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Sign-changing solutions for discrete second-order periodic boundary value problems

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Abstract

In this paper, we study the existence of sign-changing solutions and positive solutions for second-order nonlinear difference equations on a finite discrete segment with periodic boundary condition provided that the nonlinearity is asymptotically linear at infinity. The critical point theory and variational methods are employed to discuss this problem.

Keywords: Periodic boundary value problem; Difference equation; Asymptotically linear; Invariant sets of descending flow; Sign-changing solution

2010 *Mathematics Subject Classification:* 39A10

1. Introduction

We are concerned with the existence of sign-changing solutions and positive solutions for the following periodic boundary value problem (BVP for short)

$$\begin{cases} -\Delta[p(t-1)\Delta u(t-1)]+q(t)u(t)=f(t,u(t)), & t\in[1,T], \\ u(0)=u(T), \qquad \Delta u(0)=\Delta u(T), \end{cases} \tag{1.1}$$

where T is a fixed positive integer, $[1,T]:=\{1,2,\cdots,T\}$; $f:[1,T]\times\mathbf{R}\to\mathbf{R}$ is continuous in the second variable; $p:[0,T]\to(0,+\infty)$ with $p(0)=p(T)$, $q:[1,T]\to[0,+\infty)$ with $\max_{t\in[1,T]}q(t)>0$; Δ is the forward difference operator defined by $\Delta u(t)=u(t+1)-u(t)$.

Many authors have contributed to the study of Problem (1.1). They made use of various methods to investigate (1.1) and obtained some interesting results. For example, Atici and Guseinov [3], using fixed point theorem and the properties of the Green's function, obtained the existence of positive solutions of BVP (1.1). Based on the methods of upper and lower solutions, Atici and Cabada [2] investigated the existence and uniqueness of periodic solutions of BVP (1.1). By minimax principle, Guo and Yu studied the existence of periodic solutions to BVP (1.1) with $p\equiv 1$ and $q\equiv 0$, where the nonlinearity is of sublinear growth and superlinear growth at infinity in [8, 9], respectively. By applying the critical point theory, there are also some existence results for periodic solutions of BVP (1.1), see [10, 12, 13, 19, 21]. Very recently, by considering the sublinearity, superlinearity of f and using the fixed point index theory, He and Xu [11] obtained the existence, multiplicity and nonexistence results for positive solutions of BVP (1.1). However, to the authors' knowledge, there were few papers that considered the sign-changing solutions for BVP (1.1). For results on nonlinear difference equations with other boundary

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conditions, see [1, 6] and reference therein.

Invariant sets of descending flow defined by a pseudogradient vector field of a functional in a Banach space plays an important role in the existence of sign-changing solutions. For the properties of invariant sets of descending flow and applications, one can refer to [15, 17, 18]. The Dirichlet boundary value problem

$$\begin{cases} -\Delta u = f(x,u), & x \in \Omega, \\ u\big|_{\partial\Omega} = 0 \end{cases}$$

was studied in [15] by invariant sets of descending flow, and sign-changing solutions was obtained. In this direction, one can find more results in [4, 5, 7]. Still by means of invariant sets of descending flow, Zhang and Perera [22] got sign-changing solutions of the nonlocal Kirchhoff type problem

$$\begin{cases} -(a+b\int_{\Omega}|\nabla u|^2)\Delta u = f(x,u) & \text{in } \Omega, \\ u = 0 \quad \text{on } \partial\Omega, \end{cases}$$

under the 4-sublinear case, asymptotically 4-linear case and 4-superlinear case. For the recent progress of such problems, one can refer to [16, 20]. It is natural for us to think that invariant sets of descending flow may be also applied to prove the existence of sign-changing solutions of difference equations. The main purpose of this paper is to give some sufficient conditions for the existence of sign-changing solutions and positive solutions to BVP (1.1) via invariant sets of descending flow and variational techniques. Let us point out that the nonlinearity satisfies locally Lipschitz continuity in [7, 16, 20, 22] or Ambrosett-Rabinowitz type condition in [4, 5, 15], which are unnecessary in our results. Moreover, comparing the results in [3, 11] with the results of this paper, we allow the nonlinearity to change sign.

To state our main results, we define the linear eigenvalue problem

$$\begin{cases} -\Delta[p(t-1)\Delta u(t-1)] + q(t)u(t) = \lambda u(t), & t \in [1,T], \\ u(0) = u(T), \quad \Delta u(0) = \Delta u(T). \end{cases} \tag{1.2}$$

Denote by $\lambda_1 < \lambda_2 \le \lambda_3 \le \cdots \le \lambda_T$ the eigenvalues of (1.2) and by $\varphi_1, \varphi_2, \cdots, \varphi_T$ corresponding eigenfunctions. We will see in the next section that $\lambda_1, \lambda_2, \cdots, \lambda_T$ are also the eigenvalues of an appropriate matrix.

We assume that

(H₁) $f^0 < \lambda_1$, where $f^0 = \max\limits_{t\in[1,T]} \limsup\limits_{x\to 0} \left|\dfrac{f(t,x)}{x}\right|$.

(H₂) $\lim\limits_{|x|\to\infty} \dfrac{f(t,x)}{x} = l$ uniformly in $t \in [1,T]$ where $l \in (0,+\infty)$ is a constant, or $l = +\infty$ with $s > 2$ and $C > 0$ such that

$$|f(t,x)| \le C(1+|x|^{s-1}), \quad t \in [1,T], \ x \in \mathbf{R}.$$

(H₃) Either

(i) $\lim\limits_{|x|\to\infty}[2F(t,x) - xf(t,x)] = +\infty$, uniformly for $t \in [1,T]$,

or

(ii) $\lim\limits_{|x|\to\infty}[2F(t,x) - xf(t,x)] = -\infty$, uniformly for $t \in [1,T]$, where $F(t,x)=\int_0^x f(t,s)\mathrm{d}s$.

Here are the main results.

Theorem 1.1. Assume that (H_1) and (H_2). Suppose that $l>\lambda_2$. Then the following hold:

(i) If $l\in(\lambda_2,+\infty]$ is not an eigenvalue of (1.2), then Problem (1.1) has at least a positive solution, a negative solution and a sign-changing solution.

(ii) If condition (H_3) is satisfied, then the conclusion of (i) holds even if l is an eigenvalue of (1.2).

Theorem 1.2. Assume that $\limsup_{x\to 0} f(t,x)/x<\lambda_1$ and $\liminf_{|x|\to\infty} f(t,x)/x>\lambda_1$ uniformly for $t\in[1,T]$. Then Problem (1.1) has at least a positive solution and a negative solution.

The rest of the paper is arranged as follows. Section 2 presents some preliminaries. In Sections 3 and 4, we prove the existence of sign-changing solutions and positive solutions of Problem (1.1), respectively.

2. Preliminaries

Let $H=\{u:u=(u(0),u(1),\cdots,u(T),u(T+1)),u(0)=u(T),u(1)=u(T+1)\}$. For any given $m\geq 0$ we define new inner product of H as follows

$$\langle u,v\rangle_m=\sum_{t=1}^{T}[p(t-1)\Delta u(t-1)\Delta v(t-1)+(q(t)+m)u(t)v(t)].$$

The inner product induces the norm

$$\|u\|_m=(\sum_{t=1}^{T}[p(t-1)|\Delta u(t-1)|^2+(q(t)+m)|u(t)|^2])^{\frac{1}{2}}.$$

Let E be the T-dimensional Hilbert space. We denote by $(\cdot,\cdot)$ and $\|\cdot\|$ the usual inner product and the usual norm in E. It is easy to see that H is isomorphic to E. In the following, when we say $u=(u(1),\cdots,u(T))\in E$, we always imply that $u=(u(0),u(1),\cdots,u(T),u(T+1))\in H$. Clearly the norm $\|\cdot\|_m$ is equivalent to the norm $\|\cdot\|$. Let $u^+=\max\{u,0\}$, $u^-=\min\{u,0\}$. Then for any $u\in E$, $\langle u^+,u^-\rangle_m\geq 0$.

Now define the functional $J:E\to\mathbf{R}$ as

$$J(u)=\frac{1}{2}\sum_{t=1}^{T}[p(t-1)|\Delta u(t-1)|^2+q(t)|u(t)|^2]-\sum_{t=1}^{T}F(t,u(t)). \tag{2.1}$$

Note that the norm $\|\cdot\|_m$ is a part of the functional J, which makes it convenient to estimate J. By a direct computation, we can rewrite $J(u)$ as

$$J(u)=\frac{1}{2}((P+Q)u,u)-\sum_{t=1}^{T}F(t,u(t)) \tag{2.2}$$

for any $u=(u(1),u(2),\cdots,u(T))^{\tau}\in E$, where α^{τ} is the transpose of the vector α on E and P, Q are $T\times T$ identity matrices:

$$P=\begin{pmatrix} p(T)+p(1) & -p(1) & 0 & \cdots & 0 & -p(T) \\ -p(1) & p(1)+p(2) & -p(2) & \cdots & 0 & 0 \\ 0 & -p(2) & p(2)+p(3) & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & p(T-2)+p(T-1) & -p(T-1) \\ -p(T) & 0 & 0 & \cdots & -p(T-1) & p(T-1)+p(T) \end{pmatrix},$$

$$Q=\begin{pmatrix} q(1) & 0 & 0 & \cdots & 0 & 0 \\ 0 & q(2) & 0 & \cdots & 0 & 0 \\ 0 & 0 & q(3) & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & q(T-1) & 0 \\ 0 & 0 & 0 & \cdots & 0 & q(T) \end{pmatrix}.$$

We consider the following BVP

$$\begin{cases} -\Delta[p(t-1)\Delta u(t-1)]+(q(t)+m)u(t)=h(t), & t\in[1,T], \\ u(0)=u(T), \quad \Delta u(0)=\Delta u(T), \end{cases} \tag{2.3}$$

where $h:t\in[1,T]\to\mathbf{R}$ and $m\geq 0$. It is easy to see that BVP (2.3) is equivalent to the system of linear algebra equations $(P+Q+mI)u=h$, where I is the $T\times T$ identity matrix. By matrix theory, $P+Q$ is positive definite. Then the unique solution of BVP (2.3) is

$$u=(P+Q+mI)^{-1}h, \tag{2.4}$$

and $\lambda_1,\lambda_2,\cdots,\lambda_T$ are also the eigenvalues of matrix $P+Q$. It follows from Lemma 2.3 in [11] that the first eigenvalue $\lambda_1>0$ is simple and the corresponding eigenfunction φ_1 satisfies $\varphi_1>0$ in $[1,T]$. On the other hand, we have

Lemma 2.1 ([3])**.** The unique solution of BVP (2.3) is given by

$$u(t)=\sum_{s=1}^{T}G_m(t,s)h(s), \quad t\in[0,T+1],$$

where

$$G_m(t,s)=\frac{\psi(T)}{D}\varphi(t)\varphi(s)-\frac{p(T)\Delta\varphi(T)}{D}\psi(t)\psi(s)$$

$$+\begin{cases} \dfrac{p(T)\Delta\psi(T)-1}{D}\varphi(t)\psi(s)-\dfrac{\varphi(T)-1}{D}\varphi(s)\psi(t), 0\le s\le t\le T+1, \\ \dfrac{p(T)\Delta\psi(T)-1}{D}\varphi(s)\psi(t)-\dfrac{\varphi(T)-1}{D}\varphi(t)\psi(s), 0\le t\le s\le T+1, \end{cases}$$

and $\{\varphi(t)\}_{t=0}^{T+1}$, $\{\psi(t)\}_{t=0}^{T+1}$ are the solutions of the corresponding homogeneous equation

$$-\Delta[p(t-1)\Delta u(t-1)]+(q(t)+m)u(t)=0, \quad t\in[1,T],$$

under the initial conditions that $\varphi(0)=\varphi(1)=1$; $\psi(0)=0$, $p(0)\psi(1)=1$.

Lemma 2.2 ([3]). $G_m(t,s)=G_m(s,t)>0$, $t,s\in[0,T]$.

Define operators $K_m,\mathbf{f}_m,A_m:E\to E$, respectively, by

$$(K_m u)(t)=\sum_{k=1}^{T}G_m(t,k)u(k), \quad u\in E, \quad t\in[1,T];$$

$$(\mathbf{f}_m u)(t)=f(t,u(t))+mu(t), \quad u\in E, \quad t\in[1,T];$$

$$A_m=K_m\mathbf{f}_m.$$

According to [3, Lemmas 3.1 and 3.2], $A_m:E\to E$ is a completely continuous operator. It follows from (2.4) and Lemma 2.1, we have

$$K_m=(P+Q+mI)^{-1}. \tag{2.5}$$

Remark 2.1. By Lemma 2.1, it is easy to see that $u=\{u(t)\}_{t=1}^{T}\in E$ is a fixed point of the operator A_m if and only if $\{u(t)\}_{t=0}^{T+1}$ is a solution of BVP (1.1), where $u(0)=u(T)$, $u(T+1)=u(1)$.

Lemma 2.3. The functional J defined by (2.1) is Fréchet differentiable on E and $J'(u)=u-K_m\mathbf{f}_m u$ for all $u\in E$.

Proof. It follows from the mean value theorem that for any $u,v\in E$,

$$J(u+v)-J(u)=\sum_{t=1}^{T}[p(t-1)\Delta u(t-1)\Delta v(t-1)+q(t)u(t)v(t)-f(t,u(t)+\theta(t)v(t))v(t)]$$

$$+\frac{1}{2}\sum_{t=1}^{T}[p(t-1)|\Delta v(t-1)|^2+q(t)|v(t)|^2],$$

where $0<\theta(t)<1$, $t\in[1,T]$. Since $f:[1,T]\times\mathbf{R}\to\mathbf{R}$ is continuous in the second variable, we have

$$J(u+v)-J(u)-\langle u,v\rangle_m+\sum_{t=1}^{T}(f(t,u(t))+mu(t))v(t)$$

$$=\sum_{t=1}^{T}[f(t,u(t))-f(t,u(t)+\theta(t)v(t))]v(t)+\frac{1}{2}\|v\|_m^2-\frac{1}{2}m\|v\|^2=\|v\|_m o(1).$$

Hence J is Fre′chet differentiable on E and

$$\langle J'(u),v\rangle_m=\langle u,v\rangle_m-\sum_{t=1}^{T}(f(t,u(t))+mu(t))v(t).$$

On the other hand, the T periodicity of $v=\{v(t)\}$ and $w=\{w(t)\}$ in t implies that

$$\sum_{t=1}^{T}w(t)\Delta v(t-1)=-\sum_{t=1}^{T}v(t)\Delta w(t).$$

This together with the definition of inner product and Lemma 2.1 gives that

$$\langle u-K_m\mathbf{f}_m u,v\rangle_m=\langle u,v\rangle_m-\sum_{t=1}^{T}[p(t-1)\Delta(K_m\mathbf{f}_m u)(t-1)\Delta v(t-1)+(q(t)+m)(K_m\mathbf{f}_m u)(t)v(t)]$$

$$=\langle u,v\rangle_m-\sum_{t=1}^{T}\{-\Delta[p(t-1)\Delta(K_m\mathbf{f}_m u)(t-1)]+(q(t)+m)(K_m\mathbf{f}_m u)(t)\}v(t)$$

$$=\langle u,v\rangle_m-\sum_{t=1}^{T}(f(t,u(t))+mu(t))v(t).$$

Therefore, $\langle J'(u),v\rangle_m=\langle u-K_m\mathbf{f}_m u,v\rangle_m$ for all $u,v\in E$. Then $J'(u)=u-K_m\mathbf{f}_m u$ for any $u\in E$. The proof is complete. □

Remark 2.2. By Lemma 2.4 and Remark 2.1, we reduce the existence of solutions to Problem (1.1) to the existence of critical points of the functional J defined on E.

The following Lemma will be useful in the proofs of our main results.

Lemma 2.4 ([15])**.** Let E be a Hilbert space and J be a C^1 functional defined on E. Assume that J satisfies the (PS) condition on E and $J'(u)$ has the expression $J'(u)=u-Au$ for $u\in E$. Assume that D_1 and D_1 are two open convex subsets of E with the properties that $A(\partial D_1)\subset D_1$, $A(\partial D_2)\subset D_2$ and $D_1\cap D_2\neq\varnothing$. If there exists a path $h:[0,1]\to E$ such that

$$h(0)\in D_1\setminus D_2,\quad h(1)\in D_2\setminus D_1,$$

and

$$\inf_{u\in\overline{D_1\cap D_2}}J(u)>\sup_{\tau\in[0,1]}J(h(\tau)),$$

then J has at least four critical points, one in $D_1\cap D_2$, one in $D_1\setminus\overline{D_2}$, one in $D_2\setminus\overline{D_1}$, and one in

$E \setminus (\overline{D_1} \cup \overline{D_2})$.

Remark 2.3. It follows from Theorem 5.1 in [14] that if the usual (PS) condition is replaced by the weaker (C) condition, Lemma 2.5 still holds.

3. Existence of sign-changing solutions of Problem (1.1)

In the following we consider the convex cones $\Lambda = \{u \in E : u \geq 0\}$ and $-\Lambda = \{u \in E : u \leq 0\}$. The distance in E with respect to $\|\cdot\|_m$ is denoted by dist$_m$. For $\varepsilon > 0$, we define

$$D_\varepsilon^+ = \{u \in E : \mathrm{dist}_m(u, \Lambda) < \varepsilon\}, \qquad D_\varepsilon^- = \{u \in E : \mathrm{dist}_m(u, -\Lambda) < \varepsilon\}.$$

Obviously, $D_\varepsilon^+ \cap D_\varepsilon^- \neq \varnothing$. Note that D_ε^+ and D_ε^- are open convex subsets of E. Moreover, $E \setminus (D_\varepsilon^+ \cup D_\varepsilon^-)$ contains only sign changing functions.

Lemma 3.1. Assume that (H_1) and (H_2) hold. Then there exist $m \geq 0$ and $\varepsilon_0 > 0$ such that for $0 < \varepsilon < \varepsilon_0$, one has

(i) $A_m(\partial D_\varepsilon^-) \subset D_\varepsilon^-$, and if $u \in D_\varepsilon^-$ is a nontrivial critical point of J, then u is a negative solution of Problem (1.1);

(ii) $A_m(\partial D_\varepsilon^+) \subset D_\varepsilon^+$, and if $u \in D_\varepsilon^+$ is a nontrivial critical point of J, then u is a positive solution of Problem (1.1).

Proof. (i) By assumptions (H_1) and (H_2), we may fixed $m \geq 0$ such that

$$x(f(t,x) + mx) > 0 \tag{3.1}$$

for all $x \neq 0$ and $t \in [1, T]$. For $u \in E$, we denote $v = A_m(u)$ and $u^+ = \max\{u, 0\}$, $u^- = \min\{u, 0\}$. Then

$$\|u^+\| = \inf_{w \in -P} \|u - w\| \leq \frac{1}{\sqrt{m + \lambda_1}} \inf_{w \in -P} \|u - w\|_m = \frac{1}{\sqrt{m + \lambda_1}} \mathrm{dist}_m(u, -\Lambda). \tag{3.2}$$

It follows from (H_1) and (H_2) that there exist $\tau > 0$, $C > 0$ and $s > 2$ such that

$$|f(t,x) + mx| \leq (m + \lambda_1 - \tau)|x| + C|x|^{s-1}, \quad \forall (t,x) \in [1,T] \times \mathbf{R}. \tag{3.3}$$

Since $u \in E$ is finite-dimensional, there exists $D > 0$ such that

$$|u|_s := \left(\sum_{t=1}^{T} |u(t)|^s\right)^{\frac{1}{s}} \leq D \min\{\|u\|, \|u\|_m\}, \quad \forall u \in E. \tag{3.4}$$

Obviously, $|u|_2 = \|u\|$. The fact that $v^+ = v - v^-$ and $v^- \in -\Lambda$ implies $\mathrm{dist}_m(v, -\Lambda) \leq \|v - v^-\|_m = \|v^+\|_m$. Then by (3.3), (3.4) and (3.2), we have,

$$\operatorname{dist}_m(v,-\Lambda)\|v^+\|_m \le \|v^+\|_m^2 \le \langle v,v^+\rangle_m = \sum_{t=1}^{T}[f(t,u(t))+mu(t)]v^+(t)$$

$$\le \sum_{t=1}^{T}[f(t,u^+(t))+mu^+(t)]v^+(t)$$

$$\le (m+\lambda_1-\tau)\|u^+\|\|v^+\|+C|u^+|_s^{s-1}|v^+|_s$$

$$\le \left(\frac{m+\lambda_1-\tau}{\sqrt{m+\lambda_1}}\|u^+\|+CD^s\|u^+\|^{s-1}\right)\|v^+\|_m$$

$$\le \left(\frac{m+\lambda_1-\tau}{m+\lambda_1}\operatorname{dist}_m(u,-\Lambda)+\frac{CD^s}{\sqrt{(m+\lambda_1)^{s-1}}}(\operatorname{dist}_m(u,-\Lambda))^{s-1}\right)\|v^+\|_m$$

Hence

$$\operatorname{dist}_m(v,-\Lambda)\le \frac{m+\lambda_1-\tau}{m+\lambda_1}\operatorname{dist}_m(u,-\Lambda)+C_1(\operatorname{dist}_m(u,-\Lambda))^{s-1},$$

where $C_1=\dfrac{CD^s}{\sqrt{(m+\lambda_1)^{s-1}}}$. So there exists $\varepsilon_0>0$ such that for any $u\in D_\varepsilon^-$ with $0<\varepsilon<\varepsilon_0$,

$$\operatorname{dist}_m(A_m(u),-\Lambda)\le \frac{2(m+\lambda_1)-\tau}{2(m+\lambda_1)}\operatorname{dist}_m(u,-\Lambda). \tag{3.5}$$

Then

$$A_m(u)\in D_\varepsilon^-,\ \ \forall u\in\partial D_\varepsilon^-.$$

If $u\in D_\varepsilon^-$ is a nontrivial critical point of J, then $J'(u)=u-A_mu=0$, that is, $A_mu=u$. By (3.5), $u\in-\Lambda\setminus\{0\}$. According to (3.1) and Lemma 2.2, $u(t)<0$ in $[1,T]$. Hence, u is a negative solution of Problem (1.1).

The proof of (ii) is similar and omitted. □

Lemma 3.2. Assume that one of the following condition holds.

(i) $l=+\infty$ or

(ii) $l<+\infty$ is not an eigenvalue of Problem (1.2),

where l is defined as in condition (H_2). Then the functional J (see (2.2)) satisfies the (PS) condition, i.e., for any sequence $\{u_n\}$ such that $J(u_n)$ is bounded and $J'(u_n)\to 0$ as $n\to\infty$, there exists a subsequence of $\{u_n\}$ which is convergent in E.

Proof. First suppose that $l=+\infty$. Recall that E is a finite dimensional Hilbert space. Consequently, in order to prove that J satisfies the (PS) condition, we only need to prove that $\{u_n\}$ is bounded. Let $\{u_n\}$ be a sequence in E such that $J(u_n)$ is bounded and $J'(u_n)\to 0$ as $n\to\infty$. By $l=+\infty$, we know that there exists a constant $a_1 > 0$ such that for any $t \in [1,T]$ and $x \in \mathbf{R}$, $F(t,x) \geq \lambda_T x^2 - a_1$. Thus,

$$J(u_n) = \frac{1}{2}((P+Q)u_n, u_n) - \sum_{t=1}^{T} F(t, u_n(t)) \leq \frac{1}{2}\lambda_T \|u_n\|^2 - \lambda_T \|u_n\|^2 + a_1 T. \tag{3.6}$$

Since $J(u_n)$ is bounded, the above inequality implies that $\{u_n\}$ is a bounded sequence, and the (PS) condition is verified.

Now, suppose that $l < +\infty$ is not an eigenvalue of Problem (1.2). We claim that $\{u_n\}$ is bounded. Suppose the contrary, then there exists a subsequence of $\{u_n\}$ (still denoted by $\{u_n\}$) such that $\rho_n = \|u_n\| \to +\infty$ as $n \to \infty$ and for each $t \in [1,T]$, either $u_n(t) \to +\infty$ or $\{u_n(t)\}$ is bounded. Set $v_n = \frac{u_n}{\rho_n}$. Obviously, $\|v_n\| = 1$. Therefore, there exist a subsequence of $\{v_n\}$ (still denoted by $\{v_n\}$) and $v \in E$ such that $v_n \to v$ as $n \to \infty$. Put $w_n = (\frac{f(1,u_n(1))}{u_n(1)} v_n(1), \cdots, \frac{f(T,u_n(T))}{u_n(T)} v_n(T))^{\tau}$. Since $\lim_{|x| \to +\infty} \frac{f(t,x)}{x} = l$ uniformly in $t \in [1,T]$, we have

$$\frac{J'(u_n)}{\rho_n} = v_n - \frac{1}{\rho_n} K_0 \mathbf{f}_0 u_n = v_n - K_0 w_n \to v - lK_0 v.$$

Bearing in mind that $\frac{J'(u_n)}{\rho_n} \to 0$ as $n \to \infty$, we get that $v - lK_0 v = 0$. According to Lemma 2.3, l is an eigenvalue of the matrix $P+Q$, contrary to assumption. Hence, $\{u_n\}$ is bounded. The proof is complete. □

Lemma 3.3. Assume that (H_3). Then J satisfies the Cerami condition ((C) condition for short), i.e., if any sequence $\{u_n\}$ such that $J(u_n)$ is bounded and $(1 + \|u_n\|_m)\|J'(u_n)\|_m \to 0$ as $n \to \infty$ has a convergent subsequence in E.

Proof. First suppose that (H_3)(i) holds. Let $\{u_n\} \subset E$ be a sequence such that $J(u_n)$ is bounded and $(1 + \|u_n\|_m)\|J'(u_n)\|_m \to 0$ as $n \to \infty$. Then there exists a constant $R_1 > 0$ such that

$$|J(u_n)| \leq R_1, \quad (1 + \|u_n\|_m)\|J'(u_n)\|_m \leq R_1.$$

This gives that

$$-3R_1 \leq 2J(u_n) - (1 + \|u_n\|_m)\|J'(u_n)\|_m \leq 2J(u_n) - \langle J'(u_n), u_n \rangle_m$$

$$= \sum_{t=1}^{T} [u_n(t) f(t, u_n(t)) - 2F(t, u_n(t))]. \tag{3.7}$$

Then, $\{u_n\}$ is bounded. In fact, if $\{u_n\}$ is unbounded, there exists a subsequence of $\{u_n\}$ (still denoted by $\{u_n\}$) and some $t_0 \in [1,T]$ such that $|u_n(t_0)| \to +\infty$ as $n \to \infty$. By (H$_4$)(i), we have

$$u_n(t_0) f(t_0, u_n(t_0)) - 2F(t_0, u_n(t_0)) \to -\infty \quad \text{as} \quad n \to \infty.$$

By (H$_3$)(i) again and the continuity of f, there exists a constant $R_2 > 0$ such that for any $t \in [1,T]$ and $x \in \mathbf{R}$, $xf(t,x)) - 2F(t,x) \le R_2$. Thus,

$$\sum_{t=1}^{T} [u_n(t) f(t,u_n(t)) - 2F(t,u_n(t))] \le [u_n(t_0) f(t_0, u_n(t_0) - 2F(t_0, u_n(t_0))] + (T-1)R_2 \to -\infty,$$

which contradicts to (3.7). Thus, $\{u_n\}$ is bounded in E and J satisfies the condition (C).

Now, suppose that (H$_4$)(ii) holds. By a similar argument as above, we know that J satisfies the condition (C). The proof is complete. □

Lemma 3.4. Assume that $l > \lambda_2$. Then $J(u) \to -\infty$ as $\|u\|_m \to +\infty$, where $u \in E_2 = \operatorname{span}\{\varphi_1, \varphi_2\}$, and φ_1, φ_2 are eigenfunctions corresponding to eigenvalues λ_1, λ_2 of problem (1.2).

Proof. First suppose that $l = +\infty$. By (3.6), we know that for any $u \in E$, $J(u) \to -\infty$ as $\|u\|_m \to +\infty$. Now, suppose that $l \in (\lambda_2, +\infty)$. For $u \in E_2$, $u = \varepsilon_1 \varphi_1 + \varepsilon_2 \varphi_2$. Without loss of generality, we can suppose that φ_1 and φ_2 are orthogonal, i.e., $(\varphi_1, \varphi_2) = 0$. Then $\|u\|_m^2 = \varepsilon_1^2 \|\varphi_1\|^2 + \varepsilon_2^2 \|\varphi_2\|^2$. Choose ε such that $0 < \varepsilon < \min\{l - \lambda_1, l - \lambda_2\}$. By $\lim_{|x| \to +\infty} \frac{f(t,x)}{x} = l$, we have that there exists $a_2 > 0$ such that for any $t \in [1,T]$ and $x \in \mathbf{R}$, $F(t,x) \ge \frac{l-\varepsilon}{2} x^2 - a_2$. Hence, for $u \in E_2$,

$$\begin{aligned} J(u) &= \frac{1}{2}((P+Q)u,u) - \sum_{t=1}^{T} F(t,u(t)) \le \frac{1}{2}(\lambda_1 \varepsilon_1^2 \|\varphi_1\|^2 + \lambda_2 \varepsilon_2^2 \|\varphi_2\|^2) - \frac{l-\varepsilon}{2}\|u\|^2 + a_2 T \\ &= \frac{1}{2}(\lambda_1 - l + \varepsilon)\varepsilon_1^2 \|\varphi_1\|^2 + \frac{1}{2}(\lambda_2 - l + \varepsilon)\varepsilon_2^2 \|\varphi_2\|^2 + a_2 T \to -\infty \end{aligned}$$

as $\|u\|_m \to +\infty$. The proof is complete. □

Proof of Theorem 1.1. (i) Our aim is to apply Lemma 2.4. By (3.3), we have

$$F(t,x) + \frac{m}{2}|x|^2 \le (m + \lambda_1 - \tau)\frac{1}{2}|x|^2 + \frac{C}{s}|x|^s.$$

This together with (3.4) gives that

$$J(u) = \frac{1}{2}\langle u,u \rangle_m - \sum_{t=1}^{T} [F(t,u(t)) + \frac{m}{2}|u(t)|^2] \ge \frac{1}{2}\|u\|_m^2 - \frac{m + \lambda_1 - \tau}{2}\|u\|^2 - \frac{C}{s}|u|_s^s$$

$$\geq \frac{\tau}{2(m+\lambda_1)}\|u\|_m^2 - \frac{CD^s}{s}\|u\|_m^s.$$

We note that (3.2) implies that for any $u \in \overline{D_\varepsilon^+} \cap \overline{D_\varepsilon^-}$, $\|u^\pm\| \leq \frac{1}{\sqrt{m+\lambda_1}} \mathrm{dist}_m(u, \mp\Lambda) \leq \frac{1}{\sqrt{m+\lambda_1}} \varepsilon_0$. Thus there exists $c_0 > -\infty$ such that $\inf_{u \in \overline{D_\varepsilon^+} \cap \overline{D_\varepsilon^-}} J(u) = c_0$. Lemma 3.4 yields that there exists $R > 2\varepsilon_0$ such that $J(u) < c_0 - 1$ for $u \in E_2$ and $\|u\|_m = R$. Define a path $h:[0,1] \to E_2$ as

$$h(s) = R\frac{\cos(\pi s)\varphi_1 + \sin(\pi s)\varphi_2}{\|\cos(\pi s)\varphi_1 + \sin(\pi s)\varphi_2\|_m}.$$

Then $h(0) = R\frac{\varphi_1}{\|\varphi_1\|_m} \in D_\varepsilon^+ \setminus D_\varepsilon^-$, $h(1) = -R\frac{\varphi_1}{\|\varphi_1\|_m} \in D_\varepsilon^- \setminus D_\varepsilon^+$ and

$$\inf_{u \in \overline{D_\varepsilon^+} \cap \overline{D_\varepsilon^-}} J(u) > \sup_{\tau \in [0,1]} J(h(\tau)).$$

According to Lemmas 3.1, 3.2 and 2.4, there exists a critical point in $E \setminus (\overline{D_\varepsilon^+} \cup \overline{D_\varepsilon^-})$, which is a sign-changing solution of Problem (1.1). Also we have a critical point in $D_\varepsilon^+ \setminus \overline{D_\varepsilon^-}$ and a critical point in $D_\varepsilon^- \setminus \overline{D_\varepsilon^+}$, which correspond to a positive solution and a negative solution of Problem (1.1), respectively. This completes the proof for (i). Note that Lemma 3.3 and Remark 2.3. The proof for (ii) is similar and thus omitted. □

Now we give an example to illustrate the conclusion of Theorem 1.1.

Example 3.1. Let $p(t) \equiv 1$, $q(t) \equiv 1$ and $f(t,x) = \frac{|x|-b}{|x|+1}ax$, where $a > 1 + 4\sin^2\frac{\pi}{T}$, $0 < b < \frac{1}{a}$. Then,

$$F(t,x) = \begin{cases} \frac{ax^2}{2} - a(b+1)(x - \ln(1+x)), & x \geq 0, \\ \frac{ax^2}{2} + a(b+1)(x + \ln(1-x)), & x < 0. \end{cases}$$

By simple calculation, we have

$$\lim_{|x|\to\infty}[2F(t,x) - xf(t,x)] = -\infty, \text{ uniformly for } t \in [1,T]$$

It is easy to see that $\lambda_1 = 1$ and $\lambda_2 = 1 + 4\sin^2\frac{\pi}{T}$. In addition, $f_0 = ab < \lambda_1$, $l = a > \lambda_2$. Hence by Theorem 1.1, Problem (1.1) has at least a positive solution, a negative solution and a sign-changing solution.

4. Existence of positive solutions of Problem (1.1)

In this section, we will prove existence of positive solutions of Problem (1.1) by the mountain pass lemma. Let $u^+ = \max\{u, 0\}$, $u^- = \min\{u, 0\}$ and $f(t,0) = 0$ for any $t \in [1,T]$. Consider the functionals

$$J_{\pm}(u)=\frac{1}{2}\langle u,u\rangle_0-\sum_{t=1}^{T}F(t,u^{\pm}(t)),\qquad u\in E.$$

where $F(t,x)=\int_0^x f(t,s)\mathrm{d}s$. Then J_+ and J_- are continuously differentiable. The critical points of the functional J_+ (respectively J_-) correspond to positive (respectively negative) solutions of Problem (1.1).

Lemma 4.1. Suppose that $\liminf_{|x|\to\infty} f(t,x)/x>\lambda_1$ uniformly for $t\in[1,T]$, then J_+ and J_- satisfy the (PS) condition.

Proof. Let $\{u_n\}$ be a sequence in E such that $J_+(u_n)$ is bounded and $J_+'(u_n)\to 0$ as $n\to\infty$. Since E is finite dimensional, it suffices to show that $\{u_n\}$ is bounded. Let $(\mathbf{f}_+u)(t)=f(t,u^+(t))$, $u\in E$, $t\in[1,T]$. Then

$$\|u_n^-\|_0^2\le\langle u_n,u_n^-\rangle_0\le\langle u_n-K\mathbf{f}_+u_n,u_n^-\rangle_0=\langle J_+'(u_n),u_n^-\rangle_0=o(1)\|u_n^-\|_0$$

so $u_n^-\to 0$ as $n\to\infty$. We claim that $\{u_n^+\}$ is bounded. Suppose the contrary, then there exists a subsequence of $\{u_n\}$ (still denoted by $\{u_n\}$) such that $\rho_n=\|u_n^+\|_0\to+\infty$ as $n\to\infty$ and for each $t\in[1,T]$, either $u_n^+(t)\to+\infty$ or $\{u_n^+(t)\}$ is bounded. Set $v_n=\dfrac{u_n^+}{\rho_n}$. Obviously, $\|v_n\|_0=1$. Therefore, there exists a subsequence of $\{v_n\}$ (still denoted by $\{v_n\}$) and $v\in E$ such that $v_n\to v$ as $n\to\infty$. Denoting $\varphi_1>0$ by the eigenfunction associated with λ_1. Then

$$\lambda_1\sum_{t=1}^{T}u_n(t)\varphi_1(t)=\sum_{t=1}^{T}[p(t-1)\Delta u_n(t-1)\Delta\varphi_1(t-1)+q(t)u_n(t)\varphi_1(t)]=\langle u_n,\varphi_1\rangle_0$$

$$=\langle K\mathbf{f}_+u_n,\varphi_1\rangle_0+J_+'(u_n)\varphi_1=\sum_{t=1}^{T}f(t,u_n^+(t))\varphi_1(t)+\langle J_+'(u_n),\varphi_1\rangle_0.$$

By divding both sides of the above equality by ρ_n, we get

$$\lambda_1\sum_{t=1}^{T}v_n(t)\varphi_1(t)=\sum_{t=1}^{T}\frac{f(t,u_n^+(t))}{u_n^+(t)}v_n(t)\varphi_1(t)+o(1).\tag{4.1}$$

If $u_n^+(t)\to+\infty$ then $\min_{t\in[1,T]}\liminf_{|x|\to\infty}f(t,x)/x>\lambda_1$ by assumption, and if $\{u_n^+(t)\}$ is bounded then $\dfrac{f(t,u_n^+(t))}{\rho_n}\to 0$ and $v(t)=0$. Since $v\neq 0$, there is a t for which $u_n^+(t)\to+\infty$ and $v(t)>0$, so passing to the limit in (4.1) yields a contradiction. Hence, $\{u_n\}$ is bounded and J_+ satisfies the (PS) condition.

By a similar argument as above, we know also that J_- satisfies the (PS) condition. The proof is complete.□

Proof of Theorem 1.2. By $\max_{t\in[1,T]} \limsup_{x\to 0} f(t,x)/x < \lambda_1$, we can choose $\varepsilon_1 > 0$ and $\delta > 0$ such that

$$F(t,x) = \int_0^x f(t,s)\mathrm{d}s \le \frac{1}{2}(\lambda_1 - \varepsilon_1)|x|^2, \quad t \in [1,T],\ |x| \le \delta .$$

Let $B_\delta = \{u \in E : \|u\|_0 < \delta\}$. Then for $u \in \partial B_\delta$, we have

$$J_+(u) = \frac{1}{2}\langle u,u\rangle_0 - \sum_{t=1}^{T} F(t,u^+(t)) \ge \frac{1}{2}\|u\|_0^2 - \frac{1}{2}(\lambda_1 - \varepsilon_1)\|u\|^2 \ge \frac{1}{2\lambda_1}\varepsilon_1\delta^2 := \rho .$$

By $\min_{t\in[1,T]} \liminf_{x\to\infty} f(t,x)/x > \lambda_1$, we can choose $\varepsilon_2 > 0$ such that $\min_{t\in[1,T]} \liminf_{x\to\infty} f(t,x)/x > \lambda_1 + \varepsilon_2$. Then there is some $C > 0$ such that

$$F(t,x) \ge \frac{1}{2}(\lambda_1 + \varepsilon_2)x^2 - C, \ \forall (t,x) \in [1,T]\times \mathbf{R} .$$

Therefore

$$J_+(s\varphi_1) \le \frac{s^2}{2}\|\varphi_1\|_0^2 - \frac{s^2}{2\lambda_1}(\lambda_1 + \varepsilon_2)\|\varphi_1\|_0^2 + C \le -\frac{s^2\varepsilon_2}{2\lambda_1}\|\varphi_1\|_0^2 + C < 0$$

for s large enough. Lemma 4.1 and the mountain pass lemma now give that there exists $w \in E$ with $J_+'(w) = 0$ and $J_+(w) \ge \rho > 0$. Then

$$\|w^-\|_0^2 \le \langle w, w^-\rangle_0 \le \langle w - K\mathbf{f}_+ w, w^-\rangle_0 = \langle J_+'(w), w^-\rangle_0 = 0 .$$

Thus $w^- = 0$, and so $w \ge 0$. If $w(t) = 0$ for some $t \in [1,T]$ then

$$-p(t)w(t+1) - p(t-1)w(t-1) = -\Delta[p(t-1)\Delta w(t-1)] + q(t)w(t) = f(t,w(t)) = 0 ,$$

so $w(t\pm 1) = 0$, and it follows that if w is zero somewhere in $[1,T]$ then it vanishes identically. Then by $J_+(w) > 0$, $w > 0$ in $[1,T]$. Hence, w is a positive solution of Problem (1.1). Similar, a negative solution can be obtained for the case of J_-. The proof is completed.

From the proof of Theorem 1.2, we can easily get the following corollary.

Corollary 4.1. Assume that $f(t,0) = 0$ for any $t \in [1,T]$, then the following hold:

(i) If $\limsup_{x\to 0^+} f(t,x)/x < \lambda_1$ and $\liminf_{x\to +\infty} f(t,x)/x > \lambda_1$ uniformly for $t \in [1,T]$, then Problem (1.1) has at least a positive solution.

(ii) If $\limsup_{x\to 0^-} f(t,x)/x < \lambda_1$ and $\liminf_{x\to -\infty} f(t,x)/x > \lambda_1$ uniformly for $t \in [1,T]$, then Problem (1.1) has at least a negative solution.

Example 4.1. Let $p(t) \equiv 1$, $q(t) \equiv 1$ and $f(t,x) = x^5 - (1+\sin t)x^4 + ax$, where $a < 1$. Then $\lambda_1 = 1$. In addition, $f(t,0) = 0$ for all $t \in [1,T]$, $\limsup_{x\to 0} f(t,x)/x < \lambda_1$ and $\liminf_{|x|\to\infty} f(t,x)/x > \lambda_1$ uniformly for $t \in [1,T]$. Hence by Theorem 1.2, Problem (1.1) has at least a positive solution and a negative solution.

Remark 4.1. It is easy to see that $f(t,x)$ is unbounded from below and sign-changing for $x \ge 0$ in Examples 4.1 and 3.1. The existence of positive solutions could not be obtained by any theorems in [3, 11].

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