Then

$$\phi(a, a) = \phi(a^2, 1) \quad \textit{and} \quad \phi(a, 1) = \phi(1, a) \quad a, b \in \mathcal{A}$$

and there exists a continuous linear map $T : \mathcal{A} \to \mathcal{X}$ such that

$$\phi(a, b) + \phi(b, a) = T(a \circ b), \quad a, b \in \mathcal{A}.$$

Proof. Let a be in $\mathcal{A}$. For each scalar $\lambda \in \mathbb{C}$, we have $\phi(exp(\lambda a), exp(-\lambda a)) = \phi(1,1)$, since $exp(\mathcal{A}) \subseteq Inv_0(\mathcal{A})$. Thus

$$\begin{aligned}
\phi(1,1) &= \phi(exp(\lambda a), exp(-\lambda a)) \\
&= \phi(exp(\lambda a), \Sigma_{m=0}^{\infty} \frac{(-1)^m \lambda^m a^m}{m!}) \\
&= \Sigma_{m=0}^{\infty} \frac{(-1)^m \lambda^m}{m!} \phi(exp(\lambda a), a^m) \\
&= \Sigma_{m=0}^{\infty} \frac{(-1)^m \lambda^m}{m!} \phi(\Sigma_{n=0}^{\infty} \frac{\lambda^n a^n}{n!}, a^m) \\
&= \Sigma_{m=0}^{\infty} \Sigma_{n=0}^{\infty} \frac{(-1)^m \lambda^{m+n}}{m!n!} \phi(a^n, a^m) \\
&= \phi(1,1) + \Sigma_{k=1}^{\infty} \lambda^k (\sum_{m+n=k} \frac{(-1)^m}{m!n!} \phi(a^n, a^m)),
\end{aligned}$$

since ϕ is a continuous bilinear map. Therefore $\Sigma_{k=1}^{\infty} \lambda^k (\sum_{m+n=k} \frac{(-1)^m}{m!n!} \phi(a^n, a^m)) = 0$ for any $\lambda \in \mathbb{C}$. Consequently,

$$\sum_{m+n=k} \frac{(-1)^m}{m!n!} \phi(a^n, a^m) = 0 \tag{2.1}$$

for all $a \in \mathcal{A}$ and $k \in \mathbb{N}$. Let $k = 1$, we find that $\phi(a,1) - \phi(1,a) = 0$ and hence

$$\phi(a,1) = \phi(1,a) \tag{2.2}$$

for all $a \in \mathcal{A}$. Now taking $k = 2$ in (2.1), we obtain $\frac{1}{2}\phi(a^2,1) - \phi(a,a) + \frac{1}{2}\phi(1,a^2) = 0$ for any $a \in \mathcal{A}$. So by (2.2) we have

$$\phi(a,a) = \phi(a^2,1), \quad a \in \mathcal{A}. \tag{2.3}$$

For any $a, b \in \mathcal{A}$, replacing a by $a + b$ in (2.3), we get that

$$\phi(a,b) + \phi(b,a) = \phi(ab + ba, 1).$$

If we define the linear map $T : \mathcal{A} \to \mathcal{X}$ by $T(a) = \phi(a,1)$, then T is continuous and

$$\phi(a,b) + \phi(b,a) = T(a \circ b)$$

for all $a, b \in \mathcal{A}$. □

Corollary 2.2. *Let $\mathcal{X}$ be a Banach space and let $\phi : \mathcal{A} \times \mathcal{A} \to \mathcal{X}$ be a continuous bilinear map. If ϕ satisfies any of the following conditions;*

(i) $a \in Inv(\mathcal{A}) \Rightarrow \phi(a, a^{-1}) = \phi(1,1)$,
(ii) $a, b \in \mathcal{A}$, $ab = 1 \Rightarrow \phi(a,b) = \phi(1,1)$,
(iii) $a, b \in \mathcal{A}$, $a \circ b = 1 \Rightarrow \phi(a,b) = \frac{1}{2}\phi(1,1)$,

then

$$\phi(a,b) + \phi(b,a) = \phi(a \circ b, 1)$$

for all $a, b \in \mathcal{A}$.

Proof. In the cases (i) and (ii) the result is clear from Theorem 2.1. Let the condition (iii) holds. For each $a \in Inv(\mathcal{A})$, we have $(\frac{1}{2}a) \circ a^{-1} = 1$. So $\phi(a, a^{-1}) = \phi(1,1)$ and hence by Theorem 2.1, the result is true in this case, too. □

Recall that a bilinear map $\phi : \mathcal{A} \times \mathcal{A} \to \mathcal{X}$ is called *symmetric* if $\phi(a,b) = \phi(b,a)$ holds for all $a, b \in \mathcal{A}$.

By Theorem 2.1 and Corollary 2.2, the following corollary is obvious.

Corollary 2.3. *Let $\mathcal{X}$ be a Banach space and let $\phi : \mathcal{A} \times \mathcal{A} \to \mathcal{X}$ be a continuous symmetric bilinear map. Then the following conditions are equivalent:*

(i) $a \in Inv_0(\mathcal{A}) \Rightarrow \phi(a, a^{-1}) = \phi(1, 1)$;
(ii) $a \in Inv(\mathcal{A}) \Rightarrow \phi(a, a^{-1}) = \phi(1, 1)$;
(iii) $a, b \in \mathcal{A}$, $a \circ b = 1 \Rightarrow \phi(a, b) = \frac{1}{2}\phi(1, 1)$;
(iv) $\phi(a, b) = \frac{1}{2}\phi(a \circ b, 1)$, $a, b \in \mathcal{A}$.

The Theorems 2.4 and 2.5 are our main results.

Theorem 2.4. *Let $\mathcal{M}$ be a unital Banach $\mathcal{A}$-bimodule and $\varphi : \mathcal{A} \to \mathcal{M}$ be a continuous linear map. Then the following conditions are equivalent:*

(i) $\varphi(a) = a\varphi(1)$ *for all* $a \in \mathcal{A}$, *i.e.* φ *is a right centralizer;*
(ii) $a, b \in \mathcal{A}$, $ab = 1 \Rightarrow a\varphi(b) = \varphi(1)$;
(iii) $a \in Inv(\mathcal{A}) \Rightarrow a\varphi(a^{-1}) = \varphi(1)$;
(iv) $a \in Inv_0(\mathcal{A}) \Rightarrow a\varphi(a^{-1}) = \varphi(1)$.

Proof. $(i) \Rightarrow (ii)$, $(ii) \Rightarrow (iii)$ and $(iii) \Rightarrow (iv)$ are clear.

$(iv) \Rightarrow (i)$ Define a continuous bilinear map $\phi : \mathcal{A} \times \mathcal{A} \to \mathcal{M}$ by $\phi(a, b) = a\varphi(b)$. Then $\phi(a, a^{-1}) = \phi(1, 1)$ for all $a \in Inv_0(\mathcal{A})$. By applying Theorem 2.1, we obtain $\phi(1, a) = \phi(a, 1)$ for all $a \in \mathcal{A}$. So

$$\varphi(a) = a\varphi(1)$$

all $a \in \mathcal{A}$. □

Left centralizer analogs of Theorem 2.4 can be obtained with the same argument.

Theorem 2.5. *Let $\mathcal{M}$ be a unital Banach $\mathcal{A}$-bimodule and $\varphi : \mathcal{A} \to \mathcal{M}$ be a continuous linear map with the property that*

$$a \in Inv_0(\mathcal{A}) \Rightarrow a \bullet \varphi(a^{-1}) = 2\varphi(1).$$

Then

$$2\varphi(a^2) = a \bullet \varphi(a) \quad a \in \mathcal{A}.$$

Proof. Define a continuous bilinear map $\phi : \mathcal{A} \times \mathcal{A} \to \mathcal{M}$ by $\phi(a, b) = a \bullet \varphi(b)$. Then $\phi(a, a^{-1}) = \phi(1, 1)$ for all $a \in Inv_0(\mathcal{A})$. By applying Theorem 2.1, we obtain $\phi(a, b) + \phi(b, a) = \phi(a \circ b, 1) = \phi(1, a \circ b)$ for all $a, b \in \mathcal{A}$. So

$$a \bullet \varphi(b) + b \bullet \varphi(a) = 2\varphi(a \circ b), \quad a \in \mathcal{A}.$$

Letting $b = a$, we obtain

$$2\varphi(a^2) = a \bullet \varphi(a)$$

for all $a \in \mathcal{A}$. □

In this theorem if φ satisfies

$$a, b \in \mathcal{A}, \quad a \circ b = 1 \Rightarrow a \bullet \varphi(b) = \varphi(1),$$

then it is obvious that $a \bullet \varphi(a^{-1}) = 2\varphi(1)$ for all $a \in Inv(\mathcal{A})$. Hence the theorem holds in this case, too.

Let us mention an example of a Banach algebra where identity $a \bullet \varphi(b) = \varphi(1)$ for all $a, b \in \mathcal{A}$ with $a \circ b = 1$ does not imply that φ is a centralizer.

Example 2.6. Consider an algebra of the form

$$\mathcal{A} = \left\{ \begin{pmatrix} a & b & c \\ 0 & a & d \\ 0 & 0 & a \end{pmatrix} \,\middle|\, a, b, c, d \in \mathbb{C} \right\}$$

under the usual matrix operations. Then the algebra $\mathcal{A}$ is a Banach algebra with respect to the norm defined by

$$\left\| \begin{pmatrix} a & b & c \\ 0 & a & d \\ 0 & 0 & a \end{pmatrix} \right\| = |a| + |b| + |c| + |d|, \quad a, b, c, d \in \mathbb{C}.$$

Let $X = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ and define a continuous linear map $\varphi : \mathcal{A} \to \mathcal{A}$ by $\varphi(A) = AX + XA$. By a straightforward calculation one can prove that

$$BAX + XAB = BXA + AXB, \quad A, B \in \mathcal{A}$$

So we have $\varphi(A \circ B) = A \circ \varphi(B)$ for each $A, B \in \mathcal{A}$ and hence $A \circ \varphi(B) = \varphi(\mathbf{1})$ for all $A, B \in \mathcal{A}$ with $A \circ B = \mathbf{1}$. If we consider $A = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$, then $\varphi(A) \neq \mathbf{0}$ and $A\varphi(\mathbf{1}) = \mathbf{0}$, where $\mathbf{0}$ and $\mathbf{1}$ are the zero matrix and identity matrix, respectively. Thus φ is not a centralizer.

In view of Theorem 2.5, this example shows that there exist a Banach algebra $\mathcal{A}$ and a linear map $\varphi : \mathcal{A} \to \mathcal{A}$ such that $2\varphi(a^2) = a \circ \varphi(a)$ for all $a \in \mathcal{A}$ but φ is not a centralizer.

Remark 2.7. Let $\mathcal{M}$ be a unital Banach $\mathcal{A}$-bimodule and $\varphi : \mathcal{A} \to \mathcal{M}$ be a linear map satisfying the relation $2\varphi(a^2) = a \bullet \varphi(a)$ for all $a \in \mathcal{A}$. By linearizing we see that

$$2\varphi(a \circ b) = a \bullet \varphi(b) + \varphi(a) \bullet b, \quad a, b \in \mathcal{A}.$$

Replacing b by 1 in this identity, we obtain $2\varphi(a) = a\varphi(1) + \varphi(1)a$ for all $a \in \mathcal{A}$. Then obviously φ is automatically continuous.

In [14] Vukman proved that an additive mapping $\varphi : \mathcal{R} \to \mathcal{R}$, where $\mathcal{R}$ is a 2-torsion free semiprime ring, satisfying the relation $2\varphi(a^2) = a \circ \varphi(a)$ for all $a \in \mathcal{R}$ is a centralizer. So from Theorem 2.5 we have the following corollary.

Corollary 2.8. *Let $\mathcal{A}$ be a semiprime Banach algebra and $\varphi : \mathcal{A} \to \mathcal{A}$ be a continuous linear map with the property that*

$$a \in Inv_0(\mathcal{A}) \Rightarrow a \circ \varphi(a^{-1}) = 2\varphi(1).$$

Then φ is a centralizer.

The following theorem provides a class of Banach $\mathcal{A}$-bimodules $\mathcal{M}$ such that every linear map $\varphi : \mathcal{A} \to \mathcal{M}$ with the property that $2\varphi(a^2) = a \bullet \varphi(a)$ for all $a \in \mathcal{A}$, is a centralizer.

The set of idempotents of $\mathcal{A}$ is denoted by $\mathcal{I}(\mathcal{A})$ and $alg\mathcal{I}(\mathcal{A})$ denotes the subalgebra of $\mathcal{A}$ generated by $\mathcal{I}(\mathcal{A})$.

Theorem 2.9. *Let $\mathcal{M}$ be a unital Banach $\mathcal{A}$-bimodule. Suppose that there is an ideal $\mathcal{J}$ of $\mathcal{A}$ such that $\mathcal{J} \subseteq \overline{alg\mathcal{I}(\mathcal{A})}$ and*

$$\{m \in \mathcal{M} \,|\, m\mathcal{J} = \{0\} \text{ and } \mathcal{J}m = \{0\}\} = \{0\}.$$

If $\varphi : \mathcal{A} \to \mathcal{M}$ is a linear map, then:

(i) *if $2\varphi(a^2) = a \bullet \varphi(a)$ for all $a \in \mathcal{R}$, then φ is a centralizer,*

(ii) *if φ is continuous with the property that*

$$a \in Inv_0(\mathcal{A}) \Rightarrow a \bullet \varphi(a^{-1}) = 2\varphi(1),$$

then φ is a centralizer

Proof. (i) From Remark 2.7 we have

$$2\varphi(a \circ b) = a \bullet \varphi(b) + \varphi(a) \bullet b, \quad a, b \in \mathcal{A}.$$

Let p be a idempotent of $\mathcal{A}$. As $p \circ (1-p) = 0$, from above identity it follows that $p \bullet \varphi(1-p) + \varphi(p) \bullet (1-p) = 0$ and hence

$$2\varphi(p) + p\varphi(1) + \varphi(1)p = 2p\varphi(p) + 2\varphi(p)p.$$

By multiplying this identity on the left and right by p, respectively, we arrive at

$$p\varphi(1)p + \varphi(1)p = 2p\varphi(p)p,$$
$$p\varphi(1) + p\varphi(1)p = 2p\varphi(p)p,$$

which implies

$$p\varphi(1) = \varphi(1)p.$$

For idempotents $p_1, \ldots, p_n$ in $\mathcal{A}$, by applying the above identity repeatedly, we get

$$p_1 \ldots p_n\varphi(1) = \varphi(1)p_1 \ldots p_n.$$

So $x\varphi(1) = \varphi(1)x$ for all $x \in \mathcal{J} \subseteq \overline{alg\mathcal{I}(\mathcal{A})}$. Hence $a\varphi(1)x = ax\varphi(1) = \varphi(1)ax$ and $x\varphi(1)a = \varphi(1)xa = xa\varphi(1)$ for each $a \in \mathcal{A}$ and $x \in \mathcal{J}$. Therefore $(a\varphi(1) - \varphi(1)a)\mathcal{J} = \{0\}$ and $\mathcal{J}(a\varphi(1) - \varphi(1)a) = \{0\}$. By hypothesis we arrive at

$$a\varphi(1) = \varphi(1)a$$

for all $a \in \mathcal{A}$. By Remark 2.7 we have $2\varphi(a) = a\varphi(1) + \varphi(1)a$ for all $a \in \mathcal{A}$. Since $a\varphi(1) = \varphi(1)a$, it follows that $\varphi(a) = a\varphi(1) = \varphi(1)a$ for each $a \in \mathcal{A}$, i.e. φ is a centralizer.

Part (ii) is a consequence of part (i) and Theorem 2.5. □

We continue by characterizing some classes of Banach algebras and bimodules which satisfy the requirements in Theorem 2.9.

Corollary 2.10. *Suppose that $\mathcal{A} = \overline{alg\mathcal{I}(\mathcal{A})}$. Let $\mathcal{M}$ be a unital Banach $\mathcal{A}$-bimodule and $\varphi : \mathcal{A} \to \mathcal{M}$ be a linear map, then:*

(i) *if $2\varphi(a^2) = a \bullet \varphi(a)$ for all $a \in \mathcal{R}$, then φ is a centralizer,*

(ii) *if φ is continuous with the property that*

$$a \in Inv_0(\mathcal{A}) \Rightarrow a \bullet \varphi(a^{-1}) = 2\varphi(1),$$

then φ is a centralizer

Proof. Let $m \in \mathcal{M}$ such that $m\mathcal{A} = \{0\}$ and $\mathcal{A}m = \{0\}$. Since $\mathcal{A}$ is unital, it follows that $m = 0$. If we consider $\mathcal{J} = \mathcal{A}$ as an ideal of $\mathcal{A}$, then all the requirements in Theorem 2.9 hold and hence the assertion follows. □

Some examples of Banach algebras with the property that $\mathcal{A} = \overline{alg\mathcal{I}(\mathcal{A})}$, are the following:

(i) Topologically simple Banach algebras containing a nontrivial idempotent (see [2]).
(ii) The unital W^*-algebras. Indeed, the linear span of projections is norm dense in a unital W^*-algebra.

Another classes of Banach algebras $\mathcal{A}$ with the property that $\mathcal{A} = \overline{alg\mathcal{I}(\mathcal{A})}$ (not necessarily unital) are given in [2].

Let $\mathcal{X}$ be a Banach space. We denote by $\mathcal{B}(\mathcal{X})$ the algebra of all bounded linear operators on $\mathcal{X}$, and $\mathcal{F}(\mathcal{X})$ denotes the algebra of all finite rank operators in $\mathcal{B}(\mathcal{X})$. A *nest* $\mathcal{N}$ on a Banach space $\mathcal{X}$ is a chain of closed (under norm topology) subspaces of $\mathcal{X}$ which is closed under the formation of arbitrary intersection and closed linear span (denoted by $\vee$), and which includes $\{0\}$ and $\mathcal{X}$. The *nest algebra* associated to the nest $\mathcal{N}$, denoted by $Alg\mathcal{N}$, is the weak closed operator algebra of the form

$$Alg\mathcal{N} = \{T \in \mathcal{B}(\mathcal{X}) \,|\, T(N) \subseteq N \, for\, all\, N \in \mathcal{N}\}.$$

When $\mathcal{N} \neq \{\{0\}, X\}$, we say that $\mathcal{N}$ is non-trivial. It is clear that if $\mathcal{N}$ is trivial, then $Alg\mathcal{N} = \mathcal{B}(\mathcal{X})$. Denote $Alg_{\mathcal{F}}\mathcal{N} := Alg\mathcal{N} \cap \mathcal{F}(\mathcal{X})$, the set of all finite rank operators in $Alg\mathcal{N}$ and for $N \in \mathcal{N}$, let $N_- = \vee\{M \in \mathcal{N} \,|\, M \subset N\}$.

Corollary 2.11. *Let $\mathcal{N}$ be a nest on a Banach space $\mathcal{X}$ such that $N \in \mathcal{N}$ is complemented in $\mathcal{X}$ whenever $N_- = N$. If $\varphi : Alg\mathcal{N} \to \mathcal{B}(\mathcal{X})$ is a linear map, then:*

(i) *if $2\varphi(A^2) = A \bullet \varphi(A)$ for all $A \in Alg\mathcal{N}$, then φ is a centralizer,*
(ii) *if φ is continuous with the property that*

$$A \in Inv_0(Alg\mathcal{N}) \Rightarrow A \bullet \varphi(A^{-1}) = 2\varphi(1),$$

then φ is a centralizer

Proof. $Alg_{\mathcal{F}}\mathcal{N}$ is an ideal of $Alg\mathcal{N}$ and from [10], it is contained in the $\overline{alg\mathcal{I}(Alg\mathcal{N})}$. Suppose that $T \in \mathcal{B}(\mathcal{X})$ and $FT = TF = 0$ for each $F \in Alg_{\mathcal{F}}\mathcal{N}$. By [13] we have $\overline{Alg_{\mathcal{F}}\mathcal{N}}^{SOT} = Alg\mathcal{N}$. Therefore there is a net $(F_\gamma)_{\gamma\in\Gamma}$ in $Alg_{\mathcal{F}}\mathcal{N}$ converges to the identity operator I with respect to the strong operator topology. So $F_\gamma T = 0$ for each $\gamma \in \Gamma$ and hence $T = 0$. The assertion now follows directly from Theorem 2.9. $\square$

It is obvious that the nests on Hilbert spaces, finite nests and the nests having order-type $\omega+1$ or $1+\omega^*$, where ω is the order-type of the natural numbers, satisfy the condition in Corollary 2.11 automatically.

Remark 2.12. By a method similar to proof of Theorems 2.4 and 2.5, we can use the Theorem 2.1 or corollaries 2.2 or 2.3 to describe the mappings preserving identity-(Jordan) product or (Jordan) derivations through identity-products.

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