Recognizing simple K_4-groups by few special conjugacy class sizes *

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Abstract

In 1987, J. G. Thompson put forward the following conjecture: *Let G be a finite group with trivial center. If L is a finite simple group satisfying that $N(G) = N(L)$, then $G \cong L$.* The second author proved above conjecture holds for finite simple groups with non-connected prime graphes. Vasilev proved above conjecture holds for two simple groups with connected prime graphes: A_{10} and $L_4(4)$. N. Ahanjideh proved that Thompson's conjecture is true for $L_n(q)$. The authors are interested in if it is possible to weaken the conditions in the conjecture. A finite simple group is called a simple *K_n-group* if its order is divisible by exactly n distinct primes. Here, the authors prove that simple K_4-groups are characterized by their orders and few special conjugacy class sizes, which implies that Thompson's conjecture is valid for simple K_4-groups.

Key Words: Simple K_4-groups, conjugacy class size, prime graph, Thompson's conjecture.

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1 Notations and Introduction

All groups considered in this paper are finite and simple groups are finite non-abelian simple groups. Let G be a group. We denote by $N(G)$ the set of conjugacy class sizes of G and by $\pi(G)$ the set of prime divisors of $|G|$. In the middle of the 1970s, Gruenberg and O. Kegel introduced the concept of prime graph of a group G as follows: the vertices are the primes dividing the order of G, two vertices p and q are joined by an edge if and only if G contains an element of order pq (see [10]). Denote the connected components of the prime graph of group G by $T(G) = \{\pi_i(G) | 1 \leqslant i \leqslant t(G)\}$, where $t(G)$

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is the number of the prime graph components of G. If the order of G is even, we always assume that $2 \in \pi_1(G)$. In addition, for $x \in G$, $cl_G(x)$ denotes the conjugacy class in G containing x, and we denote by G_p and $Syl_p(G)$ a Sylow $p-$subgroup of G and the set of all of its Sylow $p-$subgroups for $p \in \pi(G)$, respectively. We also denote $Soc(G)$ the socle of G which is the subgroup generated by all minimal normal subgroups of G, Mult(G) the Schur multiplier of G, and $\Omega_i(G)$ a subgroup and $\Omega_i(G) = \langle g \in G | g^{p^i} = 1\rangle$ if G is a $p-$group. The other notations and terminologies in this paper are standard and the reader is referred to [7] and [13].

In the past thirty years, many mathematicians try to recognize finite groups, especially simple groups, by their quantitative characteristics. Such as quantitative characterizations by prime graph (see [18, 19, 20, 21]), by group order and element orders (see [22, 23, 24, 25]), and by non-commuting graph (see [26, 27, 28]), et al. Here we will continue to this topic. In 1987, J. G. Thompson posed the following conjecture (announced by W. J. Shi in 1989, ref. to [9, Problem 12.38]):

Conjecture 1.1 (Thompson's conjecture) *Let G be a group with trivial center. If L is a simple group satisfying that $N(G) = N(L)$, then $G \cong L$.*

In 1994, G. Y. Chen proved in his Ph. D. dissertation [2] that Thompson's conjecture holds for all simple groups with non-connected prime graph (also ref. to [3, 4, 5]). In 2009, A. V. Vasil'ev first dealt with the simple groups with connected prime graph and proved that Thompson's conjecture holds for A_{10} and $L_4(4)$ (see [9]). Later on, N. Ahanjideh in [1] proved that Thompson's conjecture is true for $L_n(q)$. Recently, G. Y. Chen and J. B. Li contributed their interests on the Thompson's conjecture under a weak condition. They only used order and one or two special conjugacy class sizes of finite simple groups, and successfully characterized sporadic simple groups (see J. B. Li's Ph. D. dissertation [14]) and simple K_3-groups (A finite simple group is called a simple K_n*-group* if its order is divisible by exactly n distinct primes) by their orders and one special conjugacy class sizes, from which the Thompson's conjecture for sporadic simple groups and simple K_3-groups follow. In fact, they provided two new ways to characterize finite simple groups, and one of them doesn't care about if prime graph of a group is non-connected. Hence it is an interesting topic to characterize simple groups by their orders and few conjugacy class sizes. In this paper, we focus our attention on finite simple K_4-groups, and characterize them by their orders and few special class sizes, by which we show Thompson's conjecture holds for simple K_4-groups as corollary. For convenience, we denote by $lcs(G)$, $slcs(G)$, $tlcs(G)$, $flcs(G)$, and $scs(G)$ the largest, the second largest, the third largest, the fourth largest, and the smallest conjugacy class size greater than one of G, respectively. Our main result is the following theorem:

Theorem 1.2 *Let G be a group, L a simple K_4-group. Then one of the following holds:*

(i) *If L is isomorphic to one of* $A_7,\ A_8,\ A_{10}, M_{12},\ L_2(16),\ L_2(25),\ L_2(49),\ L_2(81),\ L_3(5),\ L_3(7),$

$L_3(8),\ L_3(17),\ U_3(7),\ U_3(8),\ U_3(9),\ Sz(32),\ S_4(4)$, *and* $L_2(r)$*, where* $r \geq 11$ *is an odd prime satisfies*

$$r^2 - 1 = 2^a \cdot 3^b \cdot v^c$$

with $a,\ b,\ c \geq 1$*, then* $G \cong L$ *if and only if* $|G| = |L|$ *and* $scs(G) = scs(L)$*;*

(ii) *If* L *is isomorphic to one of* $M_{11},\ J_2,\ L_3(4),\ U_3(4),\ U_3(5),\ U_4(3),\ O_8^+(2),\ G_2(3),$ $Sz(8),\ L_2(2^m)$*, where* $m \geq 5$ *satisfies*

$$2^m - 1 = u,\ 2^m + 1 = 3t^b, u \text{ and } t \text{ are primes},\ t > 3,\ b \geq 1,$$

and $L_2(3^n)$*, where* $n \geq 3$ *satisfies*

$$3^n - 1 = 2u,\ 3^n + 1 = 4t^b, u \text{ and } t \text{ are primes}, b \geq 1,$$

then $G \cong L$ *if and only if* $|G| = |L|$ *and* $lcs(G) = lcs(L)$*;*

(iii) *If* L *is isomorphic to one of* $A_9,\ S_4(5),\ {}^2F_4(2'),$ *and* ${}^3D_4(2)$*, then* $G \cong L$ *if and only if* $|G| = |L|$ *and* $slcs(G) = slcs(L)$*;*

(iv) *If* $L \cong L_4(3)$*, then* $G \cong L$ *if and only if* $|G| = |L|$ *and* $tlcs(G) = tlcs(L)$*;*

(v) *If* $L \cong S_4(9)$*, then* $G \cong L$ *if and only if* $|G| = |L|$ *and* $flcs(G) = flcs(L)$*;*

(vi) *If* L *is isomorphic to one of* $S_6(2)$ *and* $U_5(2)$*, then* $G \cong L$ *if and only if* $|G| = |L|$*,* $lcs(G) = lcs(L)$*, and* $scs(G) = scs(L)$*.*

(vii) *If* $L \cong S_4(7)$*, then* $G \cong L$ *if and only if* $|G| = |L|$ *and* $5^2 \big| |cl_G(w)|$ *for every element* w *of order* $p \in \{2,\ 3,\ 7\}$ *of* G*.*

(viii) *If* $L \cong L_2(3^n)$*, where* $n \geq 5$ *satisfies*

$$3^n - 1 = 2u^c,\ 3^n + 1 = 4t, u \text{ and } t \text{ are primes}, c > 1,$$

then $G \cong L$ *if and only if* $|G| = |L|$ *and* $u^c \big| |cl_G(w)|$ *for every element* w *of order* $p \in \{2,\ 3,\ t\}$ *of* G*.*

And we have the following corollary.

Corollary 1.3 *Thompson's conjecture holds for all simple* K_4-*groups.*

Proof. If $L \ncong A_{10}$, then the prime graph of L is non-connected, and hence $|G| = |L|$ by [2]. If $L \cong A_{10}$, then $|G| = |L|$ by [9]. Therefore the corollary follows from Theorem 1.2.

2 Preliminaries

First, we prove some preliminary lemmas to be used in the proof of Theorem 1.2.

Lemma 2.1 *Let G be a group and $x \in G$. Set $\overline{G} = G/Z(G)$ and $\overline{x}$ the image of x in $\overline{G}$.*

(a) *If $|G_p| < scs(G)$ for every $p \in \pi(G)$, then every minimal normal subgroup of $\overline{G}$ is not solvable. Especially, $Soc(\overline{G}) \trianglelefteq \overline{G} \lesssim Aut(Soc(\overline{G}))$.*

(b) *For any $x \in G$, $|cl_{\overline{G}}(\overline{x})| \big| |cl_G(x)|$. Moreover, $scs(\overline{G}) \leq scs(G)$.*

(c) *If $(|x|, |Z(G)|) = 1$, then $C_{\overline{G}}(\overline{x}) = C_G(x)/Z(G)$. Moreover, $|cl_{\overline{G}}(\overline{x})| = |cl_G(x)|$.*

(d) *If x is a non-central $p-$element of G, then $|cl_{\overline{G}}(\overline{x})|_{p'} = |cl_G(x)|_{p'}$.*

Proof. (a) Assume that $\overline{N}$ is any minimal normal subgroup of $\overline{G}$ and N is the inverse image of $\overline{N}$ in G. If $\overline{N}$ is solvable, then $\overline{N}$ is an elementary abelian $p-$group. Hence N is a nilpotent normal subgroup of G, and N_p is a normal subgroup of G not contained in $Z(G)$. Thus, there is an element x of $N_p \setminus Z(G)$ satisfying that

$$1 < |cl_G(x)| = |G : C_G(x)| \leq |N_p| < scs(G),$$

violating the hypothesis. Now, every minimal normal subgroup of $\overline{G}$ is not solvable. Let $S_1,\ S_2,\ \ldots,$ and S_k be all minimal normal subgroups of $\overline{G}$, where k is a positive integer. Then $Soc(\overline{G}) = S_1 \times S_2 \times \cdots \times S_k$. We assert that $C_{\overline{G}}(Soc(\overline{G})) = 1$. Otherwise, there exists a minimal normal subgroup $\overline{S}$ of $\overline{G}$ so that $\overline{S} \leq C_{\overline{G}}(Soc(\overline{G})) \bigcap Soc(\overline{G})$. Thus $\overline{S}$ is an abelian group, a contradiction. By N/C Theorem, we have $Soc(\overline{G}) \trianglelefteq \overline{G} = \overline{G}/C_{\overline{G}}(Soc(\overline{G})) \lesssim \mathrm{Aut}(Soc(\overline{G}))$, as desired.

(b) For any $x \in G$, we have that $C_G(x)/Z(G) \leq C_{\overline{G}}(\overline{x})$, and so $|cl_{\overline{G}}(\overline{x})| \big| |cl_G(x)|$. It is easy to know that $scs(\overline{G}) \leq scs(G)$.

(c) If $(|x|, |Z(G)|) = 1$, then $C_G(x)/Z(G) = C_{\overline{G}}(\overline{x})$ by Theorem 1.6.2 of [12], and thus $|cl_{\overline{G}}(\overline{x})| = |cl_G(x)|$.

(d) Since x is a noncentral $p-$element of G, it follows that $\overline{x} \neq 1$ is a $p-$element of $\overline{G}$ and $C_{\overline{G}}(\overline{x}) \geq C_G(x)/Z(G)$.

If $C_{\overline{G}}(\overline{x})$ is a $p-$group,Then $C_G(x)/Z(G)$ is also a $p-$group. It follows that

$$|cl_{\overline{G}}(\overline{x})|_{p'} = |\overline{G} : C_{\overline{G}}(\overline{x})|_{p'} = |\overline{G}|_{p'} = |G/Z(G)|_{p'} = |G/Z(G) : C_G(x)/Z(G)|_{p'} = |cl_G(x)|_{p'},$$

as desired.

Assume that $C_{\overline{G}}(\overline{x})$ is not a $p-$group. Then, for every $p'-$element $\overline{y} \in C_{\overline{G}}(\overline{x})$, there exists a $p'-$element y of G such that $\overline{y} = yZ(G)$, and thus $[x,\ y] \in Z(G)$. Let $[x,\ y] = x^{-1}y^{-1}xy = z$. Then $x^{-1}y^{-1}x = y^{-1}z$ and $y^{-1}xy = zx$, and so $z^{|y|} = z^{|x|} = 1$. Note that $(|x|,\ |y|) = 1$, then $[x,\ y] = z = 1$ and $y \in C_G(x)$. Thus $\overline{y} \in C_G(x)/Z(G)$, and so $|C_{\overline{G}}(\overline{x})|_{p'} = |C_G(x)/Z(G)|_{p'}$. Hence

$$|cl_{\overline{G}}(\overline{x})|_{p'} = |\overline{G} : C_{\overline{G}}(\overline{x})|_{p'} = |\overline{G}|_{p'}/|\overline{C_G(x)}|_{p'} = |G/Z(G) : C_G(x)/Z(G)|_{p'} = |cl_G(x)|_{p'},$$

as claimed. So (d) holds.

Lemma 2.2 [15, Theorem 1] *If L is a simple K_3-group, then L is isomorphic to one of the following groups:*

$$A_5,\ A_6,\ L_2(7),\ L_2(8),\ L_3(3),\ U_3(3),\ U_4(2),\ L_2(17).$$

Lemma 2.3 [16, Theorem 2] *If L is a simple K_4-group, then L is isomorphic to one of the following groups:*

(a) $A_7,\ A_8,\ A_9,\ A_{10},\ M_{11},\ M_{12},\ J_2,\ L_2(16),\ L_2(25),\ L_2(49),\ L_2(81),\ L_3(4),\ L_3(5),\ L_3(7),\ L_3(8),\ L_3(17),\ L_4(3),\ S_4(4),\ S_4(5),\ S_4(7), S_4(9),\ S_6(2),\ O_8^+(2),\ G_2(3),\ U_3(4),\ U_3(5),\ U_3(7),\ U_3(8),\ U_3(9),\ U_4(3),\ U_5(2),\ Sz(8),\ Sz(32),\ {}^3D_4(2),\ {}^2F_4(2)'$.

(b) $L_2(r)$, *where r is a prime and satisfies*

$$r^2 - 1 = 2^a \cdot 3^b \cdot v^c$$

with $a,\ b,\ c \geq 1$ and a prime $v > 3$.

(c) $L_2(2^m)$, *where $m \geq 2$ satisfies*

$$2^m - 1 = u,\ 2^m + 1 = 3t^b,$$

where u and t are primes, $t > 3,\ b \geq 1$.

(d) $L_2(3^n)$, *where $n \geq 2$ satisfies*

$$3^n - 1 = 2u^c,\ 3^n + 1 = 4t$$

or

$$3^n - 1 = 2u,\ 3^n + 1 = 4t^b,$$

where u and t are odd primes, $b \geq 1,\ c \geq 1$.

In fact, we have by trivial computing that $r \geq 11,\ m \geq 5$, and $n \geq 3$ in (b), (c) and (d) of Lemma 2.3 respectively. Moreover, for any group L a group in Lemma 2.2 or Lemma 2.3 except $Sz(8)$ and $Sz(32)$, it follows that 3 divides $|L|$.

Lemma 2.4 [17, Theorem 2] *Let L be a simple K_4-group. Then L is determined uniquely up to isomorphism by the set $\pi(L)$ unless one of the following cases occurs:*

(a) $\pi(L) = \{2,\ 3,\ 5,\ p\}$ *with* $p \in \{7,\ 11,\ 13,\ 17,\ 31,\ 41\}$.

(b) $\pi(L) = \{2,\ 3,\ 7,\ p\}$ *with* $p \in \{13,\ 19\}$.

For convenience we write out exceptional groups in Lemma 2.4 in following corollary by Lemma 2.3.

Corollary 2.5 *Let L be a simple K_4-group. Then one of the following holds:*

(a) $\pi(L) = \{2,\ 3,\ 5,\ 7\}$ *if and only if L is isomorphic to one of* $A_7,\ A_8,\ A_9,\ A_{10},\ J_2,\ L_2(49), L_3(4), U_3(5),\ U_4(3),\ S_4(7),\ S_6(2),$ *and* $O_8^+(2)$.

(b) $\pi(L) = \{2,\ 3,\ 5,\ 11\}$ *if and only if L is isomorphic to one of* $M_{11},\ M_{12},\ L_2(11),$ *and* $U_5(2)$.

(c) $\pi(L) = \{2,\ 3,\ 5,\ 13\}$ *if and only if L is isomorphic to one of* $L_2(25),\ S_4(5),\ L_4(3),\ U_3(4),$ *and* ${}^2F_4(2)'$.

(d) $\pi(L) = \{2,\ 3,\ 5,\ 17\}$ *if and only if L is isomorphic to one of* $L_2(16)$ *and* $S_4(4)$.

(f) $\pi(L) = \{2,\ 3,\ 5,\ 31\}$ *if and only if L is isomorphic to one of* $L_2(31)$ *and* $L_3(5)$.

(e) $\pi(L) = \{2,\ 3,\ 5,\ 41\}$ *if and only if L is isomorphic to one of* $L_2(81)$ *and* $S_4(9)$.

(h) $\pi(L) = \{2,\ 3,\ 7,\ 13\}$ *if and only if L is isomorphic to one of* $L_2(13),\ L_2(27),\ G_2(3),$ *and* ${}^3D_4(2)$.

(i) $\pi(L) = \{2,\ 3,\ 7,\ 19\}$ *if and only if L is isomorphic to one of* $L_3(7)$ *and* $U_3(8)$.

Proof. It follows straight forward from Lemma 2.3 and [7].

Remark 2.6 *By Lemma 2.3 and Corollary 2.5, it follows that L is determined uniquely up to isomorphism by the set $\pi(L)$ if L is a group in (b), (c), and (d) of Lemma 2.3 with $r \geq 19$ but $\neq 31$, $m \geq 5$ and $n \geq 5$.*

A group G is a 2-Frobenius group if there exists a normal series $1 \lhd H \lhd K \lhd G$ such that K and G/H are Frobenius groups with kernels H and K/H, respectively. Now we quote some known results on Frobenius group or 2-Frobenius group which are useful in the sequel.

Lemma 2.7 [10, Theorem A] *Let G be a group with more than one prime graph component. Then G is one of the following holds:*

(i) *a Frobenius or 2-Frobenius group;*

(ii) *G has a normal series $1 \subseteq H \subseteq K \subseteq G$, where H is a nilpotent π_1-group, K/H is a simple group and G/K is a π_1-group such that $|G/K|$ divides the order of the outer automorphism group of K/H. Besides, each odd order component of G is also an odd order component of K/H.*

Lemma 2.8 [6, Theorem 1] *Suppose that G is a Frobenius group of even order and H, K are the Frobenius kernel and the Frobenius complement of G, respectively. Then $t(G) = 2$, $T(G) = \{\pi(H), \pi(K)\}$ and G has one of the following structures:*

(i) $2 \in \pi(H)$ *and all Sylow subgroups of* K *are cyclic;*

(ii)$2 \in \pi(K)$*,* H *is an abelian group,* K *is a solvable group, the Sylow subgroups of* K *of odd order are cyclic groups and the Sylow* 2*-subgroups of* K *are cyclic or generalized quaternion groups;*

(iii) $2 \in \pi(K)$*,* H *is abelian, and there exists a subgroup* K_0 *of* K *such that*

$$|K : K_0| \leqslant 2, K_0 = Z \times SL(2,5), (|Z|, 2 \times 3 \times 5) = 1,$$

and the Sylow subgroups of Z *are cyclic.*

Lemma 2.9 [6, Theorem 2] *Let* G *be a* 2*-Frobenius group of even order. Then* $t(G) = 2$ *and* G *has a normal series* $1 \lhd H \lhd K \lhd G$ *such that* $\pi(K/H) = \pi_2$*,* $\pi(H) \cup \pi(G/K) = \pi_1$*, the order of* G/K *divides the order of the automorphism group of* K/H*, and both* G/K *and* K/H *are cyclic. Especially,* $|G/K| < |K/H|$ *and* G *is solvable.*

Lemma 2.10 [11, Theorem 3.4.20] *Let* $R = R_1 \times \cdots \times R_k$*, where* R_i *is a direct product of* n_i *isomorphic copies of a simple group* H_i*, where* H_i *and* H_j *are not isomorphic if* $i \neq j$*. Then* $\mathrm{Aut}(R) \cong \mathrm{Aut}(R_1) \times \cdots \times \mathrm{Aut}(R_k)$ *and* $\mathrm{Aut}(R_i) \cong \mathrm{Aut}(H_i) \wr S_{n_i}$*, where in this wreath product* $\mathrm{Aut}(H_i)$ *appears in its right regular representation and the symmetric group* S_{n_i} *in its natural permutation representation. Moreover these isomorphisms induce outer automorphisms* $\mathrm{Out}R \cong \mathrm{Out}(R_1) \times \cdots \times \mathrm{Out}(R_k)$ *and* $\mathrm{Out}(R_i) \cong \mathrm{Out}(H_i) \wr S_{n_i}$*.*

Lemma 2.11 [13, Theorem 4.5.3] *Let* G *be a* $p-$*group of order* $p^n, n \geq 1$*, and* d *is the number of minimal generators of* G*. Then* $|\mathrm{Aut}(G)| \big| p^{d(n-d)}(p^d - 1)(p^d - p) \cdots (p^d - p^{d-1})$*.*

Lemma 2.12 *Let* G *be a group,* N *a normal subgroup of* G *with order* $p^n, n \geq 1$*. If* $(r, |\mathrm{Aut}(N)|) = 1$*, where* $r \in \pi(G)$*, then* G *has an element of order* pr*. Further there exists an edge connecting* r *and* p *in the prime graph of* G*.*

Proof. Let an element g of G with order r act on N. Since $(r, |\mathrm{Aut}(N)|) = 1$, this action is trivial, so that $g \in C_G(N)$, which concludes the lemma.

3 Proof of Main Theorem

In this section, we divide the proof of Theorem 1.2 into several lemmas according to classification of simple K_4-groups in Lemma 2.4 and Corollary 2.5. Since the necessity of any case in Theorem 1.2 can be checked easily, it is enough to prove the sufficiency. We divide the proof into following lemmas.

Lemma 3.1 *Let* G *be a group and* L *one of the simple* K_4-*groups with* $\pi(L) = \{2,\ 3,\ 5,\ 7\}$*.*

(I) *If L is isomorphic to one of J_2, $L_3(4)$, $U_3(5)$, $U_4(3)$, and $O_8^+(2)$, then $G \cong L$ if and only if $|G| = |L|$ and $lcs(G) = lcs(L)$;*

(II) *If L is isomorphic to one of A_7, $L_2(49)$, A_8, and A_{10}, then $G \cong L$ if and only if $|G| = |L|$ and $scs(G) = scs(L)$;*

(III) *If $L \cong A_9$, then $G \cong L$ if and only if $|G| = |L|$ and $slcs(G) = slcs(L)$;*

(IV) *If $L \cong S_4(7)$, then $G \cong L$ if and only if $|G| = |L|$ and $5^2 \big| |cl_G(w)|$ for every element w of order $p \in \{2, 3, 7\}$ of G;*

(V) *If $L \cong S_6(2)$, then $G \cong L$ if and only if $|G| = |L|$, $lcs(G) = lcs(L)$, and $scs(G) = scs(L)$.*

Proof. (I) In [14], the case of J_2 have been done. So we only prove the remaining cases in (I).

Case 1. If $L \cong L_3(4)$, then it follows by [7] that $|G| = |L| = 2^6 \cdot 3^2 \cdot 5 \cdot 7$ and $lcs(G) = lcs(L) = 2^6 \cdot 3^2 \cdot 7$. Hence G has an element x of order 5 such that $lcs(G) = |G : C_G(x)| = 2^6 \cdot 3^2 \cdot 7$. This means that $C_G(x) = \langle x \rangle$, and $C_G(x)$ is a Sylow 5-subgroup of G. By Sylow Theorem, we have that $C_G(y) = \langle y \rangle$ for any element $y \in G$ of order 5 . Thus $\{5\}$ is a prime graph component of G and $t(G) \geqslant 2$, i. e., 5 is an isolated vertex of the prime graph of G.

We first show that G is neither a Frobenius group nor a 2-Frobenius group.

Suppose that G is a Frobenius group with kernel H and complement K. If $5 \in \pi(H)$, then H is a Sylow 5-subgroup of G and $\pi(K) = \{2, 3, 7\}$ by Lemma 2.8. Considering Sylow 5$-$subgroup of H and a prime $7 \in \pi(G)$, one can see that 5 is connected to 7 in prime graph of G by Lemma 2.12, a contradiction. If $5 \in \pi(K)$, then, considering the Sylow 7-subgroup of H and $5 \in \pi(K)$, we come to a contradiction by Lemma 2.12. Hence G is not a Frobenius group.

Assume that G is a 2-Frobenius group. By Lemma 2.9, we have that G has a normal series $1 \lhd H \lhd K \lhd G$ such that $\pi(K/H) = \pi_2$, $\pi(H) \cup \pi(G/K) = \pi_1$, and $|G/K| \big| |\mathrm{Aut}(K/H)|$. It forces that $|K/H| = 5$ and $7 \in \pi(H)$. Because of $(5, |\mathrm{Aut}(H_7)|) = 1$, it is a contradiction by Lemma 2.12. Therefore G is not a 2-Frobenius group.

Now, by Lemma 2.7, G has a normal series $1 \subseteq H \subseteq K \subseteq G$, where H is a nilpotent π_1-group, K/H is a simple group, G/K is a π_1-group such that $|G/K|$ divides the order of the outer automorphism group of K/H and each odd order component of G is also an odd order component of K/H. It follows that 5 is an isolated vertex of prime graph of K/H. By Lemma 2.2 and (a) of Corollary 2.5, we have that K/H is isomorphic to one of the following simple groups:

$$A_5,\ A_6,\ A_7,\ A_8,\ L_3(4).$$

If $K/H \cong A_5$ or A_6, then $7 \in \pi(H)$ since $|G/K|$ divides 2^2 by $|\mathrm{Out}(A_5)| = 2$ and $|\mathrm{Out}(A_6)| = 2^2$. It follows that 5 and 7 are connected by Lemma 2.12, a contradiction.

If K/H is isomorphic to A_7, then $|\mathrm{Out}(A_7)| = |\mathrm{Out}(K/H)| = 2$, and so $|G/K| = 1$ or 2. Hence H is a group of order 2^2 or 2^3. Since $(5, |\mathrm{Aut}(H)|) = 1$ by Lemma 2.11, we have that 5 and 2 are connected in the prime graph of G by Lemma 2.12, a contradiction.

If K/H is isomorphic to A_8. Then $G \cong A_8$ by $|G| = |A_8|$, but $lcs(A_8) = 2^5 \cdot 3 \cdot 5 \cdot 7$ by [7], a contradiction.

Hence K/H must be isomorphic to $L_3(4)$, which immediately implies that $G \cong L_3(4)$.

Case 2. If $L \cong U_3(5)$, then $|G| = |L| = 2^4 \cdot 3^2 \cdot 5^3 \cdot 7$ and $lcs(G) = lcs(L) = 2^4 \cdot 3^2 \cdot 5^3$. Therefore there exsits an element x of order 7 in G satisfying that $lcs(G) = |G : C_G(x)|$ and $C_G(x) = \langle x \rangle$, from which 7 is an isolated vertex of prime graph of G and $t(G) \geqslant 2$.

Suppose that G is a Frobenius group with kernel H and complement K. If $7 \in \pi(H)$, then H is a Sylow $7-$subgroup of G by Lemma 2.8. Since an element of K of order 5 acts trivially on H, 5 and 7 are connected, a contradiction. If $7 \in \pi(K)$, then H_3, the Sylow $3-$ group of H, is a normal Sylow 3-subgroup of G by nilpotency of H. Hence G has an element of order 21 for $(7, |\mathrm{Aut}(H_3)|) = 1$ by Lemma 2.12, a contradiction. Therefore G is not a Frobenius group.

Assume that G is a 2-Frobenius group. Then, by Lemma 2.9, G has a normal series $1 \lhd H \lhd K \lhd G$ such that the Sylow 3-subgroup of H is of order 3 or 3^2, and is normal in G. As above, we see that 3 and 7 are connected for $(7, |\mathrm{Aut}(H_3)|) = 1$, a contradiction. Thus G is not a 2-Frobenius group.

Hence G has a normal series $1 \subseteq H \subseteq K \subseteq G$ by Lemma 2.7, where H is a nilpotent π_1-group, K/H is a simple group and G/K is π_1-group such that $|G/K|$ divides the order of the outer automorphism group of K/H. Furthermore, $\{7\}$ is a component of K/H. By Lemma 2.2 and (a) of Corollary 2.5, It follows that K/H is isomorphic to one of the following groups:

$$L_2(7),\ L_2(8),\ A_7,\ U_3(5).$$

If K/H is isomorphic to $L_2(7)$. Since $(|G/K|, 3) = 1$ by $|\mathrm{Out}(K/H)| = 2$, it follows that G has an element of order 21, a contradiction.

If K/H is isomorphic to $L_2(8)$, then G has an element of order 14 by Lemma 2.12 since $7 \in \pi(G)$ and the Sylow 2-subgroup of H of order 2 is normal in G, a contradiction;

If K/H is isomorphic to A_7, then similarly, one can get that G has an element of order 35 as the Sylow 5-subgroup of H is of order 5^2, a contradiction.

Therefore, K/H must be isomorphic to $U_3(5)$, which implies that $G \cong U_3(5)$ by $|G| = |U_3(5)|$.

Similar to Case 1 and Case 2, we can prove that the lemma holds for $U_4(3)$ and $O_8^+(2)$. Hence (I) follows..

(II) Let $\overline{G} = G/Z(G)$. If L is one of A_7, $L_2(49)$, A_8 and A_{10}, then the order of Sylow $p-$subgroup of G is less than $scs(G)$ for any prime $p \in \pi(G)$ by the hypothesis and [7]. By (a) of Lemma 2.1, every minimal normal subgroup of $\overline{G}$ is non-solvable and $Soc(\overline{G}) \trianglelefteq \overline{G} \leq \mathrm{Aut}(Soc(\overline{G}))$. Let $M = Soc(\overline{G})$ and $S_1,\ S_2,\ \ldots,$ and $S_k (k \geq 1)$ be all minimal normal subgroups of $\overline{G}$. Then

$M = Soc(\overline{G}) = S_1 \times S_2 \times \cdots \times S_k$ and S_i is a direct product of some isomorphic simple groups for $i = 1,\ 2, \ldots,\ k$. Now, we continue the argument case by case.

Case 1. If $L \cong A_7$, then $|G| = |L| = 2^3 \cdot 3^2 \cdot 5 \cdot 7$ and $scs(G) = scs(L) = 2 \cdot 5 \cdot 7$ by [7]. By the hypothesis, there exists an element x in G such that $scs(G) = |G : C_G(x)| = 2 \cdot 5 \cdot 7$. Hence 5 and 7 do not divide the order of $Z(G)$ because of $Z(G) \leq C_G(x)$.

We assert that $5 \in \pi(M)$. Otherwise, M is a simple K_3-group such that $7 \in \pi(M)$ and $5 \in \pi(\mathrm{Out}(M))$. Checking the order of M, M may be isomorphic to $L_2(7)$ or $L_2(8)$ by Lemma 2.2. But $|\mathrm{Out}(L_2(7))| = 2$ and $|\mathrm{Out}(L_2(8))| = 3$, a contradiction.

If $7 \notin \pi(M)$, then M is a simple K_3-group such that $\pi(M) = \{2,\ 3,\ 5\}$ and $7 \in \pi(\mathrm{Out}(M))$. By Lemma 2.2, M may be isomorphic to one of A_5, and A_6, but $|\mathrm{Out}(A_5)| = 2$ and $|\mathrm{Out}(A_6)| = 4$, a contradiction to $7 \in \pi(\mathrm{Out}(M))$. Hence $7 \in \pi(M)$. Suppose that $5 \in \pi(S_i)$, and $7 \in \pi(S_j)$ for $i,\ j \in \{1, 2, \ldots, k\}$.

If $i \neq j$, then S_i and S_j are simple K_3-groups with $\pi(S_i) = \{2,\ 3,\ 5\}$, $\pi(S_j) = \{2,\ 3,\ 7\}$, respectively. By checking their orders, one can see that it is impossible by Lemma 2.2.

Hence $i = j$, and we obtain that $k = 1$ and M is isomorphic to A_7, which implies that $G \cong A_7$, as desired.

Case 2. If $L \cong L_2(49)$, then $|G| = |L| = 2^4 \cdot 3 \cdot 5^2 \cdot 7^2$ and $scs(G) = scs(L) = 2^4 \cdot 3 \cdot 5^2$. Set $x \in G$ such that $scs(G) = |cl_G(x)| =2^4 \cdot 3 \cdot 5^2$. Since $Z(G)$ is contained in $C_G(x)$ for any $x \in G$, we have that $Z(G)$ is a proper subgroup of G and $2,\ 3$ and $5 \notin \pi(Z(G))$.

It is clear that $2,\ 3 \in \pi(M)$. If 5 is not in $\pi(M)$, then $5 \in \pi(\mathrm{Out}(M))$ and M is a simple K_3-group. Therefore M may be isomorphic to $L_2(7)$ by Lemma 2.2, but $|\mathrm{Out}(L_2(7))| = 2$, a contradiction. Thus $5 \in \pi(M)$. Checking the order of M, we see that M is isomorphic to A_5 or $L_2(49)$. If $M \cong A_5$, then $5 \in \pi(Z(G))$ by comparing three orders of M, $\overline{G}$ and $\mathrm{Aut}(\overline{G})$, which is a contradiction.

Hence $M \cong L_2(49)$, and thus $G \cong L_2(49)$, as desired.

Case 3. If $L \cong A_8$, then $|G| = |L| = 2^6 \cdot 3^2 \cdot 5 \cdot 7$ and $scs(G) = scs(L) = 3 \cdot 5 \cdot 7$ by the hypothesis. It follows that there is an element x of G such that $scs(G) = |G : C_G(x)| = 3 \cdot 5 \cdot 7$. Hence 5 and 7 do not divide the order of $Z(G)$ by $Z(G) \leq C_G(x)$.

Similar to Case 1 and Case 2, we can obtain that 5 and 7 are contained in $\pi(M)$. Assume that $5 \in \pi(S_i)$ and $7 \in \pi(S_j)$ for $i,\ j \in \{1,\ 2,\ \ldots,\ k\}$.

Assume that $i \neq j$. Then S_i and S_j are simple K_3-groups such that $\pi(S_i) = \{2,\ 3,\ 5\}$ and $\pi(S_j) = \{2,\ 3,\ 7\}$, respectively. Checking the order of M, we come to that M is isomorphic to $A_5 \times L_2(7)$ by Lemma 2.2. Hence $\mathrm{Aut}(M)$=$A_5 \cdot 2 \times L_2(7) \cdot 2$ by Lemma 2.10 and [7]. It follows that $|Z(G)|\big|2$. Therefore there exists an element w of order 5 in G such that $C_G(w)/Z(G) = C_{\overline{G}}(\overline{w}) \geq$

$\langle \overline{w} \rangle \times L_2(7)$ by (c) of Lemma 2.1, where $\overline{w}$ is the image of w in $\overline{G}$. Hence $1 < |cl_G(w)| < scs(G)$, a contradiction.

Hence $i = j$, and then $k = 1$ and M is isomorphic to A_7, $L_3(4)$, and A_8 by checking the possible order of M. Recall that $M \trianglelefteq \overline{G} \lesssim \mathrm{Aut}(M)$.

If $M \cong A_7$, then $\overline{G} \cong A_7$ or $A_7 \cdot 2$, and thus $|Z(G)| \big| 2^3$. It follows that there exists an element w of order 3 such that $|cl_{\overline{G}}(\overline{w})| = |cl_G(w)| = 2 \cdot 5 \cdot 7 < scs(G)$ by (c) of Lemma 2.1, a contradiction.

If $M \cong L_3(4)$, then $G \cong L_3(4)$ by $|G| = |L_3(4)|$, but $scs(L_3(4)) > 3 \cdot 5 \cdot 7$ by [7], a contradiction.

Therefore $M \cong A_8$ implies that $G \cong A_8$, as claimed.

Case 4. If $L \cong A_{10}$, then $|G| = |L| = 2^7 \cdot 3^4 \cdot 5^2 \cdot 7$ and $scs(G) = scs(L) = 2^4 \cdot 3 \cdot 5$. It is clear that $Z(G) \leqslant C_G(x)$ for any $x \in G$. Set $y \in G$ such that $scs(G) = |G : C_G(y)| = 2^4 \cdot 3 \cdot 5$. Hence $|C_G(y)| = 2^3 \cdot 3^3 \cdot 5 \cdot 7$ and 7 is not contained in $\pi(Z(G))$ by the hypothesis.

It is clear that 2 and $3 \in \pi(M)$. We assert that $7 \in \pi(M)$. Otherwise, then $\pi(M) = \{2,\ 3,\ 5\}$ and $7 \in \pi(\mathrm{Out}(M))$. By checking the order of M, M may be isomorphic to one of following groups:

$$A_5,\ A_6,\ U_4(2),\ A_5 \times A_5,\ A_5 \times A_6,\ A_6 \times A_6.$$

By [7], we see that outer automorphism groups of these groups above are $2-$groups, contradicting the fact that $7 \in \pi(\mathrm{Out}(M))$. Hence $7 \in \pi(M)$. It follows that M may be isomorphic to one of groups: $L_2(7),\ L_2(8),\ U_3(3),\ A_7,\ A_8,\ L_3(4),\ A_9,\ J_2,\ A_{10},\ A_5 \times L_2(7),\ A_5 \times L_2(8),\ A_5 \times U_3(3),\ A_5 \times A_7,$ $L_2(7) \times A_6$, and $L_2(8) \times A_6$.

Notice that $M \trianglelefteq \overline{G} \lesssim \mathrm{Aut}(M)$. If M is isomorphic to one of $L_2(7)$, $L_2(8)$ and $U_3(3)$, then $5^2 \big| |Z(G)|$, so $5^2 \big| |C_G(w)|$ for any element w of G. But by $5^2 \parallel |G|$, one has that $5 \nmid |cl_G(w)|$, a contradiction to the hypothesis.

If $M \cong A_7$, then $\overline{G} \cong A_7$ or $A_7 \cdot 2$ by $|\mathrm{Out}(A_7)| = 2$. Assume that $\overline{G} \cong A_7$, then $|Z(G)| = 2^4 \cdot 3^2 \cdot 5$, which implies $2^4 \big| |C_G(w)|$ for every element w of G. Hence $|cl_G(w)|_2 \big| 2^3$, a contradiction to $2^4 \big| scs(G)$. We assert that $\overline{G}$ is not isomorphic to $A_7 \cdot 2$. Otherwise, $|Z(G)| = 2^3 \cdot 3^2 \cdot 5$ and there exists a noncentral element w in G of order 2 satisfying that $|cl_G(w)|_{2'} = |cl_{\overline{G}}(\overline{w})|_{2'} = |cl_{\overline{G}}(\overline{w})| = 3 \cdot 7$ by (d)of Lemma 2.1 and [7], where $\overline{w}$ is image of w in $\overline{G}$. By $2^4 \big| |\langle w, Z(G) \rangle| \big| |C_G(w)|$, we have that $|cl_G(w)| \leq 2^3 \cdot 3 \cdot 7 < scs(G)$, a contradiction.

If $M \cong A_8$, then $\overline{G} \cong A_8$ or S_8 by $|\mathrm{Out}(A_8)| = 2$. Assume that $\overline{G} \cong A_8$. Then G is a central extensive of A_8 by $Z(G)$ and $|Z(G)| = 2 \cdot 3^2 \cdot 5$. If the extension is split, then $G \cong A_8 \times Z(G)$. Otherwise, $G \cong 2 \cdot A_8 \times Z(G)_{2'}$by $|\mathrm{Mult}(A_8)| = 2$, where $Z(G)_{2'}$ is a $2'$-Hall subgroup of $Z(G)$. Since $scs(A_8) = scs(2 \cdot A_8) = 3 \cdot 5 \cdot 7$ by [7], we come to that $scs(G) = 3 \cdot 5 \cdot 7$, a contradiction to the hypothesis. We also assert that $\overline{G}$ is not isomorphic to $A_8 \cdot 2$. Otherwise, $|Z(G)| = 3^2 \cdot 5$, and there exists a noncentral element w in G of order 2 satisfying with $|cl_G(w)| = |cl_{\overline{G}}(\overline{w})| = 2^2 \cdot 7 < scs(G)$ by (c) of Lemma 2.1 and [7], where $\overline{w}$ is image of w in $\overline{G}$, a contradiction.

If $M \cong L_3(4)$, then $\overline{G}$ is isomorphic to one of $L_3(4)$, $L_3(4){\cdot}2_1$, $L_3(4){\cdot}2_2$, $L_3(4){\cdot}2_3$, $L_3(4){\cdot}3$, $L_3(4){\cdot}6$, $L_3(4)\cdot 3\cdot 2_2$, and $L_3(4)\cdot 3\cdot 2_3$ by $\mathrm{Aut}(L_3(4)) = L_3(4)\cdot D_{12}$. By (b) of Lemma 2.1 and [7], we have that $scs(\overline{G}) \geq 280 > scs(G)$ if $\bar{G} \ncong L_3(4)\cdot 2_2$, a contradiction. If $\overline{G} \cong L_3(4)\cdot 2_2$, then $|Z(G)| = 3^2\cdot 5$ and there exists a noncentral element w in G of order 2 such that $|cl_G(w)| = |cl_{\overline{G}}(\overline{w})| = 120 < scs(G)$ by (a) of Lemma 2.1 and [7], a contradiction.

If $M \cong A_9$, then $\overline{G} \cong A_9$ or $A_9\cdot 2$ for $|\mathrm{Out}(A_9)| = 2$. If $\overline{G} \cong A_9$, then $|Z(G)| = 2\cdot 5$ and $G \cong A_9 \times Z(G)$ or $2\cdot A_9 \times Z(G)_{2'}$ since $|\mathrm{Mult}(A_9)| = 2$, where $Z(G)_{2'}$ is a $2'$-Hall subgroup of $Z(G)$. But $scs(A_9) = 2^3\cdot 3\cdot 7$ and $scs(2\cdot A_9) = 2^4\cdot 3\cdot 7$ by [7], it follows that $scs(G) = 2^3\cdot 3\cdot 7$ or $2^4\cdot 3\cdot 7$, a contradiction. If $\overline{G} \cong A_9\cdot 2$, then $|Z(G)| = 5$, and there exists a noncentral element w in G of order 3 satisfying that $|cl_G(w)| = |cl_{\overline{G}}(\overline{w})| = 2^3\cdot 3\cdot 7 < scs(G)$ by (c) of Lemma 2.1 and [7], a contradiction.

If $M \cong J_2$, then $\overline{G} \cong J_2$ or $J_2\cdot 2$ by $|\mathrm{Out}(J_2)| = 2$. Comparing the orders of M, $\overline{G}$ and $\mathrm{Aut}(M)$, we get that $|Z(G)| = 3$ and $\overline{G} \cong J_2$. But $|\mathrm{Mult}(J_2)| = 2$, so G is a split extension of J_2 by $Z(G)$ such that $G = J_2 \times Z(G)$. It follows that $scs(G) = scs(J_2) = 3^2\cdot 5\cdot 7$ by [7], a contradiction.

If $M \cong A_5 \times L_2(7)$, then $3^2{\cdot}5\big||Z(G)|$ and $\pi(Z(G)) \subseteq \{2,3,5\}$ by comparing the orders of M, $\overline{G}$ and $\mathrm{Aut}(G)$. Hence there exists a noncentral element w in G of order 7 such that $C_G(w)/Z(G) = C_{\overline{G}}(\overline{w}) \geq A_5 \times \langle\overline{w}\rangle$ by (c) of Lemma 2.1. It follows that $|C_G(w)| \geq |Z(G)||A_5 \times \langle\overline{w}\rangle| = 2^2\cdot 3^3\cdot 5^2\cdot 7 > |C_G(y)|$, a contradiction.

If $M \cong A_5 \times L_2(8)$. When $\overline{G} \ncong A_5\cdot 2 \times L_2(8)\cdot 3$, we have that $2^2\cdot 5 \leq |Z(G)|$ and $7 \notin \pi(Z(G))$. By (c) of Lemma 2.1, it follows that there exists a noncentral element w in G of order 7 such that $|C_G(w)| \geq |Z(G)||A_5 \times \langle\overline{w}\rangle| = 2^4\cdot 3\cdot 5^2\cdot 7 > |C_G(y)|$, a contradiction. When $\overline{G} \cong A_5\cdot 2 \times L_2(8)\cdot 3$, it follows that $|Z(G)| = 2\cdot 5$. Then there exists a noncentral element z in G of order 5 satisfying that $|cl_G(z)|_{5'} = |cl_{\overline{G}}(\overline{z})|_{5'} = 12$ by (d) of Lemma 2.1 and [7], and so $|cl_G(w)| \leq 12\cdot 5 < scs(G)$, a contradiction.

If $M \cong A_5 \times U_3(3)$, then $|Z(G)| = 5$ and $\overline{G} \cong A_5 \times U_3(3)$. It follows that there exists a noncentral element w in G of order 5 satisfying that $|cl_G(w)|_{5'} = |cl_{\overline{G}}(\overline{w})|_{5'} = 12$ by (d) of Lemma 2.1 and [7], and thus $|cl_G(w)| \leq 12\cdot 5 < scs(G)$, a contradiction.

If $M \cong A_5 \times A_7$, then $3\big||Z(G)|$ and $\pi(Z(G)) \subseteq \{2,3\}$. Hence there exists a noncentral element w in G of order 5 such that $|C_G(w)| \geq |Z(G)||A_7 \times \langle\overline{w}\rangle| = 2^3\cdot 3^3\cdot 5^2\cdot 7 > |C_G(y)|$ by (c) of Lemma 2.1 and [7], a contradiction.

If $M \cong A_6 \times L_2(7)$, then $15\big||Z(G)|$ and $\pi(Z(G)) \subseteq \{2,3,5\}$. It follows that there exists a noncentral element w in G of order 7 satisfying that $|cl_G(w)| = |cl_{\overline{G}}(\overline{w})| = 24 < scs(G)$ by (c) of Lemma 2.1 and [7], a contradiction.

If $M \cong A_6 \times L_2(8)$, then $5\big||Z(G)|$ and $\pi(Z(G)) \subseteq \{2,5\}$. Similarly, there exists a noncentral element w in G of order 7 such that $|cl_G(w)| = |cl_{\overline{G}}(\overline{w})| = 72 < scs(G)$, a contradiction.

Hence $M \cong A_{10}$ and so G must be isomorphic to A_{10} by $|G| = |A_{10}|$.

(III) If $L \cong A_9$, then $|G| = 2^6 \cdot 3^4 \cdot 5 \cdot 7$ and $slcs(G) = 2^6 \cdot 3^4 \cdot 5$. It follows that 7 is an isolated vertex in the prime graph of G and $t(G) \geqslant 2$.

Suppose that G is a Frobenius group with kernel H and complement K. If $7 \in \pi(H)$, then H has a Sylow $7-$ subgroup of order 7, which is normal in G. Hence 5 and 7 are connected in the prime graph of G by Lemma 2.12, a contradiction. If $7 \in \pi(K)$, then G has an element of order 35 by Lemma 2.12 and similar reasoning, a contradiction. Therefore G is not a Frobenius group.

Suppose that G is a 2-Frobenius group. By Lemma 2.9, G has a normal series $1 \lhd H \lhd K \lhd G$ such that the Sylow 5-subgroup of H is of order 5. It follows that 5 and 7 are connected in the prime graph of G by Lemma 2.12, a contradiction. Thus, G is not a 2-Frobenius group.

Therefore G has a normal series $1 \subseteq H \subseteq K \subseteq G$, where H is a nilpotent π_1-group, K/H is a simple group and G/K is π_1-group such that $|G/K|$ divides the order of the outer automorphism group of K/H. Moreover, $\{7\}$ is a prime graph component of K/H. By Lemma 2.2 and (a) of Corollary 2.5, K/H may be isomorphic to one of the following simple groups:

$$L_2(7),\ L_2(8),\ A_7,\ U_3(3),\ A_8,\ L_3(4),\ A_9.$$

If K/H is isomorphic to one of $L_2(7)$, $L_2(8)$, and $U_3(3)$, then G has an element of order 35 because the Sylow 5-subgroup of H is of order 5 by $|\mathrm{Out}(K/H)|\big|6$, and an element of order 7 of K acts trivially on H_5, a contradiction.

If K/H is isomorphic to one of A_7, A_8, and $L_3(4)$, then the Sylow 3-subgroup of H is of order 3 or 3^2 by $|\mathrm{Out}(A_7)|$=$|\mathrm{Out}(A_8)|$=2 and $|\mathrm{Out}(L_3(4))| = 12$, which implies that G has an element of order 21, a contradiction.

Therefore K/H must be isomorphic to A_9, which concludes $G \cong A_9$.

(IV) If $L \cong S_4(7)$, then, by hypothesis, $|G| = 2^8 \cdot 3^2 \cdot 5^2 \cdot 7^4$ and 5 is an isolated vertex in the prime graph of G.

Assume that G is a Frobenius group with kernel H and complement K. If $5 \in \pi(H)$, then the $5-$Sylow subgroup of H is of order 5^2 and is normal in G, hence 5 and 7 are connected by Lemma 2.12, a contradiction. If $5 \in \pi(K)$, then G has an element of order 15 by $(5, |\mathrm{Aut}(H_3)|) = 1$ by Lemma 2.12, a contradiction. Therefore G is not a Frobenius group.

Suppose that G is a 2-Frobenius group. By Lemma 2.9, G has a normal series $1 \lhd H \lhd K \lhd G$ such that $|K/H| = 5^2$, $\pi(H) \cup \pi(G/K) = \{2, 3, 7\}$, and $|G/K|\big||\mathrm{Aut(K/H)}| = 20$. Hence the Sylow 3-subgroup of H is of order 3^2 and a normal subgroup of K,. It follows that 5 and 3 are connected by Lemma 2.12, a contradiction. Thus, G is not a 2-Frobenius group.

Therefore G has a normal series $1 \subseteq H \subseteq K \subseteq G$, where H is a nilpotent π_1-group, K/H is a simple group and G/K is π_1-group such that $|G/K|$ divides the order of the outer automorphism group of K/H. Moreover, $\{5\}$ is a prime graph component of K/H. Checking the order of K/H, we come to that K/H may be isomorphic to one of $L_2(49)$ and $S_4(7)$.

If $K/H \cong L_2(49)$, then $|\mathrm{Out}(K/H)| = 2^2$, so $|G/K| \leq 4$. Hence the Sylow 3-subgroup of H is of order 3 and a normal subgroup of G. It follows that 5 and 3 are connected by Lemma 2.12, a contradiction.

Hence K/H must be isomorphic to $S_4(7)$, so that $G \cong S_4(7)$ by $|G| = |S_4(7)|$, as claimed.

(V) If $L \cong S_6(2)$, then $|G| = |L| = 2^9 \cdot 3^4 \cdot 5 \cdot 7$, $lcs(G) = lcs(L) = 2^9 \cdot 3^4 \cdot 5$, and $scs(G) = scs(L) = 3^2 \cdot 7$ by [7], from which 7 is an isolated vertex in the prime graph of G and $t(G) \geqslant 2$.

By the same reasoning as previous cases, one can show that G is not a Frobenius group and a 2-Frobenius group. So it follows by Lemma 2.7 that G has a normal series $1 \subseteq H \subseteq K \subseteq G$, where H is a nilpotent π_1-group, K/H is a simple group and G/K is π_1-group such that $|G/K|$ divides the order of the outer automorphism group of K/H. Hence $\{7\}$ is a prime component of K/H. By Lemma 2.2 and (a) of Corollary 2.5, we know that K/H is isomorphic to one of the following groups:

$$L_2(7),\ L_2(8),\ U_3(3),\ A_7,\ A_8,\ A_9,\ L_3(4),\ S_6(2).$$

Suppose that K/H is isomorphic to one of $L_2(7)$, $L_2(8)$, and $U_3(3)$. then $(|G/K|, 5) = 1$ by $|\mathrm{Out}(L_2(7))| = 2$, $|\mathrm{Out}(L_2(8))| = 3$ and $|\mathrm{Out}(U_3(3))| = 2$, Hence the Sylow 5-subgroup of H is of order 5 and is normal in G. Since $(7,\ |\mathrm{Aut}(H_5)|) = 1$, we have that G has an element of order 35 by Lemma 2.12, a contradiction.

If K/H is isomorphic to one of A_7, A_8, and $L_3(4)$, then G has an element of order 21 by Lemma 2.12 as the Sylow 3-subgroup of H is of order 3 or 9, a contradiction;

If K/H is isomorphic to A_9, then $G/H \cong A_9$ or $A_9 \cdot 2$, and thus H is a group of order 4 or 8. If H is not an elementary abelian group of order 8, then G has an element of order 14 by Lemma 2.12, a contradiction. If H is an elementary abelian group of order 8, then $G/H \cong A_9$ and $scs(A_9) = 2^3 \cdot 3 \cdot 7$. Note that $scs(G) = scs(L) = 3^2 \cdot 7$, then $scs(G/H) > scs(G)$, a contradiction. Hence K/H is not isomorphic to A_9.

Now we have that $K/H \cong S_6(2)$, which concludes $G \cong S_6(2)$.

Remark 3.2 (a) *It is an interesting fact that A_{10} is unique one having connected prime graph among simple K_4-groups. By our approach, we successfully characterize A_{10} by its order and smallest conjugacy class sizes greater than 1.*

(b) *In the proofs (I), (III) and (IV), there is a crucial step to show that the prime graph of G is non-connected by a special conjugacy class size. But some simple groups which have the same order and the same one conjugacy class size are not isomorphic. For example, A_8 and $L_3(4)$ have the same order and the same second largest class size by [7], but they are not isomorphic. In fact, the following counter example is true:*

A_8 and $L_3(4)$ are of order $2^6 \cdot 3^2 \cdot 5 \cdot 7$, moreover and $slcs(A_8) = slcs(L_3(4)) = 2^6 \cdot 3^2 \cdot 5$.

Lemma 3.3 *Let G be a group and L one of the simple K_4-groups with $\pi(L) = \{2, 3, 5, 11\}$, then one of the following holds:*

(I) *If L is isomorphic to one of M_{12} and $L_2(11)$, then $G \cong L$ if and only if $|G| = |L|$ and $scs(G) = scs(L)$;*

(II) *If $L \cong M_{11}$, then $G \cong L$ if and only if $|G| = |L|$ and $lcs(G) = lcs(L)$;*

(III) *If $L \cong U_5(2)$, then $G \cong L$ if and only if $|G| = |L|$, $lcs(G) = lcs(L)$, and $scs(G) = scs(L)$.*

Proof. Since the cases of M_{11} and M_{12} have been done in [14], (II) follows. Moreover it is enough to show the sufficiency of the remaining groups L.

(I) If $L \cong L_2(11)$, then $|G| = |L| = 2^2 \cdot 3 \cdot 5 \cdot 11$ and $scs(G) = scs(L) = 5 \cdot 11$ by [7]. It follows that 5 and 11 do not divide the order of $Z(G)$ because of $Z(G) \leq C_G(x)$ for any $x \in G$. Let $\overline{G} = G/Z(G)$. Then the order of Sylow $p-$subgroup of G is less than $scs(G)$ for any prime $p \in \pi(G)$. By (a) of Lemma 2.1, we know that $M \trianglelefteq \overline{G} \leq \mathrm{Aut}(M)$, where $M = Soc(\overline{G}) = S_1 \times S_2 \times \cdots \times S_k$ and S_i is a direct product of some isomorphic simple groups for $i = 1, 2, \ldots, k$.

Since $|G| = 2^2 \cdot 3 \cdot 5 \cdot 11$, M may be isomorphic to A_5 or $L_2(11)$ by [7]. If $M \cong A_5$, then $\overline{G}$ is isomorphic to A_5 or $A_5 \cdot 2$ by $|\mathrm{Out}(A_5)| = 2$, and thus 11 is a prime divisor of $|Z(G)|$, a contradiction.

Hence $M \cong L_2(11)$, and $G \cong L_2(11)$ as desired.

(III) If $L \cong U_5(2)$, then $|G| = 2^{10} \cdot 3^5 \cdot 5 \cdot 11$, $lcs(G) = 2^{10} \cdot 3^5 \cdot 5$ and $scs(G) = 3 \cdot 5 \cdot 11$ by the hypothesis and [7]. Hence 11 is an isolated vertex in the prime graph of G and $t(G) \geqslant 2$.

Suppose that G is a Frobenius group with kernel H and complement K. If $11 \in \pi(H)$, then by Lemma 2.8, K has an element of order 5 acts trivially on H. It follows that 3 and 11 are connected in the prime graph of G by Lemma 2.12, a contradiction. If $11 \in \pi(K)$, then the $5-$Sylow subgroup of H is of order 5 and normal in G by nilpotency of H. Hence G has an element of order 55 by Lemma 2.12, a contradiction. Therefore G is not a Frobenius group.

Assume that G is a 2-Frobenius group. Then G has a normal series $1 \lhd H \lhd K \lhd G$ such that $\pi(K/H) = \pi_2=\{11\}$, $\pi(H) \cup \pi(G/K) = \pi_1=\{2,\ 3,\ 5\}$, and $|G/K|\big||\mathrm{Aut}(K/H)| = 10$. So $|G/K| = 2,\ 5$, or 10.

If $|G/K| = 2$, then the Sylow 5-subgroup H_5 of H is normal in G. Hence 5 and 11 are connected in the prime graph of G by Lemma 2.12, a contradiction.

If $|G/K| = 10$, then the Sylow 2-subgroup H_2 of H is of order 2^9 and normal in G. Considering the characteristic subgroup $\Omega_1(Z(H_2))$ of H_2, which is normal in G. Since 11 is prime to $2^i - 1$ for $i = 0, \ldots, 9$, $|\mathrm{Aut}(\Omega_1(Z(H_2)))|$ and 11 are co-prime by Lemma 2.11. It follows that G has an element of order 22 by Lemma 2.12, a contradiction.

If $|G/K| = 5$. Then the Sylow 2-subgroup H_2 and Sylow 3-subgroup H_3 of H are normal in G. It is difficult to reach a contradiction according to method above. We need to apply the hypothesis

$scs(G) = 3 \cdot 5 \cdot 11$. If one of H_2 and H_3 is not an elementary abelian subgroup, then we can get a contradiction by Lemma 2.12 as above. If H_2 and H_3 are two elementary abelian subgroups, then H is an ableian subgroup, and so there is an element y of order 2 in H satisfying that $|cl_G(y)|$ is less than $5 \cdot 11$, a contradiction to $scs(G) = 3 \cdot 5 \cdot 11$. Thus G is not a 2-Frobenius group.

Now G has a normal series $1 \subseteq H \subseteq K \subseteq G$ by Lemma 2.7, where H is a nilpotent π_1-group, K/H is a simple group and G/K is a π_1-group such that $|G/K|$ divides the order of the outer automorphism group of K/H. One has that $\{11\}$ is a prime graph component of K/H. By checking the order of K/H, we know that K/H is isomorphic to one of the following groups:

$$L_2(11),\ M_{11},\ M_{12},\ U_5(2).$$

If K/H is isomorphic to $L_2(11)$. Then $(|G/K|, 3) = 1$ since $|Out(L_2(11))| = 2$, so the Sylow 3-subgroup H_3 of H is of order 3^4 and normal in G. Hence G has an element of order 33 by Lemma 2.12, which is a contradiction.

If K/H is isomorphic to M_{11} or M_{12}, then, by [7], G has an element of order 33 by Lemma 2.12 as the Sylow 3-subgroup of H is of order 3^3 or 3^2, and normal in G, a contradiction.

Therefore, K/H is isomorphic to $U_5(2)$, so that $G \cong U_5(2)$ by $|G| = |U_5(2)|$, as claimed.

Lemma 3.4 *Let G be a group and L one of the simple K_4-groups with $\pi(L) = \{2, 3, 5, 13\}$, then one of the following holds:*

(I) *If $L \cong U_3(4)$, then $G \cong L$ if and only if $|G| = |L|$ and $lcs(G) = lcs(L)$;*

(II) *If $L \cong L_2(25)$, then $G \cong L$ if and only if $|G| = |L|$ and $scs(G) = scs(L)$;*

(III) *If L is isomorphic to one of $S_4(5)$ and ${}^2F_4(2)'$, then $G \cong L$ if and only if $|G| = |L|$ and $slcs(G) = slcs(L)$;*

(IV) *If $L \cong L_4(3)$, then $G \cong L$ if and only if $|G| = |L|$ and $tlcs(G) = tlcs(L)$.*

Proof. It is enough to show the sufficiency of the lemma. By [7] and the hypothesis, we have the following statements:

If $L \cong L_2(25)$, then the order of Sylow $p-$subgroup of G is less than $scs(G)$ for any prime $p \in \pi(G)$;

If L is isomorphic to one of $U_3(4),\ S_4(5),\ L_4(3)$, and ${}^2F_4(2)'$, then 13 is an isolated vertex of the prime graph of G, so that $t(G) \geq 2$.

In the next, we can finish the proof by similar reasoning as in proofs of Lemma 3.1 and 3.3, here we omit the process.

Lemma 3.5 *Let G be a group and L one of the simple K_4-groups with $\pi(L) = \{2, 3, 7, 13\}$, then one of the following holds:*

(I) *If L is isomorphic to one of $L_2(27)$ and $G_2(3)$, then $G \cong L$ if and only if $|G| = |L|$ and $lcs(G) = lcs(L)$;*

(II) *If $L \cong L_2(13)$, then $G \cong L$ if and only if $|G| = |L|$ and $scs(G) = scs(L)$;*

(III) *If $L \cong {}^3D_4(2)'$, then $G \cong L$ if and only if $|G| = |L|$ and $slcs(G) = slcs(L)$.*

Proof. Similar to Lemma 3.4, we have the following:

If L is isomorphic to one of $L_2(27)$ and ${}^3D_4(2)'$, then $\{13\}$ is a prime graph component of G, and $t(G) \geq 2$;

If $L \cong G_2(3)$, then $\{7\}$ is a prime graph component of G, and $t(G) \geq 2$;

If $L \cong L_2(13)$, then 7, $13 \notin \pi(Z(G))$, and the order of Sylow $p-$subgroup of G is less than $scs(G)$ for any prime $p \in \pi(G)$.

Therefore, it is easy to prove (I) and (III) by Lemma 2.7, and (II) by (a) of Lemma 2.1. The details are omitted.

Lemma 3.6 *Let G be a group, and L one of the simple K_4-groups with $\pi(L) = \{2, 3, 5, 17\}$, then one of the following holds:*

(I) *If $L \cong L_2(16)$, then $G \cong L$ if and only if $|G| = |L|$ and $scs(G) = scs(L)$;*

(II) *If $L \cong S_4(4)$, then $G \cong L$ if and only if $|G| = |L|$ and $scs(G) = scs(L)$.*

Proof. Similar to Lemma 3.4 and 3.5, we have the following:

If $L \cong L_2(16)$, then $|G| = 2^4 \cdot 3 \cdot 5 \cdot 17$ and $scs(G) = 2^4 \cdot 3 \cdot 5$ by the hypothesis, and thus $\{17\}$ is a prime graph component of G and $t(G) \geqslant 2$;

If $L \cong S_4(4)$, then $|G| = 2^8 \cdot 3^2 \cdot 5^2 \cdot 17$ and $tlcs(G) = 2^8 \cdot 3^2 \cdot 5^2$, and also $\{17\}$ is a prime graph component of G and $t(G) \geqslant 2$.

Hence, it is easy to prove (I) and (II) by using a similar way as in Lemma 3.1, and the details are omitted.

Lemma 3.7 *Let G be a group and L one of $L_2(31)$ and $L_3(5)$. Then $G \cong L$ if and only if $|G| = |L|$ and $scs(G) = scs(L)$.*

Proof. The lemma can be proved by similar reasoning as in the proof for (II) of Lemma 3.1.

Lemma 3.8 *Let G be a group, and L one of the simple K_4-groups with $\pi(L) = \{2, 3, 5, 41\}$, then one of the following holds in the class of simple K_4 groups:*

(I) *If $L \cong L_2(81)$, then $G \cong L$ if and only if $|G| = |L|$ and $scs(G) = scs(L)$;*

(II) *If $L \cong S_4(9)$, then $G \cong L$ if and only if $|G| = |L|$ and $flcs(G) = flcs(L)$.*

Proof. The necessity of is obvious, and so we only prove the sufficiency. By the hypothesis and [7], we have the following:

If $L \cong L_2(81)$, then $|G| = 2^4 \cdot 3^4 \cdot 5 \cdot 41$ and $scs(G) = 3^4 \cdot 41$. It follows that the order of Sylow $p-$subgroup of G is less than $scs(G)$ for any $p \in \pi(G)$.

If $L \cong S_4(9)$, then $|G| = 2^8 \cdot 3^8 \cdot 5^2 \cdot 41$ and $flcs(G) = 2^8 \cdot 3^8 \cdot 5^2$, and so $\{41\}$ is a prime graph component of G and $t(G) \geqslant 2$.

We can apply (a) of Lemma 2.1 to prove (I), and apply Lemma 2.7 to prove (II). The details are omitted for the processes are similar.

Lemma 3.9 *Let G be a group and L one of $L_3(7)$ and $U_3(8)$. Then $G \cong L$ if and only if $|G| = |L|$ and $scs(G) = scs(L)$.*

Proof. Because for each case of L the order of Sylow $p-$subgroup of G is less than $scs(G)$ for any $p \in \pi(G)$, we can prove this lemma with similar approach as the proof of (II) in Lemma 3.1. Hence we omit the detail.

Lemma 3.10 *Let G be a group and L a simple K_4-group. Then one of the following holds:*

(I) *If L is one of $L_3(8), L_3(17), U_3(7), U_3(9), Sz(32)$ and $L_2(r)$, where $r \geq 19$ is an odd prime $\neq 31$ and satisfies*

$$r^2 - 1 = 2^a \cdot 3^b \cdot v^c$$

with $a,\ b,\ c \geq 1$ and a prime $v > 3$, then $G \cong L$ if and only if $|G| = |L|$ and $scs(G) = scs(L)$;

(II) *If L is one of $Sz(8)$, $L_2(2^m)$, where $m \geq 5$ satisfies*

$$2^m - 1 = u,\ 2^m + 1 = 3t^b, u \text{ and } t \text{ are primes},\ t > 3,\ b \geq 1,$$

and $L_2(3^n)$, where $n \geq 5$ satisfies

$$3^n - 1 = 2u,\ 3^n + 1 = 4t^b, u \text{ and } t \text{ are primes}, b \geq 1,$$

then $G \cong L$ if and only if $|G| = |L|$ and $lcs(G) = lcs(L)$.

(III)*If $L \cong L_2(3^n)$, where $n \geq 5$ satisfies*

$$3^n - 1 = 2u^c,\ 3^n + 1 = 4t, u \text{ and } t \text{ are primes}, c > 1,$$

then $G \cong L$ if and only if $|G| = |L|$ and $u^c \big| |cl_G(w)|$ for every element w of order $p \in \{2,\ 3,\ t\}$ of G.

Proof. Similar to the preceding lemmas, we need only to prove the sufficiency of this lemma. By Lemma 2.4, we know that every simple K_4-group in this lemma is characterized by the set of prime divisors of its order. Next we prove the sufficiency of the lemma case by case.

Case 1. If L is one of $L_3(8)$, $L_3(17)$, $U_3(7)$, $U_3(9)$, and $Sz(32)$, then for each case of G, we see that the order of Sylow $p-$subgroup of G is smaller than $scs(G)$ for any prime $p \in \pi(G)$ by the hypothesis and [7]. Hence we can deal this lemma with similar method in the (II) of Lemma 3.1 such that we omit it.

Case 2. If L is isomorphic to $Sz(8)$, then by the hypothesis, $|G| = 2^6 \cdot 5 \cdot 7 \cdot 13$ and $lcs(G) = 2^6 \cdot 7 \cdot 13$. It follows that $\{5\}$ is a prime graph component of G and $t(G) \geqslant 2$. Analogous to (I) of Lemma 3.1, we can apply the Lemma 2.7 to deal with this case such that we omit it.

Case 3. If $L \cong L_2(r)$, where r is an odd prime with $r \equiv 1(mod\ 4)$ and satisfies

$$r^2 - 1 = 2^a \cdot 3^b \cdot v^c$$

with $a, b, c \geq 1$, and $r \geq 19$ but $\neq 31$, then $|G| = |L| = r(r-1)(r+1)/2$ and $scs(G) = scs(L) = (r-1)(r+1)/2$ by the hypothesis, from which $\{r\}$ is a prime graph component of G and $t(G) \geqslant 2$.

Suppose that G is a Frobenius group with kernel H and complement K. If $r \in \pi(H)$, then H is a Sylow r-subgroup of G, and $\pi(K) = \{2,\ 3,\ v\}$ by Lemma 2.8. Since $r \equiv 1(mod\ 4)$ and $(r+1, r-1) = 2$, we get that $4|(r-1)$ and $((r+1)/2, r-1) = 1$. Thus, for any prime divisor p of $(r+1)/2$, r is connected to p in prime graph of G by Lemma 2.12, a contradiction. If $r \in \pi(K)$, then by Lemma 2.8, K is a Sylow r-subgroup of G and $\pi(H) = \{2,\ 3,\ v\}$, and thus $|H_2| = 2^a, a \geq 2$. Hence $2^a \leq r - 1 < r$, which implies that $(r, |\mathrm{Aut}(H_2)|) = 1$ by Lemma 2.11, consiquently 2 and r connected in the prime graph of G by Lemma 2.12, a contradiction. Hence G is not a Frobenius group.

Assume that G is a 2-Frobenius group. By Lemma 2.9, we have that G has a normal series $1 \lhd H \lhd K \lhd G$ such that $\pi(K/H) = \{r\} = \pi_2$, $\pi(H) \cup \pi(G/K) = \pi_1$, and $|G/K|\big|(r-1)$. Now, we have that K/H is of order r and $\pi(r + 1/2) \subseteq \pi(H)$. Hence $(r, |\mathrm{Aut}(H_p)|) = 1$ for any prime $p \in \pi(r + 1/2)$, r can be connected to p by Lemma 2.12, a contradiction. Therefore G is not a 2-Frobenius group either.

Now, G has a normal series $1 \subseteq H \subseteq K \subseteq G$, where H is a nilpotent π_1-group, K/H is a simple group, G/K is a π_1-group such that $|G/K|$ divides the order of the outer automorphism group of K/H, and $\{r\}$ is a prime graph component of K/H. By $\pi(K/H) \subseteq \pi(G)$, K/H is a simple K_3-group or K_4-group.

Since $r \geq 19$, it is impossible that K/H is a simple K_3-group. Therefore K/H is a simple K_4-group, which implies that $\pi(K/H) = \pi(G) = \pi(L)$. Hence K/H must be isomorphic to L such that $G \cong L$ by $|G| = |L|$, as desired.

Case 4. If $L \cong L_2(r)$, where r is an odd prime with $r \equiv 3(mod\ 4)$ and satisfies

$$r^2 - 1 = 2^a \cdot 3^b \cdot v^c,$$

where $a,\ b,\ c \geq 1$, and $r \geq 19$ but $\neq 31$. Then $|G| = |L| = r(r-1)(r+1)/2$ and $scs(G) = scs(L) = r(r-1)/2$ by the hypothesis. Set $x \in G$ such that $scs(G) = |cl_G(x)| = r(r-1)/2$. Then for any prime

$p \in \pi(r(r-1)/2)$, $p \notin \pi(Z(G))$ because of $Z(G) \leq C_G(x)$ for any $x \in G$. Consider $\overline{G} = G/Z(G)$. Since the order of Sylow $q-$subgroup of G is less than $scs(G)$ for any prime $q \in \pi(G)$, we know that $M \trianglelefteq \overline{G} \leq$Aut$(M)$ by (a) of Lemma 2.1, where $M = Soc(\overline{G}) = S_1 \times S_2 \times \cdots \times S_k$ and S_i is a direct product of some isomorphic simple groups for $i = 1, 2, \ldots, k$.

It is clear that $2,\ 3 \in \pi(M)$. If r is not in $\pi(M)$, then $r \in \pi(\mathrm{Out}(M))$, and M is s a direct product of some simple K_3-groups. Note that Out(M)=Out$(S_1)\times \cdots \times$Out$(S_k)$ by Lemma 2.10. Hence for some $i \in \{1,\ 2, \ldots, k\}$, r divides the order of Out(S_i). Suppose that S_i is a direct of t_i isomorphic simple K_3 groups S. By Lemma 2.10 again, $|\mathrm{Aut}(S_i)|$=$|\mathrm{Aut}(S)|^{t_i} \cdot t_i!$, and then $t_i \geqslant r$ by [7]. Consequently, 2^{2r} divides the order of G, a contradiction. Therefore $r \in \pi(M)$.

If v is not in $\pi(M)$, then M is a simple K_3-group with $\pi(M) = \{2,\ 3,\ r\}$. Hence $r = 5,\ 7,\ 13$, or 17, contradicting to $r \geq 19$. Therefore $v \in \pi(M)$, and so $\pi(M) = \{2,\ 3,\ v,\ r\}$.

Assume that $r \in \pi(S_i)$ for some $i \in \{1, 2, \ldots, k\}$. Then v belongs to $\pi(S_i)$. Otherwise, S_i is a simple K_3-group with $\pi(S_i) = \{2,\ 3,\ r\}$, we come to a contradiction by similar reasoning above. It follows that S_i is a simple K_4-group with $\pi(S_i) = \{2,\ 3,\ v,\ r\}$, which implies that $S_i \cong L$ by Lemma 2.4. Hence $k = 1$, and $G \cong L$ by $|G| = |L|$, which concludes the lemma.

Case 5. Assume $L \cong L_2(2^m)$, where $m \geq 5$ satisfies

$$2^m - 1 = u,\ 2^m + 1 = 3t^b,$$

where u and t are primes, $t > 3, b \geq 1$. Then by the hypothesis, $|G| = |L| = 2^m(2^m-1)(2^m+1)$ and $lcs(G) = lcs(L) = 2^m(2^m+1)$. Hence $\{u\}$ is a prime graph component of G and $t(G) \geqslant 2$.

Suppose that G is a Frobenius group with kernel H and complement K. If $u \in \pi(H)$, then H is a Sylow u-subgroup of G and $\pi(K) = \{2,\ 3,\ t\}$ by Lemma 2.8. Since $u-1 = 2^m-2 = 3t^b-3 = 3(t^b-1)$, we get that $(t,\ u-1) = 1$, and so u and t are connected in the prime graph of G, a contradiction. If $u \in \pi(K)$, then K is a Sylow u-subgroup of G and $\pi(H) = \{2,\ 3,\ v\}$ by Lemma 2.8. Since $|H_3| = 3$ and u is odd, K can act trivially on H_3, we get a contradiction by Lemma 2.12. Hence G is not a Frobenius group.

Assume that G is a 2-Frobenius group. By Lemma 2.9, we have that G has a normal series $1 \lhd H \lhd K \lhd G$ such that $\pi(K/H) = \{u\} = \pi_2$, $\pi(H) \cup \pi(G/K) = \pi_1$, and $|G/K|\big|(u-1) = 3(t^b-1)$. Thus K/H is of order u and $|H_t| = t^b,\ b \geq 1$. Since $t^b - 1 < u$ and $t < u$, it is impossible by Lemma 2.12. Therefore G is not a 2-Frobenius group.

Now G has a normal series $1 \subseteq H \subseteq K \subseteq G$, where H is a nilpotent π_1-group, K/H is a simple group, G/K is π_1-group such that $|G/K|\big||\mathrm{Out}(K/H)|$ and $\{u\}$ is a prime graph component of K/H. By $\pi(K/H) \subseteq \pi(G)$, K/H may be a simple K_3-group or K_4-group.

It is clear that K/H cannot be a simple K_3-group by $m \geq 5$. So K/H is a simple K_4-group, which implies that $\pi(K/H) = \pi(G) = \pi(L)$. Hence $K/H \cong L$ so that $G \cong L$ by $|G| = |L|$, as claimed.

Case 6. If $L \cong L_2(3^n)$, where $n \geq 5$ satisfies

$$3^n - 1 = 2u, 3^n + 1 = 4t^b,$$

where u and t are odd primes, $b \geq 1$. Then $|G| = |L| = 3^n(3^n-1)(3^n+1)/2$ and $lcs(G) = lcs(L) = 3^n(3^n+1)$ by the hypothesis. It follows that $u > t \geq 5$, and $\pi(G) = \{2,\ 3,\ t,\ u\}$, and then $\{u\}$ is a prime graph component of G and $t(G) \geqslant 2$.

Let G is a Frobenius group with kernel H and complement K. If $u \in \pi(H)$, then H is a Sylow u-subgroup of G and $\pi(K) = \{2,3,t\}$ by Lemma 2.8. By $u+1 = 2t$, it follows that $(t, u-1) = 1$, which implies that u is connected to t in prime graph of G by Lemma 2.12, a contradiction. If $u \in \pi(K)$, then by Lemma 2.8, K is a Sylow u-subgroup of G and $\pi(H) = \{2,3,t\}$. Since $|H_2| = 2^2$ and $u > 5$ is an odd prime, it follows that u and 2 are connected in the prime graph of G by Lemma 2.12, a contradiction. Hence G is not a Frobenius group.

Assume that G is a 2-Frobenius group. By Lemma 2.9, we have that G has a normal series $1 \lhd H \lhd K \lhd G$ such that $\pi(K/H) = \{u\} = \pi_2$, $\pi(H) \cup \pi(G/K) = \pi_1 = \{2,\ 3,\ t\}$, and $|G/K| \big| (u-1)$. It follows that K/H is of order u and $|H_t| = t$. By $2(t-1) = u-1$, u and t are connected by Lemma 2.12, a contradiction. Therefore G is not a 2-Frobenius group.

Hence G has a normal series $1 \subseteq H \subseteq K \subseteq G$, where H is a nilpotent π_1-group, K/H is a simple group, G/K is π_1-group such that $|G/K|$ divides the order of the outer automorphism group of K/H, and $\{u\}$ is a prime graph component of K/H. By $\pi(K/H) \subseteq \pi(G)$, We have that K/H is a simple K_3-group or K_4-group.

Assume that K/H is a simple K_3-group. Since $u > t \geq 5$ and $4 \parallel |G|$, K/H is isomorphic to A_5. It follows that $t = 5$ and $u \in \pi(H)$. Thus K has an element of order 5. But $(5, |\mathrm{Aut}(H_u)|) = (t, u-1) = 1$, it follows that G has an element of order $5u$ by Lemma 2.12, a contradiction.

Therefore K/H is a simple K_4-group. It follows that $\pi(K/H) = \pi(G) = \pi(L)$, which implies that $K/H \cong L$, and $G \cong L$ by $|G| = |L|$.

Case 7. If $L \cong L_2(3^n)$, where $n \geq 5$ satisfies

$$3^n - 1 = 2u^c,\ 3^n + 1 = 4t,$$

where u and t are odd primes, $c > 1$, then by hypothesis, $|G| = |L| = 3^n(3^n-1)(3^n+1)/2$ and $\{u\}$ is a prime graph component of G, and thus $t(G) \geqslant 2$.

Suppose that G is a Frobenius group with kernel H and complement K. If $u \in \pi(H)$, then H is a Sylow u-subgroup of G and $\pi(K) = \{2,\ 3,\ t\}$ by Lemma 2.8. By $u^c + 1 = 2t$, it follows that $(t, u^i - 1) = 1$ for $i = 1, 2, \ldots, c$, which implies that u connects to t in prime graph of G by Lemma 2.11 and 2.12, a contradiction. If $u \in \pi(K)$, then by Lemma 2.8, K is a Sylow u-subgroup of G and $\pi(H) = \{2,\ 3,\ t\}$. But $|H_2| = 2^2$ and $u \geq 5$ is an odd prime, it is impossible by Lemma 2.12, a contradiction. Hence G is not a Frobenius group.

Assume that G is a 2-Frobenius group. By Lemma 2.9, we have that G has a normal series $1 \lhd H \lhd K \lhd G$ such that $\pi(K/H) = \{u\} = \pi_2$, $\pi(H) \cup \pi(G/K) = \pi_1 = \{2, 3, t\}$, and $|G/K| \big| u^{c-1}(u-1)$. It follows that K/H is of order u^c and $|H_t| = t$. By $2(t-1) = u^c - 1$, u can be connected to t because of $(u, |\mathrm{Aut}(H_t)|) = (u, t-1) = 1$, a contradiction. Therefore G is not a 2-Frobenius group.

Thus G has a normal series $1 \subseteq H \subseteq K \subseteq G$, where H is a nilpotent π_1-group, K/H is a simple group, G/K is π_1-group such that $|G/K|$ divides the order of the outer automorphism group of K/H, and $\{u\}$ is a prime graph component of K/H. By $\pi(K/H) \subseteq \pi(G)$, K/H is a simple K_3-group or K_4-group.

If K/H is a simple K_3-group, then by $t > u \geq 5$ and $4 \parallel |G|$, K/H is isomorphic to A_5. It follows that $u = 5$ and $t \in \pi(H)$. Then there exists an element of order 5 of K can act trivially on H_t by $(5, |\mathrm{Aut}(H_t)|) = (5, t-1) = 1$, a contradiction.

Therefore K/H is a simple K_4-group. It follows that $\pi(K/H) = \pi(G) = \pi(L)$, which implies that K/H must be isomorphic to L. Hence $G \cong L$ by $|G| = |L|$, as claimed.

Proof of the Theorem 1.2. The Theorem follows from Lemma 3.1 to 3.10.

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