Erasures and equalities for fusion frames in Hilbert spaces *

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Abstract

Fusion frames are widely studied recently due to its many kinds of applications. In this paper we focus on the the erasure of a fusion frame, we use a bounded linear operator to equivalently characterize a given fusion frame to be robust to an erasure of any numbers of elements, it turns out that our result is more general and covers some important results previously obtained by Asgari, Casazza and Kutyniok. We also present a more general equality result for any fusion frame (not restricted to be tight), which improves the fusion version of a theorem obtained by Li and Sun.

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1 Introduction

Fusion frame, which we also call it a frame of subspaces, was first proposed by Casazza and Kutyniok in [9] to handle some large system which is impossible to handle effectively by just a simple frame. The essence of fusion frame is that we can first build frame in subspaces and then piece them together to obtain frames for the whole space. Due

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to this characteristic fusion frame is special suiting for applications such as distributed processing, parallel processing of large frame systems [11], and so on. Now the theory of fusion frames has been applied for optimal transmission by packet encoding [4], noise reduction in sensor networks [25], compressed sensing [7], sensor networks [12], filter bank [13], etc.

Recently fusion frame has been studied intensively with the development of kinds of applications. Now some authors such as Bodmann, Kribs, Paulsen [5] and Bodmann [4] use the Parseval fusion frames under the term *weighted projective resolution of the identity* to study erasures resilience. Also many authors focus on the theory of erasures of fusion frames, e.g. Asgari in [1] gave a sufficient condition for a fusion frame with only one element erasure to be again a fusion frame; Casazza and Kutyniok in [10] studied the optimal fusion frames when erasures occur. For erasures on other kinds of frames the reader can consult [15, 18–20], etc.

The first focus of this paper is to discuss the equivalent characterization on erasure for the fusion frames. As stated in Section 3 the author presented an equivalent characterization result on fusion frames only for one subspace (element) erasure, for study purpose we are very hope to seek a general method for equivalent characterization that given any fusion frame arbitrary numbers of elements can be erased still leaving a complete subset (fusion frame). In Section 3 we will present a general method by using a bounded linear operator relating to the frame operator of a given fusion frame to equivalently characterize the erasure of any number of subspaces.

Another focus of this paper is to study the equality for fusion frames. The equalities of a conventional frame were first found by Casazza etc in the study of the optimal decomposition of a parseval frame [3]. Later many authors such as Găvruta [16], Li and sun [23], Li and zhu [24], etc developed or improved the equalities or inequalities of the frames (including other kinds of frames). The result we present on equality for fusion frames in this paper is to generalize the fusion frame version of Theorem 2.2 in [23] to any fusion frame, that is, to make Theorem 4.1 (the fusion frame version of Theorem 2.2 in [23]) holds for any fusion frame (not restricted to be tight) with its alternate dual fusion frame. For more details see Section 4.

The paper is organized as follows. In Section 2 we mainly recall some basic properties of fusion frames and some useful lemmas. In Section 3 we mainly discuss the robustness of erasure of any given fusion frame, we give an equivalent condition for the remainder after deleting more than one elements still to be a fusion frame. In Section 4 we mainly study the equality for the fusion frames.

Throughout this paper we will adopt such notations. I is a countable index set; $\mathcal{H}$ is a Hilbert space; $I_{\mathcal{H}}$ is the identical operator for $\mathcal{H}$; if W is a closed subspace of $\mathcal{H}$, π_W is denoted by the orthogonal projection from $\mathcal{H}$ onto W; $L(H_1, H_2)$ is denoted by the collection of all the linear bounded operators from H_1 to H_2, where H_1, H_2 are Hilbert spaces, denote $L(H_1, H_2)$ by $L(H_1)$ if $H_1 = H_2$; $R(T)$ is denoted by the range of T if T is a bounded linear operator.

2 Preliminaries

Definition 2.1 *[9] Let $\{W_i\}_{i\in I}$ be a sequence of closed subspaces in $\mathcal{H}$, $\{v_i\}_{i\in I}$ be a family of weights, i.e., $v_i > 0$ for all $i \in I$. $\{(W_i, v_i)\}_{i\in I}$ is called a fusion frame for $\mathcal{H}$, if there exist two positive constants A, B such that*

$$A\|f\|^2 \leq \sum_{i\in I} v_i^2 \|\pi_{W_i}(f)\|^2 \leq B\|f\|^2, \quad \forall f \in \mathcal{H}. \tag{2.1}$$

We call A, B the lower and upper fusion frame bounds for $\{(W_i, v_i)\}_{i\in I}$, respectively. $\{(W_i, v_i)\}_{i\in I}$ is called a λ-tight fusion frame if $A = B = \lambda$, moreover, if $\lambda = 1$, $\{(W_i, v_i)\}_{i\in I}$ is called a Parseval fusion frame. If only the right-hand inequality of (2.1) holds, then we call $\{(W_i, v_i)\}_{i\in I}$ the fusion-Bessel sequence.

Suppose that $\{(W_i, v_i)\}_{i\in I}$ is a fusion-Bessel sequence for $\mathcal{H}$, with Bessel bound B, then we can define its analysis operator U, synthesis operator T and frame operator S as follows:

$$U : \mathcal{H} \to l^2(\{W_i\}_{i\in I}), \quad Uf = \{v_i \pi_{W_i} f\}_{i\in I}, \tag{2.2}$$

$$T : l^2(\{W_i\}_{i\in I}) \to \mathcal{H}, \quad T(\{f_i\}_{i\in I}) = \sum_{i\in I} v_i f_i, \tag{2.3}$$

$$S : \mathcal{H} \to \mathcal{H}, \quad Sf = \sum_{i\in I} v_i^2 \pi_{W_i} f, \tag{2.4}$$

where the representation space $l^2(\{W_i\}_{i\in I})$ is defined as follows

$$l^2(\{W_i\}_{i\in I}) = \{\{f_i\}_{i\in I} | f_i \in W_i \text{ and} \sum_{i\in I} \|f_i\|^2 < \infty\},$$

with the inner product

$$\langle \{f_i\}_{i\in I}, \{g_i\}_{i\in I} \rangle = \sum_{i\in I} \langle f_i, g_i \rangle.$$

It's trivial to check that $l^2(\{W_i\}_{i\in I})$ is a Hilbert space and $T^* = U, S = TU$. Moreover, from [9] we know that S is a positive, self-adjoint and invertible operator on $\mathcal{H}$ if $\{(W_i, v_i)\}_{i\in I}$ is a fusion frame for $\mathcal{H}$ with frame bounds A, B. From the definition of fusion frame we can get $AI_{\mathcal{H}} \leq S \leq BI_{\mathcal{H}}$, it follows that

$$B^{-1} I_{\mathcal{H}} \leq S^{-1} \leq A^{-1} I_{\mathcal{H}}. \tag{2.5}$$

In case $\{(W_i, v_i)\}_{i\in I}$ is a tight (resp. Parseval) fusion frame for $\mathcal{H}$ with frame bound A (resp. $A = 1$), then $S = AI_{\mathcal{H}}$ (resp. $S = I_{\mathcal{H}}$). If $\{(W_i, v_i)\}_{i\in I}$ is a fusion frame for $\mathcal{H}$, we can have the following standard reconstruction formula

$$\begin{aligned} f &= S^{-1}Sf = \sum_{i\in I} v_i^2 S^{-1}\pi_{W_i} f \\ &= SS^{-1}f = \sum_{i\in I} v_i^2 \pi_{W_i}(S^{-1}f), \quad \forall f \in \mathcal{H}. \end{aligned} \tag{2.6}$$

In [17] the author gives a more general alternate dual reconstruction formula, that is, given a fusion frame $\{(W_i, w_i)\}_{i\in I}$ with frame operator S and a Bessel sequence $\{(V_i, v_i)\}_{i\in I}$, there is

$$f = \sum_{i\in I} v_i w_i \pi_{V_i} S^{-1} \pi_{W_i} f, \quad \forall f \in \mathcal{H}.$$

In this case we also call $\{(V_i, v_i)\}_{i\in I}$ is an alternate dual of $\{(W_i, w_i)\}_{i\in I}$.

Lemma 2.2 *Suppose that X is a Banach space and $Q \in L(X)$. If $\|Q\| < 1$, then $I_X - Q$ is invertible on X, where I_X is the identical operator for X. Moreover, we have $\|(I_X - Q)^{-1}\| \leq \frac{1}{1-\|Q\|}$.*

Lemma 2.3 *[23] Suppose that $L_1, L_2 \in L(H)$ and $L_1 + L_2 = I_{\mathcal{H}}$, then there is*

$$L_1 - L_1^* L_1 = L_2^* - L_2^* L_2.$$

3 Erasures for fusion frames

In [10] the authors gave an equivalent statement for a tight fusion frame to be a fusion frame only for one element erasure. To understand more clearly let me recall this statement.

Proposition 3.1 *Suppose that $\{(W_i, v_i)\}_{i\in I}$ is a tight fusion frame for $\mathcal{H}$ with frame bounds A. Then the following are equivalent:*

(i) $v_{j_0}^2 < A$;

(ii) $\{(W_i, v_i)\}_{i\in I\setminus\{j_0\}}$ *is a fusion frame for* $\mathcal{H}$.

From this proposition we can see clearly the impact of weight to decide a remainder by deleting one element from a tight fusion frame to be still a fusion frame. But the condition for this proposition is a little strict, it needs that the fusion frame is tight, and the result is only for one element erasure. So it's natural to ask that whether there is a general version for erasure of Proposition 3.1? That is to say, the general result for erasure should satisfy the following two demands:

- the fusion frame is arbitrary, not only for tight fusion frames;

- the number of elements to be erased should be any ($< |I|$).

In this section we will give such a erasure result (see Theorem 3.2), we will show that Proposition 3.1 and part of Theorem 4.3 obtained by Asgari in [1] are the special cases of our result. To do that we need to introduce a bounded linear operator in $\mathcal{H}$.

Let $J \subset I$ and let $\{(W_i, v_i)\}_{i\in I}$ be a fusion-Bessel sequence in $\mathcal{H}$, we now define S_J as follows

$$S_J : \mathcal{H} \to \mathcal{H}, \quad S_J f = \sum_{i\in J} v_i^2 \pi_{W_i} f. \tag{3.1}$$

It's trivial to show that S_J is a bounded linear operator in $\mathcal{H}$.

Theorem 3.2 *Let $J \subset I$. Suppose that $\{(W_i, v_i)\}_{i\in I}$ is a fusion frame for $\mathcal{H}$ with frame bounds A, B. Then the following statements are equivalent:*

(i) $I_{\mathcal{H}} - S^{-1}S_J$ *is invertible on* $\mathcal{H}$*;*

(ii) $I_{\mathcal{H}} - S_J S^{-1}$ *is invertible on* $\mathcal{H}$*;*

(iii) $\{(W_i, v_i)\}_{i\in I\setminus J}$ *is a fusion frame for* $\mathcal{H}$*,*

where S_J is defined as in (3.1), S is the frame operator for $\{(W_i, v_i)\}_{i\in I}$.

In addition, if (i) or (ii) is satisfied, then the fusion frame $\{(W_i, v_i)\}_{i\in I\setminus J}$ has the lower fusion frame bounds $\frac{A}{\|(I_{\mathcal{H}}-S^{-1}S_J)^{-1}\|^2}$.

Proof. *(i)* ⇔ *(ii)* It's trivial to check that the operator S_J is self-adjoint. So we have

$$(I_{\mathcal{H}} - S^{-1}S_J)^* = I_{\mathcal{H}} - (S^{-1}S_J)^* = I_{\mathcal{H}} - S_J^*(S^{-1})^* = I_{\mathcal{H}} - S_J S^{-1},$$

hence $I_{\mathcal{H}} - S^{-1}S_J$ is invertible on $\mathcal{H}$ iff $I_{\mathcal{H}} - S_J S^{-1}$ is invertible on $\mathcal{H}$.

(i) ⇔ *(iii)* Denote the frame operator of fusion frame $\{(W_i, v_i)\}_{i\in I\setminus J}$ by $S_{I\setminus J}$. Since $S_{I\setminus J} = S - S_J = S(I - S^{-1}S_J)$, we have

$$\begin{aligned}\{(W_i, v_i)\}_{i\in I\setminus J} \text{ is a fusion frame for } \mathcal{H} &\iff S_{I\setminus J} \text{ is boundedly invertible}\\ &\iff S(I - S^{-1}S_J) \text{ is boundedly invertible}\\ &\iff I - S^{-1}S_J \text{ is boundedly invertible.}\end{aligned}$$

Next we show the "In addition" part. Assume that $I_{\mathcal{H}} - S^{-1}S_J$ is invertible on $\mathcal{H}$. Since $\{(W_i, v_i)\}_{i\in I}$ is a fusion frame for $\mathcal{H}$ with frame bounds A, B, so for any $f \in \mathcal{H}$ we get

$$\begin{aligned} f &= S^{-1}Sf\\ &= S^{-1}\Big(\sum_{i\in J} v_i^2 \pi_{W_i} f + \sum_{i\in I\setminus J} v_i^2 \pi_{W_i} f\Big)\\ &= S^{-1}S_J f + \sum_{i\in I\setminus J} v_i^2 S^{-1}\pi_{W_i} f.\end{aligned}$$

Hence we have

$$(I_{\mathcal{H}} - S^{-1}S_J)f = \sum_{i\in I\setminus J} v_i^2 S^{-1}\pi_{W_i} f, \quad \forall f \in \mathcal{H}. \tag{3.2}$$

Therefore we obtain

$$\begin{aligned}
\|(I_{\mathcal{H}} - S^{-1}S_J)f\| &= \left\| \sum_{i\in I\setminus J} v_i^2 S^{-1}\pi_{W_i} f \right\| \\
&= \sup_{g\in\mathcal{H},\|g\|=1} \left| \left\langle \sum_{i\in I\setminus J} v_i^2 S^{-1}\pi_{W_i} f, g \right\rangle \right| \\
&= \sup_{g\in\mathcal{H},\|g\|=1} \left| \sum_{i\in I\setminus J} v_i^2 \langle \pi_{W_i} f, S^{-1}g\rangle \right| \\
&= \sup_{g\in\mathcal{H},\|g\|=1} \left| \sum_{i\in I\setminus J} v_i^2 \langle \pi_{W_i} f, \pi_{W_i}(S^{-1}g)\rangle \right| \\
&\le \sup_{g\in\mathcal{H},\|g\|=1} \sum_{i\in I\setminus J} v_i^2 \|\pi_{W_i} f\| \cdot \|\pi_{W_i}(S^{-1}g)\| \\
&\le \sup_{g\in\mathcal{H},\|g\|=1} \left(\sum_{i\in I\setminus J} v_i^2 \|\pi_{W_i} f\|^2 \right)^{\frac{1}{2}} \cdot \left(\sum_{i\in I\setminus J} v_i^2 \|\pi_{W_i}(S^{-1}g)\|^2 \right)^{\frac{1}{2}} \\
&\le \sup_{g\in\mathcal{H},\|g\|=1} \left(\langle S(S^{-1}g), (S^{-1}g)\rangle \right)^{\frac{1}{2}} \cdot \left(\sum_{i\in I\setminus J} v_i^2 \|\pi_{W_i} f\|^2 \right)^{\frac{1}{2}} \\
&\le \sqrt{A^{-1}} \left(\sum_{i\in I\setminus J} v_i^2 \|\pi_{W_i} f\|^2 \right)^{\frac{1}{2}},
\end{aligned} \tag{3.3}$$

where the last inequality is deduced by (2.5). It follows that $I_{\mathcal{H}} - S^{-1}S_J$ is well defined in $\mathcal{H}$. If $I_{\mathcal{H}} - S^{-1}S_J$ is invertible on $\mathcal{H}$, then for any $f \in \mathcal{H}$ we have

$$\begin{aligned}
\|f\| &= \|(I_{\mathcal{H}} - S^{-1}S_J)^{-1}(I_{\mathcal{H}} - S^{-1}S_J)f\| \\
&\le \|(I_{\mathcal{H}} - S^{-1}S_J)^{-1}\| \cdot \|(I_{\mathcal{H}} - S^{-1}S_J)f\|,
\end{aligned}$$

it follows that

$$\frac{\|f\|}{\|(I_{\mathcal{H}} - S^{-1}S_J)^{-1}\|} \le \|(I_{\mathcal{H}} - S^{-1}S_J)f\|, \quad \forall f \in \mathcal{H}. \tag{3.4}$$

From (3.3) and (3.4) we have

$$\frac{A}{\|(I_{\mathcal{H}} - S^{-1}S_J)^{-1}\|^2} \|f\|^2 \le \sum_{i\in I\setminus J} v_i^2 \|\pi_{W_i} f\|^2, \quad \forall f \in \mathcal{H}.$$

□

Combining with Lemma 2.2 and Theorem 3.2 we can have the following sufficient conditions to judge the remainder after an erasure to be still a fusion frame.

Corollary 3.3 *Let $J \subset I$. Suppose that $\{(W_i, v_i)\}_{i\in I}$ is a fusion frame for $\mathcal{H}$ with frame bounds A, B. If $\|S^{-1}S_J\| < 1$, then $\{(W_i, v_i)\}_{i\in I\setminus J}$ is a fusion frame for $\mathcal{H}$, with frame bounds $A(1 - \|S^{-1}S_J\|)^2$ and B, where S_J is defined as in (3.1).*

Proof. We only prove the lower frame bound. Since $\|S^{-1}S_J\| < 1$, by Lemma 2.2 we know that $I_{\mathcal{H}} - S^{-1}S_J$ is invertible on $\mathcal{H}$. So for any $f \in \mathcal{H}$, we have

$$f = (I_{\mathcal{H}} - S^{-1}S_J)^{-1}(I_{\mathcal{H}} - S^{-1}S_J)f,$$

it follows that $\|f\| \le \|(I_{\mathcal{H}} - S^{-1}S_J)^{-1}\| \cdot \|(I_{\mathcal{H}} - S^{-1}S_J)f\|$, by Lemma 2.2 we get

$$(1 - \|S^{-1}S_J\|)\|f\| \le \frac{\|f\|}{\|(I_{\mathcal{H}} - S^{-1}S_J)^{-1}\|} \le \|(I_{\mathcal{H}} - S^{-1}S_J)f\|, \tag{3.5}$$

combining (3.3) we obtain

$$A(1 - \|S^{-1}S_J\|)^2\|f\|^2 \le \sum_{i\in I\setminus J} v_i^2\|\pi_{W_i}f\|^2.$$

□

Corollary 3.4 *Let $J \subset I$. Suppose that $\{(W_i, v_i)\}_{i\in I}$ is a tight fusion frame for $\mathcal{H}$ with frame bound A. If $\|S_J\| < A$, then $\{(W_i, v_i)\}_{i\in I\setminus J}$ is a fusion frame for $\mathcal{H}$, with frame bounds $\frac{(A-\|S_J\|)^2}{A}$ and A, where S_J is defined as in (3.1).*

Proof. Since $\{(W_i, v_i)\}_{i\in I}$ is a tight fusion frame for $\mathcal{H}$ with frame bound A, then its frame operator $S = AI_{\mathcal{H}}$, so $S^{-1} = \frac{1}{A}I_{\mathcal{H}}$, then the result follows from Corollary 3.3.

Corollary 3.5 *Let $J \subset I$. Suppose that $\{(W_i, v_i)\}_{i\in I}$ is a Parseval fusion frame for $\mathcal{H}$. If $\|S_J\| < 1$, then $\{(W_i, v_i)\}_{i\in I\setminus J}$ is a fusion frame for $\mathcal{H}$, with frame bounds $(1 - \|S_J\|)^2$ and 1, where S_J is defined as in (3.1).*

Moreover, if $J = \{j_0\}, j_0 \in I$, from Theorem 3.2 we can easily have the following corollaries. Note that the part $(ii) \Rightarrow (iii)$ in Corollary 3.6 was first stated in Theorem 4.3 in [1].

Corollary 3.6 *Suppose that $\{(W_i, v_i)\}_{i\in I}$ is a fusion frame for $\mathcal{H}$ with frame bounds A, B. Then the following statements are equivalent:*

(i) $I_{\mathcal{H}} - v_{j_0}^2 S^{-1}\pi_{W_{j_0}}$ *is invertible on* $\mathcal{H}$*;*

(ii) $I_{\mathcal{H}} - v_{j_0}^2 \pi_{W_{j_0}} S^{-1}$ *is invertible on* $\mathcal{H}$*;*

(iii) $\{(W_i, v_i)\}_{i\in I\setminus\{j_0\}}$ *is a fusion frame for* $\mathcal{H}$.

Moreover, if (i) or (ii) is satisfied, then the fusion frame $\{(W_i, v_i)\}_{i\in I\setminus\{j_0\}}$ *has fusion frame bounds* $\frac{A}{\|(I_{\mathcal{H}}-v_{j_0}^2 S^{-1}\pi_{W_{j_0}})^{-1}\|^2}$ *and* B.

Combining with Corollary 3.7 and Proposition 3.1 (see also Corollary 3.4 in [10]) we can get the following corollary.

Corollary 3.7 *Suppose that* $\{(W_i, v_i)\}_{i\in I}$ *is a tight fusion frame for* $\mathcal{H}$ *with frame bounds* A. *Then the following statements are equivalent:*

(i) $v_{j_0}^2 < A$;

(ii) $I_{\mathcal{H}} - \frac{1}{A}v_{j_0}^2\pi_{W_{j_0}}$ *is invertible on* $\mathcal{H}$;

(iii) $\{(W_i, v_i)\}_{i\in I\setminus\{j_0\}}$ *is a fusion frame for* $\mathcal{H}$.

Moreover, if (i) or (ii) is satisfied, then the fusion frame $\{(W_i, v_i)\}_{i\in I\setminus\{j_0\}}$ *has fusion frame bounds* $\frac{1}{A\|(AI_{\mathcal{H}}-v_{j_0}^2\pi_{W_{j_0}})^{-1}\|^2}$ *and* A.

4 Equalities for fusion frames

In [23] Theorem 2.2 the authors presented an equality for tight g-frames. In this section we will generalize the version of tight fusion frame for Theorem 2.2 [23] to a pair of alternate dual fusion frames. For this we need first to restate the Theorem 2.2 in [23] using the setup of fusion frame.

Theorem 4.1 *(The fusion frame version for Theorem 2.2 in [23]) Suppose that* $\{(W_i, w_i)\}_{i\in I}$ *is a tight fusion frame for* $\mathcal{H}$ *with frame bound* A*, then for any* $f \in H$ *and* $\{a_i\}_{i\in I} \in l^\infty(I)$*, there is*

$$\Big\|\sum_{i\in I}(1-a_i)w_i^2\pi_{W_i}f\Big\|^2 - \Big\|\sum_{i\in I}a_iw_i^2\pi_{W_i}f\Big\|^2 = A\sum_{i\in I}(1-a_i)w_i^2\|\pi_{W_i}f\|^2 - A\sum_{i\in I}\overline{a}_iw_i^2\|\pi_{W_i}f\|^2,$$

where $\overline{a}_i$ *is the conjugate of* a_i, $l^\infty(I) = \{\{a_i\}_{i\in I}|\sup|a_i| < \infty, i \in I\}$.

This theorem tells us that there is a good equality with coefficient $\{a_i\}_{i\in I} \in l^\infty(I)$ for tight fusion frames. The defect is that the fusion frames needs to be tight. So it's natural to consider whether it holds for any fusion frame? In the following Theorem 4.2 we will prove that Theorem 4.1 can be held for any fusion frame with its alternate dual fusion frames.

Theorem 4.2 *Suppose that $\{(W_i,w_i)\}_{i\in I}$ is a fusion frame for $\mathcal{H}$ with frame operator S, $\{(V_i,v_i)\}_{i\in I}$ is an alternate dual fusion frame of $\{(W_i,w_i)\}_{i\in I}$. Then for any $f\in\mathcal{H}$ and $\{a_i\}_{i\in I}\in l^\infty(I)$, we have*

$$\begin{aligned}&\sum_{i\in I}a_iv_iw_i\langle S^{-1}\pi_{W_i}f,\pi_{V_i}f\rangle-\left\|\sum_{i\in I}a_iv_iw_i\pi_{V_i}S^{-1}\pi_{W_i}f\right\|^2\\ =&\sum_{i\in I}(1-\overline{a}_i)v_iw_i\langle\pi_{V_i}f,S^{-1}\pi_{W_i}f\rangle-\left\|\sum_{i\in I}(1-a_i)v_iw_i\pi_{V_i}S^{-1}\pi_{W_i}f\right\|^2,\end{aligned}\tag{4.1}$$

where $\overline{a}_i$ is the conjugate of a_i.

Proof. For any $f\in\mathcal{H},\{a_i\}_{i\in I}\in l^\infty(I)$, let

$$L_1f=\sum_{i\in I}a_iv_iw_i\pi_{V_i}S^{-1}\pi_{W_i}f,\quad L_2f=\sum_{i\in I}(1-a_i)v_iw_i\pi_{V_i}S^{-1}\pi_{W_i}f.$$

Next we prove that the operators L_1,L_2 are well defined in $\mathcal{H}$. In fact, for any subset $J\subset I$ and any $f\in\mathcal{H}$ we have

$$\begin{aligned}\left\|\sum_{i\in J}a_iv_iw_i\pi_{V_i}S^{-1}\pi_{W_i}f\right\|^2 &= \sup_{g\in H,\|g\|=1}\left|\left\langle\sum_{i\in J}a_iv_iw_i\pi_{V_i}S^{-1}\pi_{W_i}f,g\right\rangle\right|^2\\ &= \sup_{g\in H,\|g\|=1}\left|\sum_{i\in J}a_i\langle w_iS^{-1}\pi_{W_i}f,v_i\pi_{V_i}g\rangle\right|^2\\ &\le \sup_{g\in H,\|g\|=1}\left(\sum_{i\in J}|a_i\langle w_iS^{-1}\pi_{W_i}f,v_i\pi_{V_i}g\rangle|\right)^2\\ &\le \sup_{g\in H,\|g\|=1}\left(\sum_{i\in J}|a_i|\cdot|w_iS^{-1}\pi_{W_i}f|\cdot|v_i\pi_{V_i}g|\right)^2\\ &\le \sup_{g\in H,\|g\|=1}\|a\|_\infty^2\sum_{i\in J}w_i^2\|S^{-1}\pi_{W_i}(f)\|^2\cdot\sum_{i\in J}v_i^2\|\pi_{V_i}(g)\|^2\\ &\le \sup_{g\in H,\|g\|=1}\|a\|_\infty^2\|S^{-1}\|^2\sum_{i\in J}w_i^2\|\pi_{W_i}(f)\|^2\cdot\sum_{i\in J}v_i^2\|\pi_{V_i}(g)\|^2\\ &\le \sup_{g\in H,\|g\|=1}\|a\|_\infty^2\|S^{-1}\|^2D\|g\|^2\sum_{i\in J}w_i^2\|\pi_{W_i}(f)\|^2\\ &= D\|a\|_\infty^2\|S^{-1}\|^2\sum_{i\in J}w_i^2\|\pi_{W_i}(f)\|^2,\end{aligned}$$

where D is a Bessel bound for $\{(V_i,v_i)\}_{i\in I}$, it follows that L_1 is well defined. By the same way as above we can show L_2 is well defined. We can easily obtain

$$\begin{aligned}L_1f+L_2f &= \sum_{i\in I}a_iv_iw_i\pi_{V_i}S^{-1}\pi_{W_i}f+\sum_{i\in I}(1-a_i)v_iw_i\pi_{V_i}S^{-1}\pi_{W_i}f\\ &= \sum_{i\in I}v_iw_i\pi_{V_i}S^{-1}\pi_{W_i}f.\end{aligned}\tag{4.2}$$

Since $\{(V_i, v_i)\}_{i\in I}$ is an alternate dual fusion frame of $\{(W_i, w_i)\}_{i\in I}$, we then have

$$f = \sum_{i\in I} v_i w_i \pi_{V_i} S^{-1} \pi_{W_i} f,$$

combining (4.2) we get

$$L_1 + L_2 = I_{\mathcal{H}}.$$

By Lemma 2.3 it follows that

$$L_1 - L_1^* L_1 = L_2^* - L_2^* L_2,$$

so for any $f \in \mathcal{H}$ we obtain

$$\langle L_1 f, f\rangle - \langle L_1^* L_1 f, f\rangle = \langle L_2^* f, f\rangle - \langle L_2^* L_2 f, f\rangle.$$

Moreover, since

$$\begin{aligned} \langle L_1 f, f\rangle - \langle L_1^* L_1 f, f\rangle &= \left\langle \sum_{i\in I} a_i v_i w_i \pi_{V_i} S^{-1} \pi_{W_i} f, f \right\rangle - \|L_1 f\|^2 \\ &= \sum_{i\in I} a_i v_i w_i \langle S^{-1} \pi_{W_i} f, \pi_{V_i} f\rangle - \left\| \sum_{i\in I} a_i v_i w_i \pi_{V_i} S^{-1} \pi_{W_i} f \right\|^2, \end{aligned}$$

$$\begin{aligned} \langle L_2^* f, f\rangle - \langle L_2^* L_2 f, f\rangle &= \left\langle f, \sum_{i\in I} (1 - a_i) v_i w_i \pi_{V_i} S^{-1} \pi_{W_i} f \right\rangle - \|L_2 f\|^2 \\ &= \sum_{i\in I} (1 - \overline{a}_i) v_i w_i \langle \pi_{V_i} f, S^{-1} \pi_{W_i} f\rangle - \left\| \sum_{i\in I} (1 - a_i) v_i w_i \pi_{V_i} S^{-1} \pi_{W_i} f \right\|^2, \end{aligned}$$

so the conclusion follows. □

Remark 4.3 *It is easily seen that Theorem 4.1 is indeed a special case of Theorem 4.2. In fact, Theorem 4.2 also holds for g-frames which are more general than fusion frames, we leave the proof to the interested readers.*

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