# Solutions for a variable exponent Neumann boundary value problems with Hardy critical exponent*

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**Abstract**

In this paper, we deal with the existence of solutions for the following variable exponent system Neumann boundary value problem with Hardy critical exponent and approximate Sobolev critical growth condition

$$\begin{cases} -div(|\nabla u|^{p(x)-2}\nabla u)+a(x)|u|^{p(x)-2}u=F_u(x,u,v) & \text{in } \Omega, \\ -div(|\nabla v|^{q(x)-2}\nabla v)+b(x)|v|^{q(x)-2}v=F_v(x,u,v) & \text{in } \Omega, \\ \frac{\partial u}{\partial \gamma}=0=\frac{\partial v}{\partial \gamma} \quad \text{on } \partial\Omega. \end{cases}$$

We give several sufficient conditions for the existence of solutions, when $F(x,\cdot,\cdot)$ satisfies sub-$(p(x),q(x))$ growth condition, or super-$(p(x),q(x))$ growth condition and approximate Sobolev critical growth condition. Especially, we obtain the existence of infinitely many solutions, when $F(x,\cdot,v)$ satisfies sub-$p(x)$ growth condition, and $F(x,u,\cdot)$ satisfies super-$q(x)$ growth condition.

**Key words**: Variable exponent system; Variable exponent Sobolev spaces; Critical points; Hardy critical exponent

## 1 Introduction

The study of differential equations and variational problems with variable exponent has attracted intense research interests in recent years. Such problems arise from the study of electrorheological fluids, image processing, and the theory of nonlinear elasticity (see [1,10,39,52]). These problems are interesting in applications (see [25,26,28,32]). Many results have been obtained on this kind of problems, for examples [1-7,11,12,14-22,25-42,45-53]. On the existence of solutions for variable exponent elliptic systems with subcritical growth condition, we refer to [4,27,45,48]. The results to the equations with critical exponent growth conditions are rare

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(see [21,22]). In this paper, we consider the existence of solutions for the following system with Hardy critical exponent and approximate Sobolev critical growth condition

$$\begin{cases} -div(|\nabla u|^{p(x)-2}\nabla u)+a(x)\,|u|^{p(x)-2}\,u=F_u(x,u,v) \text{ in } \Omega, \\ -div(|\nabla v|^{q(x)-2}\nabla v)+b(x)\,|v|^{q(x)-2}\,v=F_v(x,u,v) \text{ in } \Omega, \\ \frac{\partial u}{\partial\gamma}=0=\frac{\partial v}{\partial\gamma} \quad \text{on } \partial\Omega, \end{cases} \tag{1}$$

where $\Omega\subset\mathbb{R}^N$ is a bounded domain and $\partial\Omega$ possesses cone property, $p,q\in C(\overline{\Omega})$ and $p(x),q(x)>1$, $-\triangle_{p(x)}u:=-div(|\nabla u|^{p(x)-2}\nabla u)$ is called the $p(x)$-Laplacian, $a,b\in L^{\infty}(\Omega)$, $\underset{x\in\Omega}{ess\inf}a(x)=a_0>0$, $\underset{x\in\Omega}{ess\inf}b(x)=b_0>0$, $\gamma$ is the outward unit normal to $\partial\Omega$. $F$ satisfies

$$F(x,s,t)=\sum_{i=1}^{m}F_i(x,s,t)=\sum_{i=1}^{m}\lambda_i a_i(x)G_i(x,s,t),\forall(x,s,t)\in\overline{\Omega}\times\mathbb{R}\times\mathbb{R}.$$

Throughout the paper, the following conditions are satisfied

**(A)** For every $i=1,\cdots,m$, $\lambda_i$ is a parameter, $a_i\in L^{r_i(\cdot)}(\Omega)$, we assume that $G_i\in C^1(\overline{\Omega}\times\mathbb{R}^2\to\mathbb{R})$ $(i=1,\cdots,m)$ and satisfies

$$\begin{aligned} |G_{i,u}(x,u,v)| &\leq C(|u|^{\alpha_i(x)-1}+|v|^{\beta_i(x)/\alpha_i^0(x)}+1), i=1,\cdots,m, \\ |G_{i,v}(x,u,v)| &\leq C(|v|^{\beta_i(x)-1}+|u|^{\alpha_i(x)/\beta_i^0(x)}+1), i=1,\cdots,m, \end{aligned}$$

where $G_{i,u}=\frac{\partial}{\partial u}G_i$, $G_{i,v}=\frac{\partial}{\partial v}G_i$, $r_i(\cdot)\equiv+\infty$ or $r_i(\cdot)\in C(\overline{\Omega})$ with $r_i(x)>1$, $\alpha_i,\beta_i\in C(\overline{\Omega})$ with $\alpha_i(x),\beta_i(x)>1$ and satisfy

$$1\leq r_i^0(x)\leq\frac{1}{\alpha_i(x)}p^*(x),1\leq r_i^0(x)\leq\frac{1}{\beta_i(x)}q^*(x),i=1,\cdots,m,$$

where the notation $\mu^0(x)$ means the conjugate function of $\mu(x)$, namely $\mu^0(x)=\begin{cases} \frac{\mu(x)}{\mu(x)-1},\mu\in C(\overline{\Omega}) \\ 1,\mu=+\infty \end{cases}$,

and

$$p^*(x)=\begin{cases} Np(x)/(N-p(x)),\ p(x)<N, \\ \infty\ ,\ p(x)\geq N. \end{cases}$$

When $p(x)\equiv p$ (a constant), $p(x)$-Laplacian becomes the usual $p$-Laplacian. The $p(x)$-Laplacian is nonhomogeneity and possesses more complicated nonlinearities than the $p$-Laplacian (see [18]). On the $p$-Laplacian problems with singular coefficients, we refer to [8,13,23,24]. But the existence of solutions for $p(x)$-Laplacian equations with singular coefficients are rare (see [19,50]). On the existence of solutions for variable exponent elliptic systems, if $F(x,\cdot,\cdot)$ satisfies the sub-$(p^-,q^-)$ growth condition, i.e. the following condition

$$\max_{x\in\overline{\Omega}}\alpha_i(x)<\min_{x\in\overline{\Omega}}p(x),\max_{x\in\overline{\Omega}}\beta_i(x)<\min_{x\in\overline{\Omega}}q(x),i=1,\cdots,m,$$

we can see that the corresponding functional is coercive, if $F(x,\cdot,\cdot)$ satisfies the super-$(p^+,q^+)$ growth condition (subcritical), i.e. the following condition

$$0 < G_i(x,s,t) \leq \frac{s}{\theta_1}\frac{\partial}{\partial s}G_i(x,s,t) + \frac{t}{\theta_2}\frac{\partial}{\partial t}G_i(x,s,t), \text{ for } x \in \overline{\Omega} \text{ and } |s|^{\theta_1} + |t|^{\theta_2} \geq 2M > 0,$$

where $M$ is a positive constant, the positive constants $\theta_1$ and $\theta_2$ satisfy

$$\max_{x\in\overline{\Omega}} p(x) < \theta_1 < \min_{x\in\overline{\Omega}} p^*(x) \text{ and } \max_{x\in\overline{\Omega}} q(x) < \theta_2 < \min_{x\in\overline{\Omega}} q^*(x),$$

we can see that the corresponding functional satisfies Palais-Smale conditions. On the variable exponent equations, many results are focused on the case of $F(x,\cdot,\cdot)$ satisfy sub-$(p^-,q^-)$ growth condition or super-$(p^+,q^+)$ growth condition (see [4,27,45,47]). If $F(x,\cdot,\cdot)$ satisfy subcritical growth condition, but it does not satisfy the sub-$(p^-,q^-)$ growth condition or super-$(p^+,q^+)$ growth condition, it is difficult to testify the corresponding functional be coercive or satisfying Palais-Smale conditions, the results on this case are rare. This paper give the existence of solutions for (1), when $F(x,\cdot,v)$ satisfies sub-$p(x)$ growth condition, and $F(x,u,\cdot)$ satisfies super-$q(x)$ growth condition. This paper was motivated by [4,19,27].

Our aim is to give the existence of solutions and infinitely many solutions for (1), when $F(x,\cdot,\cdot)$ satisfies sub-$(p(x),q(x))$ growth condition i.e. the condition $\alpha_i(x) < p(x)$, $\beta_i(x) < q(x), x \in \overline{\Omega}$, or super-$(p(x),q(x))$ growth condition (subcritical) i.e. the condition

$$0 < G_i(x,s,t) \leq \frac{s}{\theta_1(x)}\frac{\partial}{\partial s}G_i(x,s,t) + \frac{t}{\theta_2(x)}\frac{\partial}{\partial t}G_i(x,s,t), \text{ for } x \in \overline{\Omega} \text{ and } |s|^{\theta_1} + |t|^{\theta_2} \geq 2M > 0,$$

where $M$ is a positive constant, the positive functions $\theta_1(x)$ and $\theta_2(x)$ satisfy

$$p(x) < \theta_1(x) < p^*(x) \text{ and } q(x) < \theta_2(x) < q^*(x), x \in \overline{\Omega},$$

and our results permit some $G_i$ satisfies the following approximate Sobolev critical growth condition

$$G_i(x,s,t) = (|u|^{p^*(x)} + |v|^{q^*(x)})/\ln(1+|u|+|v|),$$

and the principle of concentration compactness should be used in the discussions. This paper partly generalized the results of [4,17,19,21,27,45].

This paper is organized as four sections. In Section 2, we introduce some basic properties of the variable exponent Lebesgue-Sobolev spaces. In Section 3, several important properties of $p(x)$-Laplacian and variational principle are presented. In Section 4, we give the existence of solutions for problem (1).

## 2 Preliminary results and notations

Throughout this paper, the letters $c$, $c_i$, $C_i$, $i = 1, 2, \cdots$, denote positive constants which may vary from line to line but are independent of the terms which will take part in any limit process.

In order to discuss problem (1), we need some theories on space $W^{1,p(\cdot)}(\Omega)$ which we call variable exponent Sobolev space. Firstly, we state some basic properties of spaces $W^{1,p(\cdot)}(\Omega)$ and $p(x)$-Laplacian which we will use later (for details, see [14,17,19-21]). Write

$$C_+(\overline{\Omega}) = \left\{h \left| h \in C(\overline{\Omega}),\, h(x) > 1 \text{ for } x \in \overline{\Omega}\right.\right\},$$

$$h^+ = ess\sup_{x\in\Omega} h(x),\ h^- = ess\inf_{x\in\Omega} h(x), \text{ for any } h \in L^\infty(\Omega),$$

$$S(\Omega) = \{u \mid u \text{ is a real-valued measurable function on } \Omega\},$$

$$L^{p(\cdot)}(\Omega) = \left\{u \in S(\Omega) \mid \textstyle\int_\Omega |u(x)|^{p(x)}\, dx < \infty\right\}.$$

We can introduce a norm on $L^{p(\cdot)}(\Omega)$ by

$$|u|_{p(\cdot)} = \inf\left\{\lambda > 0 \left| \int_\Omega \left|\frac{u(x)}{\lambda}\right|^{p(x)} dx \le 1\right.\right\},$$

and $(L^{p(\cdot)}(\Omega), |\cdot|_{p(\cdot)})$ becomes a Banach space, we call it variable exponent Lebesgue space.

**Proposition 1** *(see [14]) (i) The space $(L^{p(\cdot)}(\Omega), |\cdot|_{p(\cdot)})$ is a separable, reflexive, uniform convex Banach space, and its conjugate space is $L^{p^0(\cdot)}(\Omega)$, where $1/p(x) + 1/p^0(x) \equiv 1$. For any $u \in L^{p(\cdot)}(\Omega)$ and $v \in L^{p^0(\cdot)}(\Omega)$, we have*

$$\left|\int_\Omega uv dx\right| \le (\frac{1}{p^-} + \frac{1}{(p^0)^-})\, |u|_{p(\cdot)}\, |v|_{p^0(\cdot)}\,;$$

*(ii) If $p_1$, $p_2 \in C_+(\overline{\Omega})$, $p_1(x) \le p_2(x)$ for any $x \in \overline{\Omega}$, then $L^{p_2(\cdot)}(\Omega) \subset L^{p_1(\cdot)}(\Omega)$, and the imbedding is continuous.*

Denote $Y = \prod_{i=1}^{k} L^{p_i(\cdot)}(\Omega)$ with the norm

$$\| y \|_Y = \sum_{i=1}^{k} \left|y^i\right|_{p_i(\cdot)},\ \forall y = (y^1, \cdots, y^k) \in Y,$$

where $p_i(x) \in C_+(\overline{\Omega})$, $i = 1, \cdots, m$, then $Y$ is a Banach space.

**Proposition 2** *(see [9,17]) Suppose $f(x,y):\Omega\times\mathbb{R}^k\to\mathbb{R}^m$ is a Caratheodory function, i.e., $f$ satisfies*

*(i) For a.e. $x\in\Omega$, $y\to f(x,y)$ is a continuous function from $\mathbb{R}^k$ to $\mathbb{R}^m$,*

*(ii) For any $y\in\mathbb{R}^k$, $x\to f(x,y)$ is measurable.*

If there exist $\beta(x),p_1(x),\cdots,p_k(x)\in C_+(\overline{\Omega})$, $\rho(x)\in L^{\beta(\cdot)}(\Omega)$ and positive constant $c>0$ such that

$$|f(x,y)|\leq\rho(x)+c\sum_{i=1}^{k}|y_i|^{p_i(x)/\beta(x)}\text{ for any }x\in\Omega,y\in\mathbb{R}^k,$$

then the Nemytsky operator from $Y$ to $(L^{\beta(\cdot)}(\Omega))^m$ defined by $(N_fu)(x)=f(x,u(x))$ is continuous and bounded.

The space $W^{1,p(\cdot)}(\Omega)$ is defined by

$$W^{1,p(\cdot)}(\Omega)=\left\{u\in L^{p(\cdot)}(\Omega)\,\middle|\,|\nabla u|\in L^{p(\cdot)}(\Omega)\right\},$$

and it can be endowed with the norm

$$\|u\|_{p(\cdot)}=|u|_{p(\cdot)}+|\nabla u|_{p(\cdot)},\forall u\in W^{1,p(\cdot)}(\Omega).$$

Denote

$$\begin{aligned}\|u\|'_{p(\cdot)}&=\inf\left\{\lambda>0\,\middle|\,\int_\Omega\left|\frac{\nabla u}{\lambda}\right|^{p(x)}dx+\int_\Omega a(x)\left|\frac{u(x)}{\lambda}\right|^{p(x)}dx\leq1\right\},\\ \|u\|'_{q(\cdot)}&=\inf\left\{\lambda>0\,\middle|\,\int_\Omega\left|\frac{\nabla u}{\lambda}\right|^{q(x)}dx+\int_\Omega b(x)\left|\frac{u(x)}{\lambda}\right|^{q(x)}dx\leq1\right\}.\end{aligned}$$

Since $a,b\in L^\infty(\Omega)$, $\underset{x\in\Omega}{ess\inf}a(x)=a_0>0$, $\underset{x\in\Omega}{ess\inf}b(x)=b_0>0$, we can easily see that the norm $\|\cdot\|'_{p(\cdot)}$ is equivalent to $\|\cdot\|_{p(\cdot)}$ on $W^{1,p(\cdot)}(\Omega)$, and $\|\cdot\|'_{q(\cdot)}$ is equivalent to $\|\cdot\|_{q(\cdot)}$ on $W^{1,q(\cdot)}(\Omega)$. In the following, we will use $\|\cdot\|'_{p(\cdot)}$ to replace $\|\cdot\|_{p(\cdot)}$ on $W^{1,p(\cdot)}(\Omega)$, and use $\|\cdot\|'_{q(\cdot)}$ to replace $\|\cdot\|_{q(\cdot)}$ on $W^{1,q(\cdot)}(\Omega)$.

We denote by $W_0^{1,p(\cdot)}(\Omega)$ the closure of $C_0^\infty(\Omega)$ in $W^{1,p(\cdot)}(\Omega)$.

**Proposition 3** *(see [14]) (i) $W^{1,p(\cdot)}(\Omega)$ is a separable reflexive Banach space;*

*(ii) If $\beta\in C_+\left(\overline{\Omega}\right)$ and $\beta(x)<p^*(x)$ for any $x\in\overline{\Omega}$, then the imbedding from $W^{1,p(\cdot)}(\Omega)$ to $L^{\beta(\cdot)}(\Omega)$ is compact and continuous.*

Let $\beta \in C_+(\overline{\Omega}), \mu \in S(\Omega)$, and $\mu(x) > 0$ for $a.e.$ $x \in \Omega$. Define

$$L^{\beta(\cdot)}_{\mu(\cdot)}(\Omega) = \left\{ u \mid u \in S(\Omega),\ \int_\Omega \mu(x) \left|u(x)\right|^{\beta(x)} dx < \infty \right\},$$

with the norm

$$|u|_{L^{\beta(\cdot)}_{\mu(\cdot)}(\Omega)} = |u|_{(\beta(\cdot),\mu(\cdot))} = \inf \left\{ \lambda > 0 \left| \int_\Omega \mu(x) \left| \frac{u(x)}{\lambda} \right|^{\beta(x)} dx \leq 1 \right. \right\},$$

then $L^{\beta(\cdot)}_{\mu(\cdot)}(\Omega)$ is a Banach space.

**Proposition 4** *(see [19]) Assume that the boundary of $\Omega$ possesses the cone property and $1 < p \in C(\overline{\Omega})$. Suppose that $\mu \in L^{r(\cdot)}(\Omega)$, $a(x) > 0$ for a.e.$x \in \Omega$, $r \in C_+(\overline{\Omega})$. If $\beta \in C_+(\overline{\Omega})$ and*

$$1 \leq \beta(x) < \frac{r(x) - 1}{r(x)} p^*(x), \forall x \in \overline{\Omega},$$

*then there is a compact continuously embedding $W^{1,p(\cdot)}(\Omega) \hookrightarrow L^{\beta(\cdot)}_{\mu(\cdot)}(\Omega)$.*

Denote $X = W^{1,p(\cdot)}(\Omega) \times W^{1,q(\cdot)}(\Omega)$. Let us endow the norm $\|\cdot\|$ on $X$ as

$$\|(u, v)\| = \max\{\|u\|_{p(\cdot)}, \|v\|_{q(\cdot)}\}.$$

The dual space of $X$ will be denoted as $X^*$, then for any $H \in X^*$, there exist $f \in (W^{1,p(\cdot)}(\Omega))^*$, $g \in (W^{1,q(\cdot)}(\Omega))^*$ such that $H(u, v) = f(u) + g(v)$. If we denote $\|\cdot\|_*, \|\cdot\|_{*,p(\cdot)}$ and $\|\cdot\|_{*,q(\cdot)}$ the norms of $X^*, (W^{1,p(\cdot)}(\Omega))^*$ and $(W^{1,q(\cdot)}(\Omega))^*$, respectively, then

$$\|H\|_* = \|f\|_{*,p(\cdot)} + \|g\|_{*,q(\cdot)},$$

and $X^* = (W^{1,p(\cdot)}(\Omega))^* \times (W^{1,q(\cdot)}(\Omega))^*$. Therefore

$$\|J'(u, v)\|_* = \|D_1 J(u, v)\|_{*,p(\cdot)} + \|D_2 J(u, v)\|_{*,q(\cdot)}.$$

For every $(u, v)$ and $(\varphi, \psi)$ in $X$, set

$$\begin{aligned}
\Phi_1(u) &= \int_\Omega \frac{1}{p(x)} |\nabla u|^{p(x)} dx + \int_\Omega \frac{a(x)}{p(x)} |u|^{p(x)} dx, \\
\Phi_2(v) &= \int_\Omega \frac{1}{q(x)} |\nabla v|^{q(x)} dx + \int_\Omega \frac{b(x)}{q(x)} |v|^{q(x)} dx, \\
\Phi(u, v) &= \Phi_1(u) + \Phi_2(v), \Psi(u, v) = \int_\Omega F(x, u, v) dx,
\end{aligned}$$

then

$$\begin{aligned}\Phi'(u,v)(\varphi,\psi) &= D_1\Phi(u,v)(\varphi)+D_2\Phi(u,v)(\psi),\\ \Psi'(u,v)(\varphi,\psi) &= D_1\Psi(u,v)(\varphi)+D_2\Psi(u,v)(\psi),\end{aligned}$$

where

$$\begin{aligned}D_1\Phi(u,v)(\varphi) &= \int_\Omega |\nabla u|^{p(x)-2}\nabla u\nabla\varphi dx+\int_\Omega a(x)\,|u|^{p(x)-2}u\varphi dx=\Phi_1'(u)(\varphi),\\ D_2\Phi(u,v)(\psi) &= \int_\Omega |\nabla v|^{q(x)-2}\nabla v\nabla\psi dx+\int_\Omega b(x)\,|v|^{q(x)-2}v\psi dx=\Phi_2'(v)(\psi),\\ D_1\Psi(u,v)(\varphi) &= \int_\Omega \frac{\partial}{\partial u}F(x,u,v)\varphi dx, D_2\Psi(u,v)(\psi)=\int_\Omega\frac{\partial}{\partial v}F(x,u,v)\psi dx.\end{aligned}$$

The integral functional associated with the problem (1) is

$$J(u,v)=\Phi(u,v)-\Psi(u,v).$$

It is easy to see that $J\in C^1(X,\mathbb{R})$ (see [9]). Without loss of generality, we may assume that $G_i(x,0,0)=0,\forall x\in\overline{\Omega},i=1,\cdots,m$. Obviously, we have

$$G_i(x,u,v)=\int_0^1[u\partial_2G_i(x,tu,tv)+v\partial_3G_i(x,tu,tv)]dt, i=1,\cdots,m,$$

where $\partial_j$ denotes the partial derivative of $G$ w.r.t. its $j$-th variable, then the condition **(A)** holds

$$|G_i(x,u,v)|\le c(|u|^{\alpha_i(x)}+|v|^{\beta_i(x)}+1),\forall x\in\overline{\Omega},i=1,\cdots,m. \tag{2}$$

From Proposition 2 and condition **(A)**, it is easy to see that $J\in C^1(X,\mathbb{R})$ and satisfies

$$J'(u,v)(\varphi,\psi)=D_1J(u,v)(\varphi)+D_2J(u,v)(\psi),$$

where

$$\begin{aligned}D_1J(u,v)(\varphi) &= D_1\Phi(u,v)(\varphi)-D_1\Psi(u,v)(\varphi),\\ D_2J(u,v)(\psi) &= D_2\Phi(u,v)(\psi)-D_2\Psi(u,v)(\psi).\end{aligned}$$

$(u,v)\in X$ is called a critical point of $J$ if

$$J'(u,v)(\varphi,\psi)=0,\forall(\varphi,\psi)\in X.$$

**Proposition 5** *(i) If G satisfies*

$$G(x,s,t) \geq \frac{1}{\theta_1} sG_s(x,s,t) + \frac{1}{\theta_2} tG_t(x,s,t) \geq 0 \text{ for } x \in \overline{\Omega} \text{ and } |s|^{\theta_1} + |t|^{\theta_2} \geq 2M,$$

*then* $G(x,u,v) \leq c_1[(|u|^{\theta_1} + |v|^{\theta_2}) + 1], \forall (x,u,v) \in \overline{\Omega} \times \mathbb{R} \times \mathbb{R}$,

*(ii) If G satisfies*

$$0 < G(x,s,t) \leq \frac{1}{\theta_1} sG_s(x,s,t) + \frac{1}{\theta_2} tG_t(x,s,t) \text{ for } x \in \overline{\Omega} \text{ and } |s|^{\theta_1} + |t|^{\theta_2} \geq 2M,$$

*then* $G(x,u,v) \geq c_2[(|u|^{\theta_1} + |v|^{\theta_2}) - 1], \forall (x,u,v) \in \overline{\Omega} \times \mathbb{R} \times \mathbb{R}$.

**Proof** (i) Similar to the proof of [27], we omit it here. □

Let $\mathbb{M}(\overline{\Omega})$ denote the class of nonnegative Borel measures of finite total mass, and $\mu_\varepsilon \overset{*}{\rightharpoonup} \mu$ in $\mathbb{M}(\overline{\Omega})$ is defined by $\int_{\overline{\Omega}} \eta d\mu_\varepsilon \to \int_{\overline{\Omega}} \eta d\mu$ for every test function $\eta \in C(\overline{\Omega}) \cap C^\infty(\Omega)$.

**Proposition 6** *(see [21]) If $\Omega$ is an open bounded domain in $\mathbb{R}^N$, p is Lipschitz continuous on $\overline{\Omega}$ and satisfy $1 < p(x) < N$. Let $\{\omega_\varepsilon\}$ is a sequence in $W_0^{1,p(\cdot)}(\Omega)$ of norm $\|\nabla \omega_\varepsilon\|_{p(\cdot)} \leq 1$ such that*

$$\omega_\varepsilon \rightharpoonup \omega \text{ in } W_0^{1,p(\cdot)}(\Omega),\ |\nabla \omega_\varepsilon|^{p(x)} \overset{*}{\rightharpoonup} \mu \text{ in } \mathbb{M}(\overline{\Omega}),\ |\omega_\varepsilon|^{p^*(x)} \overset{*}{\rightharpoonup} \nu \text{ in } \mathbb{M}(\overline{\Omega}).$$

*Set*

$$C^*_{p^*} = \sup\{\int_\Omega |\omega_\varepsilon|^{p^*(x)} dx \,\Big|\, \omega_\varepsilon \in W_0^{1,p(\cdot)}(\Omega), |\nabla \omega_\varepsilon|_{p(\cdot)} \leq 1\}$$

*and $0 < C^*_{p^*} < +\infty$. The limit measure are of the form*

$$\begin{aligned} \mu &= |\nabla \omega|^{p(x)} + \sum_{j \in J} \mu_j \delta_{x_j} + \widetilde{\mu}, \mu(\overline{\Omega}) \leq 1, \\ \nu &= |\omega|^{p^*(x)} + \sum_{j \in J} \nu_j \delta_{x_j}, \nu(\overline{\Omega}) \leq C^*, \end{aligned}$$

*where $x_j \in \overline{\Omega}$, J is a countable set, $\widetilde{\mu} \in \mathbb{M}(\overline{\Omega})$ is nonatomic positive measure. The atoms and the regular part satisfy the generalized Sobolev inequality*

$$\begin{aligned} \nu(\overline{\Omega}) &\leq C^* \max\{\mu(\overline{\Omega})^{\frac{p^{*+}}{p^-}}, \mu(\overline{\Omega})^{\frac{p^{*-}}{p^+}}\}, \\ \nu_j &\leq C^* \max\{\mu_j^{\frac{p^{*+}}{p^-}}, \mu_j^{\frac{p^{*-}}{p^+}}\}. \end{aligned}$$

# 3 Properties of operators and variational principle

In this section, we will discuss the properties of $p(x)$-Laplacian and Nemytsky operator, and present several variational principles.

**Proposition 7** *(see [45]) (i) $\Phi$ is a convex functional;*

*(ii) $\Phi'$ is strictly monotone, i.e., for any $(u_1, v_1)$, $(u_2, v_2) \in X$ with $(u_1, v_1) \neq (u_2, v_2)$, we have*

$$(\Phi'(u_1, v_1) - \Phi'(u_2, v_2))(u_1 - u_2, v_1 - v_2) > 0,$$

*(iii) $\Phi'$ is a mapping of type $(S_+)$, i.e. if $(u_n, v_n) \rightharpoonup (u, v)$ in $X$ and*

$$\overline{\lim_{n\to\infty}}[\Phi'(u_n, v_n) - \Phi'(u, v)](u_n - u, v_n - v) \leq 0,$$

*then $(u_n, v_n) \to (u, v)$ in $X$.*

*(iV) $\Phi' : X \to X^*$ is a bounded homeomorphism.*

**Theorem 8** *(i) $\Psi \in C^1(X, \mathbb{R})$;*

*(ii) If $r_i \in C_+(\overline{\Omega})$, and*

$$1 \leq \alpha_i(x) \leq \frac{1}{r_i^0(x)} p^*(x), 1 \leq \beta_i(x) \leq \frac{1}{r_i^0(x)} q^*(x), i = 1, \cdots, m,$$

*then $\Psi_i$ and $\Psi_i'$ are weak-strong continuous, i.e., $(u_n, v_n) \rightharpoonup (u, v)$ (in $X$) implies $\Psi_i(u_n, v_n) \to \Psi_i(u, v)$ and $\Psi_i'(u_n, v_n) \to \Psi_i'(u, v)$.*

**Proof** (i) From the continuity of the Nemytsky operator, we can see that $\Psi$ and $\Psi'$ are continuous.

(ii) Since $(u_n, v_n) \rightharpoonup (u, v)$, we have $|u_n - u|_{p(\cdot)} \to 0$ and $|v_n - v|_{q(\cdot)} \to 0$. Thus $u_n \to u$ and $v_n \to v$ a.e. on $\overline{\Omega}$. Therefore $a_i(x)G_i(x, u_n(x)) \to a_i(x)G_i(x, u(x))$ a.e. on $\overline{\Omega}$. Obviously

$$\begin{aligned}
&\int_U |a_i(x)G_i(x, u_n, v_n)|\, dx \\
\leq\ & C\int_U |a_i(x)|\,(1 + |u_n|^{\alpha_i(x)} + |v_n|^{\beta_i(x)})dx \\
\leq\ & C(\int_U |a_i(x)|^{r_i(x)}\, dx)^{\frac{1}{r_i(\xi_1)}}(\int_U |u_n|^{p^*(x)}\, dx)^{\frac{r_i(\xi_2)}{r_i(\xi_2)-1}} \\
& + C(\int_U |a_i(x)|^{r_i(x)}\, dx)^{\frac{1}{r_i(\xi_3)}}(\int_U |v_n|^{q^*(x)}\, dx)^{\frac{r_i(\xi_4)}{r_i(\xi_4)-1}} + C\int_U |a_i(x)|\, dx,
\end{aligned}$$

where $U \subset \Omega$, $\xi_1, \xi_2, \xi_3, \xi_4 \in \overline{U}$, then $\{|a_i(x)G_i(x, u_n, v_n)|\}$ is uniformly integrable.

Thus $\{|a_i(x)G_i(x, u_n, v_n) - a_i(x)G_i(x, u, v)|\}$ is uniformly integrable, and then

$$\begin{aligned} & \lim_{n\to\infty} \int_e |a_i(x)G_i(x, u_n, v_n) - a_i(x)G_i(x, u, v)| \, dx \\ = & \int_e \lim_{n\to\infty} |a_i(x)G_i(x, u_n, v_n) - a_i(x)G_i(x, u, v)| \, dx = 0. \end{aligned}$$

Similarly, we can get the weak-strong continuity of $\Psi_i'$.

Since $X$ be a reflexive and separable Banach space, there are sequences $\{e_j\} \subset X$ and $\{e_j^*\} \subset X^*$ such that

$$X = \overline{span}\{e_j,\, j = 1, 2, \cdots\}, \quad X^* = \overline{span}^{w^*}\{e_j^*,\, j = 1, 2, \cdots\},$$

and $< e_j^*, e_j >= \begin{cases} 1, i = j, \\ 0, i \neq j. \end{cases}$

For convenience, we write

$$X_j = span\{e_j\},\, Y_k = \overset{k}{\underset{j=1}{\oplus}} X_j,\, Z_k = \overline{\overset{\infty}{\underset{j=k}{\oplus}} X_j}. \tag{3}$$

**Definition 9** *(i) We say $J$ satisfies $(PS)$ condition in $X$, if any sequence $\{(u_n, v_n)\} \subset X$ such that $\{J(u_n, v_n)\}$ is bounded and $\|J'(u_n, v_n)\|_* \to 0$ as $n \to \infty$, has a convergent subsequence; (ii) We say $J$ satisfies $(PS)_c^*$ condition in $X$, if any sequence $\{(u_{n_j}, v_{n_j})\} \subset X$ such that $n_j \to \infty$, $(u_{n_j}, v_{n_j}) \in Y_{n_j}$ , $J(u_{n_j}, v_{n_j}) \to c$ and $(J|_{Y_{n_j}})'(u_{n_j}, v_{n_j}) \to 0$, contains a subsequence converging to a critical point of $J$.*

**Lemma 10** *If $\{(u_n, v_n)\}$ is a bounded (PS) sequence of $J$, then there exists a small enough positive constant $C_0 < 1$ such that, if*

$$|sF_s(x, s, t)| + |tF_t(x, s, t)| \leq C(x) + C_0(|s|^{p^*(x)} + |t|^{q^*(x)}),\ \forall (x, s, t) \in \overline{\Omega} \times \mathbb{R}^2,$$

*where $C(\cdot) \in L^1(\Omega)$, then $\{u_n\}$ has a convergent subsequence in $X$.*

**Proof** Let $\{(u_n, v_n)\}$ be a bounded $(PS)$ sequence of $J$, i.e.

$$J(u_n, v_n) \to c, J'(u_n, v_n) \to 0 \text{ as } n \longrightarrow \infty.$$

Since $\{(u_n, v_n)\}$ is bounded, there exists a $(u, v) \in X$, such that $(u_n, v_n) \rightharpoonup (u, v)$ in $X$. By Proposition 6, we may assume that there exist $\mu, \nu, \mu_\#, \nu_\# \in \mathbb{M}(\overline{\Omega})$ and sequence $\{x_j\}_{j\in J}$ in $\overline{\Omega}$ such that

$$
\begin{aligned}
u_n &\rightharpoonup u \text{ in } W_0^{1,p(\cdot)}(\Omega),\\
|\nabla u_n|^{p(x)} \overset{*}{\rightharpoonup} \mu &= |\nabla u|^{p(x)} + \sum_{j\in J}\mu_j\delta_{x_j} + \widetilde{\mu}, \text{ in } \mathbb{M}(\overline{\Omega}),\\
|u_n|^{p^*(x)} \overset{*}{\rightharpoonup} \nu &= |u|^{p^*(x)} + \sum_{j\in J}\nu_j\delta_{x_j}, \text{ in } \mathbb{M}(\overline{\Omega}),\\
\nu_j &\leq C_{p^*}^* \max\{\mu_j^{\frac{p^{+*}}{p^-}}, \mu_j^{\frac{p^{-*}}{p^+}}\},
\end{aligned}
$$

where $C_{p^*}^* = \sup\{|\omega|_{p^*(\cdot)}^{p^{*+}} + 1 \Big| \omega \in W_0^{1,p(\cdot)}(\Omega), |\nabla\omega|_{p(\cdot)} \leq 1\} < +\infty$, and

$$
\begin{aligned}
v_n &\rightharpoonup v \text{ in } W_0^{1,q(\cdot)}(\Omega),\\
|\nabla v_n|^{p(x)} \overset{*}{\rightharpoonup} \mu_\# &= |\nabla v|^{q(x)} + \sum_{j\in J}\mu_{\# j}\delta_{x_j} + \widetilde{\mu}_\#, \text{ in } \mathbb{M}(\overline{\Omega}),\\
|v_n|^{p^*(x)} \overset{*}{\rightharpoonup} \nu_\# &= |v|^{q^*(x)} + \sum_{j\in J}\nu_{\# j}\delta_{x_j}, \text{ in } \mathbb{M}(\overline{\Omega}),\\
\nu_{\# j} &\leq C_{q^*}^* \max\{\mu_{\# j}^{\frac{q^{+*}}{q^-}}, \mu_{\# j}^{\frac{q^{-*}}{q^+}}\},
\end{aligned}
$$

where

$$
C_{q^*}^* = \sup\{|\omega|_{q^*(\cdot)}^{q^{*+}} + 1 \Big| \omega \in W_0^{1,q(\cdot)}(\Omega), |\nabla\omega|_{p(\cdot)} \leq 1\} < +\infty.
$$

Next we will complete the proof of this Lemma in three steps.

*Step 1.* We will prove $\mu(\{x_j\}) = \nu(\{x_j\}) = 0$ and $\mu_\#(\{x_j\}) = \nu_\#(\{x_j\}) = 0$ for all $j = 1, 2, \cdots$.

Obviously, there exists $r_n > 0$ such that

$$
\begin{aligned}
p^-(x_n) \;&:= \inf_{y\in B_r(x_n)\cap\overline{\Omega}} p(y) \leq p^+(x_n) := \sup_{y\in B_r(x_n)\cap\overline{\Omega}} p(y)\\
&< p^{*-}(x_n) := \inf_{y\in B_r(x_n)\cap\overline{\Omega}} p^*(y) \leq p^{*+}(x_n) := \sup_{y\in B_r(x_n)\cap\overline{\Omega}} p^*(y), \forall r \in (0, r_n],
\end{aligned}
$$

$$
\begin{aligned}
q^-(x_n) \;&:= \inf_{y\in B_r(x_n)\cap\overline{\Omega}} q(y) \leq q^+(x_n) := \sup_{y\in B_r(x_n)\cap\overline{\Omega}} q(y)\\
&< q^{*-}(x_n) := \inf_{y\in B_r(x_n)\cap\overline{\Omega}} q^*(y) \leq q^{*+}(x_n) := \sup_{y\in B_r(x_n)\cap\overline{\Omega}} q^*(y), \forall r \in (0, r_n].
\end{aligned}
$$

For every $\varepsilon > 0$, we set $\phi_\varepsilon(x) = \phi((x - x_1)/\varepsilon), x \in \Omega$, where $\phi \in C_0^\infty(\mathbb{R}^N)$, $0 \le \phi \le 1$, $\phi \equiv 1$ in $B_1\{0\}$ and $\phi \equiv 0$ in $\mathbb{R}^N \backslash B_2\{0\}$ and $|\nabla\phi| \le 2$. Since $J'(u_n, v_n) \to 0$ in $X^*$ as $n \longrightarrow \infty$ and $\{(u_n, v_n)\}$ is bounded, we have

$$\begin{aligned}
& \int_\Omega |\nabla u_n|^{p(x)-2} \nabla u_n \cdot \nabla(\phi_\varepsilon u_n)dx + \int_\Omega |u_n|^{p(x)-2} u_n \phi_\varepsilon u_n dx \\
= & \int_\Omega \partial_2 F(x, u_n, v_n)\phi_\varepsilon u_n dx + o(1) \\
\le & \int_\Omega [C(x) + C_0(|u_n|^{p^*(x)} + |v_n|^{q^*(x)})]\phi_\varepsilon dx + o(1),
\end{aligned}$$

which implies that

$$\begin{aligned}
& \int_\Omega \phi_\varepsilon |\nabla u_n|^{p(x)} dx + \int_\Omega |u_n|^{p(x)} \phi_\varepsilon dx + \int_\Omega u_n |\nabla u_n|^{p(x)-2} \nabla u_n \nabla\phi_\varepsilon dx \\
\le & \int_\Omega [C(x) + C_0(|u_n|^{p^*(x)} + |v_n|^{q^*(x)})]\phi_\varepsilon dx + o(1).
\end{aligned} \tag{4}$$

Since $\{(u_n, v_n)\}$ is bounded in $X$, we may assume

$$\begin{aligned}
|\nabla u_n|^{p(x)-2} \nabla u_n &\rightharpoonup T \in (L^{p^0(\cdot)}(\Omega))^N, \\
\partial_2 F(x, u_n, v_n) &\rightharpoonup g(x) \in L^{(p^*(\cdot))^0}(\Omega).
\end{aligned}$$

Since $J'(u_n, v_n) \to 0$ in $X^*$ as $n \longrightarrow \infty$, we also have

$$\int_\Omega |\nabla u_n|^{p(x)-2} \nabla u_n \cdot \nabla(\phi_\varepsilon u)dx + \int_\Omega |u_n|^{p(x)-2} u_n \phi_\varepsilon u dx = \int_\Omega \partial_2 F(x, u_n, v_n)\phi_\varepsilon u dx + o(1).$$

then

$$\int_\Omega T \cdot \nabla(\phi_\varepsilon u)dx + \int_\Omega |u|^{p(x)} \phi_\varepsilon dx = \int_\Omega \partial_2 F(x, u, v)u\phi_\varepsilon dx. \tag{5}$$

We claim

$$\int_\Omega u_n |\nabla u_n|^{p(x)-2} \nabla u_n \nabla\phi_\varepsilon dx \to \int_\Omega uT\nabla\phi_\varepsilon dx \text{ as } n \to \infty. \tag{6}$$

In fact

$$\begin{aligned}
& \int_\Omega \{u_n |\nabla u_n|^{p(x)-2} \nabla u_n \nabla\phi_\varepsilon - uT\nabla\phi_\varepsilon\}dx \\
= & \int_\Omega (u_n - u) |\nabla u_n|^{p(x)-2} \nabla u_n \nabla\phi_\varepsilon dx + \int_\Omega u\nabla\phi_\varepsilon\{|\nabla u_n|^{p(x)-2} \nabla u_n - T\}dx \to 0 \text{ as } n \to \infty.
\end{aligned}$$

It follows from (4), (5) and (6) that

$$\int_\Omega \phi_\varepsilon d\mu + \int_\Omega |u|^{p(x)} \phi_\varepsilon dx \le \int_\Omega C(x)\phi_\varepsilon dx + \int_\Omega C_0\phi_\varepsilon d\nu + \int_\Omega C_0\phi_\varepsilon d\nu_\# - \int_\Omega uT\nabla\phi_\varepsilon dx$$

$$
\begin{aligned}
&= \int_\Omega C(x)\phi_\varepsilon dx + \int_\Omega C_0\phi_\varepsilon d\nu + \int_\Omega C_0\phi_\varepsilon d\nu_\#\\
&\quad -\{\int_\Omega \partial_2 F(x,u,v)u\phi_\varepsilon dx - \int_\Omega |u|^{p(x)}\phi_\varepsilon dx - \int_\Omega \phi_\varepsilon T\cdot\nabla u dx\}.
\end{aligned}
$$

Letting $\varepsilon \to 0$, we have

$$\mu(\{x_1\}) \le C_0(\nu(\{x_1\}) + \nu_\#(\{x_1\})).$$

Similarly, we have

$$\mu_\#(\{x_1\}) \le C_0(\nu(\{x_1\}) + \nu_\#(\{x_1\})),$$

$$\mu(\{x_j\}) \le C_0(\nu(\{x_j\}) + \nu_\#(\{x_1\})), j = 2,3,\cdots,$$

$$\mu_\#(\{x_j\}) \le C_0(\nu(\{x_j\}) + \nu_\#(\{x_1\})), j = 2,3,\cdots.$$

Suppose that $\mu(\{x_j\}) + \mu_\#(\{x_j\}) > 0$ for some $j$, then $\nu(\{x_j\}) + \nu_\#(\{x_j\}) > 0$. Let $M_*$ be a constant such that

$$\int_\Omega [|u_n|^{p^*(x)} + |v_n|^{q^*(x)}]dx \le M_* < 0 \text{ for all } n. \tag{7}$$

If $\nu_j + \nu_{\#j} \ge 1$, then we have

$$
\begin{aligned}
\nu_j &\le C^*_{p^*}\max\{\mu_j^{\frac{p^{*+}(x_j)}{p^-(x_j)}}, \mu_j^{\frac{p*^-(x_j)}{p^+(x_j)}}\} \le C^*_{p^*}\max\{[C_0(\nu_j+\nu_{\#j})]^{\frac{p^{*+}(x_j)}{p^-(x_j)}}, [C_0(\nu_j+\nu_{\#j})]^{\frac{p*^-(x_j)}{p^+(x_j)}}\}\\
&\le C^*_{p^*}[C_0]^{\frac{p*^-(x_j)}{p^+(x_j)}}(\nu_j+\nu_{\#j})^{\frac{p^{*+}(x_j)}{p^-(x_j)}},
\end{aligned}
$$

$$
\begin{aligned}
\nu_{\#j} &\le C^*_{q^*}\max\{\mu_{\#j}^{\frac{q^{*+}(x_j)}{q^-(x_j)}}, \mu_{\#j}^{\frac{q^{*-}(x_j)}{q^+(x_j)}}\} \le C^*_{q^*}\max\{[C_0(\nu_j+\nu_{\#j})]^{\frac{q^{*+}(x_j)}{q^-(x_j)}}, [C_0(\nu_j+\nu_{\#j})]^{\frac{q^{*-}(x_j)}{q^+(x_j)}}\}\\
&\le C^*_{q^*}[C_0]^{\frac{q^{*-}(x_j)}{q^+(x_j)}}(\nu_j+\nu_{\#j})^{\frac{q^{*+}(x_j)}{q^-(x_j)}},
\end{aligned}
$$

which implies that

$$\nu_j + \nu_{\#j} \ge \left|\frac{1}{C^*_{p^*}[C_0]^{\frac{p*^-(x_j)}{p^+(x_j)}} + C^*_{q^*}[C_0]^{\frac{q^{*-}(x_j)}{q^+(x_j)}}}\right|^{\frac{1}{\max\{\frac{p^{*+}(x_j)}{p^-(x_j)}, \frac{q^{*+}(x_j)}{q^-(x_j)}\}-1}}.$$

Similarly, if $\nu_j + \nu_{\#j} \le 1$, then we have

$$\nu_j + \nu_{\#j} \ge \left|\frac{1}{C^*_{p^*}[C_0]^{\frac{p*^-(x_j)}{p^+(x_j)}} + C^*_{q^*}[C_0]^{\frac{q^{*-}(x_j)}{q^+(x_j)}}}\right|^{\frac{1}{\min\{\frac{p*^-(x_j)}{p^+(x_j)}, \frac{q^{*-}(x_j)}{q^+(x_j)}\}-1}}.$$

When $C_0$ is small enough, it is a contradiction to (7). Now we completed the step 1.

*Step 2.* We will show $(u_n, v_n) \to (u, v)$ strong in $L^{p^*(\cdot)}(\Omega) \times L^{q^*(\cdot)}(\Omega)$ as $n \to \infty$.

Since $|u_n|^{p^*(x)} \overset{*}{\rightharpoonup} \nu = |u|^{p^*(x)}$, we have

$$\lim_{n\to\infty} \int_\Omega |u_n|^{p^*(x)}\, dx = \int_\Omega |u|^{p^*(x)}\, dx,$$

notice that $|u_n|^{p^*(x)} \to |u|^{p^*(x)}$ in measure, then we can see $\{|u_n|^{p^*(x)}\}$ is uniformly integrable.

Since

$$|u_n - u|^{p^*(x)} \le 2^{p^*(x)}(|u_n|^{p^*(x)} + |u|^{p^*(x)}),$$

we can see that $\{|u_n - u|^{p^*(x)}\}$ is uniformly integrable. Thus

$$\lim_{n\to\infty} \int_\Omega |u_n - u|^{p^*(x)}\, dx = \int_\Omega \lim_{n\to\infty} |u_n - u|^{p^*(x)}\, dx = 0.$$

Similarly, we have $v_n \to v$ strong in $L^{q^*(\cdot)}(\Omega)$ as $n \to \infty$.

*Step 3.* We will show $(u_n, v_n) \to (u, v)$ strong in $X$ as $n \to \infty$.

Since $J'(u_n, v_n) = \Phi'(u_n, v_n) - \Psi'(u_n, v_n) \to 0$ and $(u_n, v_n) \to (u, v)$ strong in $L^{p^*(\cdot)}(\Omega) \times L^{q^*(\cdot)}(\Omega)$ as $n \to \infty$, then we can see $\Psi'(u_n, v_n) \to \Psi'(u, v)$ and

$$\Phi'(u_n, v_n) \to \Psi'(u, v) \text{ as } n \to \infty.$$

As $L = \Phi'$ is a homeomorphism, then we can see $(u_n, v_n) \to L^{-1}(\Psi'(u, v))$ in $X$ as $n \to \infty$.

$\square$

For each $i = 1, \cdots, m$, we assume $\lambda_i, a_i$ and $G_i$ satisfy one of the following conditions

**($\mathbf{B}_1$)** $\alpha_i(x) < p(x), \beta_i(x) < q(x)$ and $r_i(x) \ge (p(x)/\alpha_i(x))^0$ and $r_i(x) \ge (q(x)/\beta_i(x))^0, \forall x \in \overline{\Omega}$.

**($\mathbf{B}_2$)** $\lambda_i a_i > 0$, $r_i(\cdot) \in C_+(\overline{\Omega})$, and there exist functions $\theta_1(\cdot), \theta_2(\cdot) \in C^1(\overline{\Omega})$ (which are independent on $i$) satisfy

$$p(x) < \theta_1(x) \le \frac{1}{r_i^0(x)} p^*(x), q(x) \le \theta_2(x) \le \frac{1}{r_i^0(x)} q^*(x), \forall x \in \overline{\Omega},$$

such that $G_i$ satisfies

$$0 < G_i(x, s, t) \le \frac{s}{\theta_1} \frac{\partial}{\partial s} G_i(x, s, t) + \frac{t}{\theta_2} \frac{\partial}{\partial t} G_i(x, s, t), \forall x \in \overline{\Omega}, |s|^{\theta_1} + |t|^{\theta_2} \ge 2M > 0.$$

**($\mathbf{B}_2'$)** $\lambda_i a_i \in L_+^\infty(\Omega)$, and for the functions $\theta_1(\cdot)$ and $\theta_2(\cdot)$ in **($\mathbf{B}_2$)**, $G_i$ satisfies

$$0 < G_i(x, s, t) \le \frac{s}{\theta_1} \frac{\partial}{\partial s} G_i(x, s, t) + \frac{t}{\theta_2} \frac{\partial}{\partial t} G_i(x, s, t), \forall x \in \overline{\Omega}, |s|^{\theta_1} + |t|^{\theta_2} \ge 2M > 0,$$

and
$$\frac{\left|\frac{s}{\theta_1}\frac{\partial}{\partial s}G_i(x,s,t)\right|+\left|\frac{t}{\theta_2}\frac{\partial}{\partial t}G_i(x,s,t)\right|}{|s|^{p^*(x)}+|t|^{q^*(x)}}\to 0 \text{ uniformly as } |s|+|t|\to+\infty.$$

**($\mathbf{B}_3$)** $\lambda_i a_i<0$, and for the functions $\theta_1(\cdot)$ and $\theta_2(\cdot)$ in **($\mathbf{B}_2$)**, there exists a small positive constant $\delta$ such that $G_i$ satisfies

$$G_i(x,s,t)\geq\frac{1+\delta}{\theta_1}s\frac{\partial}{\partial s}G_i(x,s,t)+\frac{1+\delta}{\theta_2}t\frac{\partial}{\partial t}G_i(x,s,t)>0,\forall x\in\overline{\Omega},|s|^{\theta_1}+|t|^{\theta_2}\geq 2M>0.$$

Denote $\Lambda=\{1,\cdots,m\}$, and

$$\begin{aligned}\mathcal{U}_1&=\{i\in\Lambda\mid\lambda_i,a_i\text{ and }G_i\text{ satisfies }(B_1)\},\\ \mathcal{U}_2&=\{i\in\Lambda\backslash\mathcal{U}_1\mid\lambda_i,a_i\text{ and }G_i\text{ satisfies }(B_2)\text{ or }(B_2')\},\\ \mathcal{U}_3&=\{i\in\Lambda\backslash(\mathcal{U}_1\cup\mathcal{U}_2)\mid\lambda_i,a_i\text{ and }G_i\text{ satisfies }(B_3)\}.\end{aligned}$$

**Lemma 11** *If $\mathcal{U}_1\cup\mathcal{U}_3=\Lambda$, or $\mathcal{U}_2$ is nonempty and there are some $i_1,i_2\in\mathcal{U}_2$ such that $|a_{i_1}(\cdot)|^{-p(\cdot)/(\theta_1(\cdot)-p(\cdot))},|a_{i_2}(\cdot)|^{-q(\cdot)/(\theta_2(\cdot)-q(\cdot))}\in L^1(\Omega)$ then $J$ satisfies $(PS)$ conditions in $X$.*

**Proof** For any $\varepsilon>0$, it is easy to see that

$$|sF_s(x,s,t)|+|tF_t(x,s,t)|\leq C_\varepsilon(x)+\varepsilon(|s|^{p^*(x)}+|t|^{q^*(x)}),\ \forall x\in\overline{\Omega},$$

where $C_\varepsilon(\cdot)\in L^1(\Omega)$ is dependent on $\varepsilon$.

According to Lemma 10, we only need to prove that every $(PS)$ sequence of $J$ are bounded in $X$.

(i) If $\mathcal{U}_1\cup\mathcal{U}_3=\Lambda$. Let $\{(u_n,v_n)\}$ be a $(PS)$ sequence, then it is easy to see that

$$c\geq J(u_n,v_n)\geq\Phi(u_n,v_n)-\sum_{i\in\mathcal{U}_1}\int_\Omega|\lambda_i a_i(x)|\,(|u_n|^{\alpha_i(x)}+|v_n|^{\beta_i(x)})dx-C_1.$$

For any $i\in\mathcal{U}_1$, since $r_i(x)\geq(p(x)/\alpha_i(x))^0$ and $r_i(x)\geq(q(x)/\beta_i(x))^0$, from Yang inequality, we have

$$|\lambda_i a_i(x)|\,|u_n|^{\alpha_i(x)}\leq\frac{p(x)-\alpha_i(x)}{p(x)}(\frac{1}{\varepsilon}\,|\lambda_i a_i(x)|)^{(\frac{p(x)}{\alpha_i(x)})^0}+\frac{\alpha_i(x)}{p(x)}(\varepsilon\,|u_n|^{\alpha_i(x)})^{\frac{p(x)}{\alpha_i(x)}},\varepsilon>0,i=1,\cdots,m,$$

$$|\lambda_i a_i(x)|\,|v_n|^{\beta_i(x)}\leq\frac{q(x)-\beta_i(x)}{q(x)}(\frac{1}{\varepsilon}\,|\lambda_i a_i(x)|)^{(\frac{q(x)}{\beta_i(x)})^0}+\frac{\beta_i(x)}{q(x)}(\varepsilon\,|v_n|^{\beta_i(x)})^{\frac{q(x)}{\beta_i(x)}},\varepsilon>0,i=1,\cdots,m.$$

Suppose the positive number $\varepsilon$ is small enough, we can see that

$$\sum_{i\in\mathcal{U}_1}\frac{\alpha_i(x)}{p(x)}\varepsilon^{\frac{p(x)}{\alpha_i(x)}}<\frac{a_0}{2}\text{ and }\sum_{i\in\mathcal{U}_1}\frac{\beta_i(x)}{q(x)}\varepsilon^{\frac{q(x)}{\beta_i(x)}}<\frac{b_0}{2},\forall x\in\overline{\Omega}.$$

Thus

$$c \geq J(u_n, v_n) \geq \Phi(u_n, v_n) - \frac{1}{2}\Phi(u_n, v_n) - C_2 \geq \frac{1}{2}\Phi(u_n, v_n) - C_2.$$

It means that $\{(u_n, v_n)\} \subset X$ is bounded.

(ii) If $\mathcal{U}_2$ is nonempty. The conditions $(\mathbf{B}_2)$, $(\mathbf{B}_2')$ and $(\mathbf{A})$ imply that, for any $(x, s, t) \in \overline{\Omega} \times \mathbb{R}^2$, we have

$$F_i(x, s, t) \leq \frac{1}{\theta_1} s F_{i,s}(x, s, t) + \frac{1}{\theta_2} t F_{i,t}(x, s, t) + |\lambda_i a_i(x)| c_i, \forall x \in \overline{\Omega},\ i \in \mathcal{U}_2,$$

$$F_i(x, s, t) \leq \frac{1+\delta}{\theta_1} s F_{i,s}(x, s, t) + \frac{1+\delta}{\theta_2} t F_{i,t}(x, s, t) + |\lambda_i a_i(x)| c_i, \forall x \in \overline{\Omega},\ i \in \mathcal{U}_3.$$

Thus

$$\begin{aligned}
& F(x, s, t) - (\frac{1+\delta}{\theta_1} s F_s(x, s, t) + \frac{1+\delta}{\theta_2} t F_t(x, s, t)) \\
\leq & \sum_{i \in \mathcal{U}_1} [F_i(x, s, t) - (\frac{1+\delta}{\theta_1} s F_{i,s}(x, s, t) + \frac{1+\delta}{\theta_2} t F_{i,t}(x, s, t))] - \sum_{i \in \mathcal{U}_2} \delta F_i(x, s, t) + \sum_{i \in \mathcal{U}_2 \cup \mathcal{U}_3} |\lambda_i a_i(x)| c_i \\
\leq & \sum_{i \in \mathcal{U}_2 \cup \mathcal{U}_3} |\lambda_i a_i(x)| c_i + C_1 \sum_{i \in \mathcal{U}_1} |\lambda_i a_i(x)| (1 + |s|^{\alpha_i} + |t|^{\beta_i}) - \sum_{i \in \mathcal{U}_2} \delta |\lambda_i a_i(x)| (|s|^{\theta_1} + |t|^{\theta_2}).
\end{aligned}$$

Denote

$$l_1 = \min_{x \in \overline{\Omega}} (\frac{1}{p(x)} - \frac{1+\delta}{\theta_1(x)}), l_2 = \min_{x \in \overline{\Omega}} (\frac{1}{q(x)} - \frac{1+\delta}{\theta_2(x)}), \tag{8}$$

where the positive constant $\delta$ is small enough such that $l_1, l_2 > 0$.

Let $\{(u_n, v_n)\}$ be a $(PS)$ sequence, then we have

$$\begin{aligned}
& c + 1 + (\|u_n\|_{p(\cdot)} + \|v_n\|_{q(\cdot)}) \\
\geq & J(u_n, v_n) - J'(u_n, v_n)(\frac{1+\delta}{\theta_1(x)} u_n, \frac{1+\delta}{\theta_2(x)} v_n) \\
= & \int_\Omega \frac{1}{p(x)} (|\nabla u_n|^{p(x)} + a(x) |u_n|^{p(x)}) dx + \int_\Omega \frac{1}{q(x)} (|\nabla v_n|^{q(x)} + b(x) |v_n|^{q(x)}) dx - \int_\Omega F(x, u_n, v_n) dx \\
& - \int_\Omega \frac{1+\delta}{\theta_1(x)} (|\nabla u_n|^{p(x)} + a(x) |u_n|^{p(x)}) dx + \int_\Omega \frac{1+\delta}{\theta_1(x)} u_n F_u(x, u_n, v_n) dx \\
& + \int_\Omega \frac{1+\delta}{\theta_1^2(x)} u_n |\nabla u_n|^{p(x)-2} \nabla u_n \nabla \theta_1(x) dx - \int_\Omega \frac{1+\delta}{\theta_2(x)} (|\nabla v_n|^{q(x)} + b(x) |v_n|^{q(x)}) dx \\
& + \int_\Omega \frac{1+\delta}{\theta_2(x)} v_n F_v(x, u_n, v_n) dx + \int_\Omega \frac{1+\delta}{\theta_2^2(x)} v_n |\nabla v_n|^{q(x)-2} \nabla v_n \nabla \theta_2(x) dx \\
\geq & \int_\Omega (\frac{1}{p(x)} - \frac{1+\delta}{\theta_1(x)}) (|\nabla u_n|^{p(x)} + a(x) |u_n|^{p(x)}) dx + \int_\Omega (\frac{1}{q(x)} - \frac{1+\delta}{\theta_2(x)}) (|\nabla v_n|^{p(x)} + b(x) |v_n|^{p(x)}) dx \\
& + \delta \sum_{i \in \mathcal{U}_2} \int_\Omega F_i(x, u_n, v_n) dx - \int_\Omega \frac{(1+\delta) |\nabla \theta_1(x)|}{\theta_1^2(x)} |u_n| |\nabla u_n|^{p(x)-1} dx
\end{aligned}$$

$$
\begin{aligned}
&-\int_\Omega \frac{(1+\delta)\left|\nabla\theta_2(x)\right|}{\theta_2^2(x)}\left|v_n\right|\left|\nabla v_n\right|^{q(x)-1}dx - C_1\sum_{i\in\mathcal{U}_1}\int_\Omega \left|\lambda_i a_i(x)\right|(\left|u_n\right|^{\alpha_i(x)}+\int_\Omega \left|v_n\right|^{\beta_i(x)})dx - C_2\\
\geq\ & l_1\int_\Omega(\left|\nabla u_n\right|^{p(x)}+a(x)\left|u_n\right|^{p(x)})dx + l_2\int_\Omega(\left|\nabla v_n\right|^{p(x)}+b(x)\left|v_n\right|^{p(x)})dx\\
&+\delta\int_\Omega(\left|\lambda_{i_1}a_{i_1}(x)\right|\left|u_n\right|^{\theta_1(x)}+\left|\lambda_{i_2}a_{i_2}(x)\right|\left|v_n\right|^{\theta_2(x)})dx - \int_\Omega \frac{(1+\delta)\left|\nabla\theta_1(x)\right|}{\theta_1^2(x)}\left|u_n\right|\left|\nabla u_n\right|^{p(x)-1}dx\\
&-\int_\Omega \frac{(1+\delta)\left|\nabla\theta_2(x)\right|}{\theta_2^2(x)}\left|v_n\right|\left|\nabla v_n\right|^{q(x)-1}dx - C_3 - C_1\sum_{i\in\mathcal{U}_1}\int_\Omega \left|\lambda_i a_i(x)\right|(\left|u_n\right|^{\alpha_i(x)}+\left|v_n\right|^{\beta_i(x)})dx.
\end{aligned}
$$

Note that $\left|a_{i_1}(\cdot)\right|^{-p(\cdot)/(\theta_1(\cdot)-p(\cdot))}\in L^1(\Omega)$, we have

$$
\begin{aligned}
&\frac{(1+\delta)\left|\nabla\theta_1(x)\right|}{\theta_1^2(x)}\left|u_n\right|\left|\nabla u_n\right|^{p(x)-1}\\
\leq\ & C_4\frac{1}{p(x)}\frac{1}{\varepsilon_1^{p(x)}}\left|u_n\right|^{p(x)} + C_4\frac{p(x)-1}{p(x)}\varepsilon_1^{\frac{p(x)}{p(x)-1}}\left|\nabla u_n\right|^{p(x)}\\
\leq\ & C_4\frac{1}{p(x)}\frac{1}{\varepsilon_1^{p(x)}}\{\frac{\theta_1(x)-p(x)}{\theta_1(x)}[\frac{1}{\varepsilon_1^{p(x)}}a_{i_1}^{-\frac{p(x)}{\theta_1(x)}}(x)]^{\frac{\theta_1(x)}{\theta_1(x)-p(x)}} + \frac{p(x)}{\theta_1(x)}[\varepsilon_1^{p(x)}a_{i_1}^{\frac{p(x)}{\theta_1(x)}}(x)\left|u_n\right|^{p(x)}]^{\frac{\theta_1(x)}{p(x)}}\}\\
&+C_4\frac{p(x)-1}{p(x)}\varepsilon_1^{\frac{p(x)}{p(x)-1}}\left|\nabla u_n\right|^{p(x)}\\
=\ & C_4\frac{1}{p(x)}\{\frac{\theta_1(x)-p(x)}{\theta_1(x)}\varepsilon_1^{\frac{-\theta_1(x)p(x)}{\theta_1(x)-p(x)}-p(x)}a_{i_1}^{\frac{-p(x)}{\theta_1(x)-p(x)}}(x) + \frac{p(x)}{\theta_1(x)}\varepsilon_1^{\theta_1(x)-p(x)}a_{i_1}(x)\left|u_n\right|^{\theta_1(x)}\}\\
&+C_4\frac{p(x)-1}{p(x)}\varepsilon_1^{\frac{p(x)}{p(x)-1}}\left|\nabla u_n\right|^{p(x)}.
\end{aligned}
$$

Similarly, since $\left|a_{i_2}(\cdot)\right|^{-q(\cdot)/(\theta_2(\cdot)-q(\cdot))}\in L^1(\Omega)$, we have

$$
\begin{aligned}
&\frac{(1+s)\left|\nabla\theta_2(x)\right|}{\theta_2^2(x)}\left|v_n\right|\left|\nabla v_n\right|^{q(x)-1} \leq C_5\frac{q(x)-1}{q(x)}\varepsilon_2^{\frac{q(x)}{q(x)-1}}\left|\nabla v_n\right|^{q(x)}\\
&+C_5\frac{1}{q(x)}\{\frac{\theta_2(x)-q(x)}{\theta_2(x)}\varepsilon_2^{\frac{-\theta_2(x)q(x)}{\theta_2(x)-q(x)}-q(x)}a_{i_2}^{\frac{-q(x)}{\theta_2(x)-q(x)}}(x) + \frac{q(x)}{\theta_2(x)}\varepsilon_2^{\theta_2(x)-q(x)}a_{i_2}(x)\left|v_n\right|^{\theta_2(x)}\}.
\end{aligned}
$$

Suppose positive constants $\varepsilon_1$ and $\varepsilon_2$ are small enough. It follows from the definitions of $\theta_1(\cdot)$ and $\theta_2(\cdot)$ that

$$
\begin{aligned}
&c+1+(\left\|u_n\right\|_{p(\cdot)}+\left\|v_n\right\|_{q(\cdot)})\\
\geq\ & J(u_n,v_n)-J'(u_n,v_n)(\frac{1+\delta}{\theta_1(x)}u_n,\frac{1+\delta}{\theta_2(x)}v_n)\\
\geq\ & \frac{2l_1}{3}\int_\Omega(\left|\nabla u_n\right|^{p(x)}+a(x)\left|u_n\right|^{p(x)})dx + \frac{2l_2}{3}\int_\Omega(\left|\nabla v_n\right|^{p(x)}+b(x)\left|v_n\right|^{p(x)})dx - C_6\\
&-c_3\sum_{i\in\mathcal{U}_1}\int_\Omega \left|\lambda_i a_i(x)\right|(\left|u_n\right|^{\alpha_i(x)}dx+\int_\Omega \left|v_n\right|^{\beta_i(x)})dx.
\end{aligned}
$$

Similar to the proof of (i), we have

$$
c+(\left\|u_n\right\|_{p(\cdot)}+\left\|v_n\right\|_{q(\cdot)})
$$

$$
\begin{aligned}
&\geq J(u_n,v_n)-J'(u_n,v_n)(\frac{1+\delta}{\theta_1(x)}u_n,\frac{1+\delta}{\theta_2(x)}v_n) \\
&\geq \frac{l_1}{3}\int_\Omega(|\nabla u_n|^{p(x)}+a(x)\,|u_n|^{p(x)})dx+\frac{l_2}{3}\int_\Omega(|\nabla v_n|^{p(x)}+b(x)\,|v_n|^{p(x)})dx-C_7.
\end{aligned}
$$

Thus $\{\|u_n\|_{p(\cdot)}\}$ and $\{\|v_n\|_{q(\cdot)}\}$ are bounded. $\square$

**Lemma 12** *(see [17]) Assume that* $\Theta : X \to \mathbb{R}$ *is weakly-strongly continuous and* $\Theta(0,0)=0$, $\gamma>0$ *is a given number. Let*

$$\phi_k=\phi_k(\gamma)=\sup\{\Theta(u,v)\mid \|(u,v)\|\leq\gamma, u\in Z_k\},$$

*then* $\phi_k\to 0$ *as* $k\to\infty$.

**Lemma 13** *(see [19]) If* $|u|^{\varsigma(\cdot)}\in L^{s(\cdot)/\varsigma(\cdot)}(\Omega)$, *where* $s(x),\varsigma(x)\in L_+^\infty(\Omega)$, *and* $1\leq\varsigma(x)\leq s(x)$, *then* $u\in L^{s(\cdot)}(\Omega)$ *and there is a number* $\overline{\varsigma}\in[\varsigma^-,\varsigma^+]$ *such that* $||u|^{\varsigma}|_{s(\cdot)/\varsigma(\cdot)}=(|u|_{s(\cdot)})^{\overline{\varsigma}}$.

**Proposition 14** *(Fountain theorem, see [43,44]) Assume* $X$ *is a Banach space,* $J\in C^1(X,\mathbb{R})$ *is an even functional and satisfies* $(PS)$ *condition, the subspace* $X_k, Y_k$ *and* $Z_k$ *are defined by (3). If for each* $k=1,2,\cdots$, *there exist* $\gamma_k>\rho_k>0$ *such that*

*(F$_1$)* $\eta_k:=\inf\{J(u,v)\mid (u,v)\in Z_k, \|(u,v)\|=\rho_k\}\to+\infty$ $(k\to\infty)$;

*(F$_2$)* $\zeta_k:=\max\{J(u,v)|\ (u,v)\in Y_k,\|(u,v)\|=\gamma_k\}\leq 0$.

*then* $J$ *has a sequence of critical values tending to* $+\infty$.

**Proposition 15** *(Dual Fountain theorem, see [44]) Assume* $X$ *is a Banach space,* $J\in C^1(X,\mathbb{R})$ *is an even functional, the subspace* $X_k, Y_k$ *and* $Z_k$ *are defined by (3), and there is a* $k_0>0$ *such that, for each* $k\geq k_0$, *there exists* $\rho_k>\gamma_k>0$ *such that*

*(D$_1$)* $\inf\{J(u,v)\mid (u,v)\in Z_k,\|(u,v)\|=\rho_k\}\geq 0$,

*(D$_2$)* $\zeta_k:=\max\{J(u,v)|\ (u,v)\in Y_k,\|(u,v)\|=\gamma_k\}<0$,

*(D$_3$)* $\eta_k:=\inf\{J(u,v)\mid (u,v)\in Z_k,\|(u,v)\|\leq\rho_k\}\to 0$ $(k\to\infty)$,

*(D$_4$)* $J$ *satisfies* $(PS)_c^*$ *condition for every* $c\in[\eta_{k_0},0)$,

*then* $J$ *has a sequence of critical values tending to* $0$.

**Proposition 16** *(see [43,Theorem 6.3]) Suppose* $J\in C^1(X,\mathbb{R})$ *is even, and satisfies* $(PS)$ *condition. Let* $V^+$, $V^-\subset X$ *be closed subspaces of* $X$ *with codim*$V^+ + 1$=*dim* $V^-$, *and suppose there holds*

*($1^0$) $J(0,0)=0$.*

*($2^0$) $\exists\tau>0,\ \rho>0$ such that $\forall(u,v)\in V^+ : \|(u,v)\|=\rho\Rightarrow J(u,v)\geq\tau$.*

*($3^0$) $\exists R>0$ such that $\forall(u,v)\in V^- : \|(u,v)\|\geq R\Rightarrow J(u,v)\leq 0$.*

*Consider the following set:*

$$\Gamma=\{g\in C^0(X,X)\mid g \text{ is odd, } g(u,v)=(u,v) \text{ if } (u,v)\in V^- \text{ and } \|(u,v)\|\geq R\},$$

*then*

*(a) $\forall\delta>0,\ g\in\Gamma,\ S_\delta^+\cap g(V^-)\neq\varnothing$, here $S_\delta^+=\{(u,v)\in V^+\mid \|(u,v)\|=\delta\}$;*

*(b) the number $\varpi:=\inf\limits_{g\in\Gamma}\sup\limits_{(u,v)\in V^-}J(g(u,v))\geq\tau>0$ is a critical value for $J$.*

# 4 Existence of solutions

In this section, using the critical point theory, we give the existence of solutions for problem (1).

**Definition 17** *We say that $(u,v)\in X$ is a weak solution for (1), if*

$$\begin{cases} \int_\Omega |\nabla u|^{p(x)-2}\nabla u\cdot\nabla\varphi dx+\int_\Omega a(x)\,|u|^{p(x)-2}u\cdot\varphi dx=\int_\Omega F_u(x,u,v)\varphi dx,\ \forall\varphi\in W^{1,p(\cdot)}(\Omega),\\ \int_\Omega |\nabla v|^{q(x)-2}\nabla v\cdot\nabla\psi dx+\int_\Omega b(x)\,|v|^{q(x)-2}v\cdot\psi dx=\int_\Omega F_v(x,u,v)\psi dx,\ \forall\psi\in W^{1,q(\cdot)}(\Omega).\end{cases}$$

*It is easy to see that the critical point of $J$ is a solution for (1).*

**Theorem 18** *If **(A)** is satisfied, and **($B_1$)** is satisfied for $i=1,\cdots,m$, then problem (1) has a solution. Furthermore, if $F$ satisfies the following properties*

*(i) $F(x,-s,-t)=F(x,s,t)$, $\forall(x,s,t)\in\Omega\times\mathbb{R}^2$,*

*(ii) There exist constants $\sigma,\delta>0$, an open bounded subset $\Omega_0$ of $\Omega$, such that*

$$F(x,s,t)\geq\sigma(s^{\epsilon_1(x)}+t^{\epsilon_2(x)}),\forall(x,s,t)\in\overline{\Omega}_0\times(0,\delta)\times(0,\delta),$$

*where $1<\epsilon_1(x)<p(x), 1<\epsilon_2(x)<q(x)$ on $\overline{\Omega}_0$,*

*then the problem (1) has a sequence of solutions $\{\pm(u_k,v_k)\mid k=1,2,\cdots\}$ such that $J(\pm(u_k,v_k))<0$ and $J(\pm(u_k,v_k))\to 0$ as $k\to\infty$.*

**Proof** At first, let's prove that $J$ is coercive on $X$. According to (2), similar to the proof of Lemma 11, we have

$$\int_\Omega |\Psi(u,v)|\,dx\leq\frac{1}{2}\Phi(u,v)+c_0.$$

Therefore

$$J(u,v) \geq \frac{1}{2}\Phi(u,v) - c_0 \geq \frac{1}{2p^+}\|u\|_{p(\cdot)}^{p^-} + \frac{1}{2q^+}\|u\|_{q(\cdot)}^{q^-} - c_0 \to +\infty, \text{ as } \|(u,v)\| \to +\infty.$$

From Theorem 8, it is easy to see that $J$ is weak lower semi-continuous. Then $J$ can achieve its infimum in $X$, this provides a solution for (1).

From Lemma 11, we know that $J$ satisfies $(PS)$ condition on $X$. From condition (i), $J$ is an even functional. Denote by $\gamma(A)$ the genus of $A$ (see [9]). Set

$$\begin{aligned}
\Re &= \{A \subset X\backslash\{0\} \mid A \text{ is compact and } A = -A\},\\
\Re_k &= \{A \in \Re \mid \gamma(A) \geq k\},\\
b_k &= \inf_{A\in\Re_k}\sup_{(u,v)\in A} J(u,v), k = 1, 2, \cdots,
\end{aligned}$$

we have

$$-\infty < b_1 \leq b_2 \leq \cdots \leq b_k \leq b_{k+1} \leq \cdots.$$

Now, let's prove that $b_k < 0$ for every $k$.

Obviously, $W_0^{1,p(\cdot)}(\Omega_0) \times W_0^{1,q(\cdot)}(\Omega_0)$ is a subspace of $X$. For any $k$, we can choose a $k$-dimensional linear subspace $E_k$ of $W_0^{1,p(\cdot)}(\Omega_0) \times W_0^{1,q(\cdot)}(\Omega_0)$ such that

$$E_k = span\{(u_1,v_1), \cdots, (u_k,v_k)\} \subset C_0^\infty(\Omega_0) \times C_0^\infty(\Omega_0)$$

satisfy $supp u_i, supp v_i \Subset \Omega_0$, $supp u_i \cap supp u_j = \varnothing$ and $supp v_i \cap supp v_j = \varnothing$ when $i \neq j$, and $\|u_i\|_{p(\cdot)} = \|v_i\|_{q(\cdot)}, i = 1, \cdots, k$. As the norms on $E_k$ are equivalent each other, there exists $\rho_k \in (0,1)$ such that $(u,v) \in E_k$ with $\|(u,v)\| \leq \rho_k$ implies $|u|_{L^\infty} + |v|_{L^\infty} \leq \delta$. Set

$$S_{\rho_k}^{(k)} = \{(u,v) \in E_k \mid \|(u,v)\| = \rho_k\}.$$

Obviously, there are real numbers $c_1, \cdots, c_k$, such that

$$(u,v) = \sum_{i=1}^{k} c_i(u_i,v_i) = \sum_{i=1}^{k}(c_iu_i, c_iv_i), \forall (u,v) \in S_{\rho_k}^{(k)}.$$

For any $(u,v) \in S_{\rho_k}^{(k)}$, from the definition of $S_{\rho_k}^{(k)}$ and the norm $\|\cdot\|$, without loss of generality, we may assume that $\|(u,v)\| = \|u\|_{p(\cdot)}$, and we have

$$\max_{1\leq i\leq k}\|c_iu_i\|_{p(\cdot)} \leq \|u\|_{p(\cdot)} = \rho_k \leq \sum_{i=1}^{k}\|c_iu_i\|_{p(\cdot)} \leq k\max_{1\leq i\leq k}\|c_iu_i\|_{p(\cdot)},$$

then

$$\frac{\rho_k}{k} \le \max_{1\le i\le k} \|c_i u_i\|_{p(\cdot)} \le \rho_k.$$

Obviously, we have

$$\max_{1\le i\le k} \|c_i v_i\|_{q(\cdot)} \le \|v\|_{q(\cdot)} \le \sum_{i=1}^{k} \|c_i v_i\|_{q(\cdot)} \le k \max_{1\le i\le k} \|c_i v_i\|_{q(\cdot)}.$$

Since $\|c_i v_i\|_{q(\cdot)} = \|c_i u_i\|_{p(\cdot)}, i = 1, \cdots, k$, we have

$$\frac{\rho_k}{k} \le \|v\|_{q(\cdot)} \le \rho_k.$$

Thus we have

$$\frac{\rho_k}{k} \le \|u\|_{p(\cdot)} \le \rho_k, \frac{\rho_k}{k} \le \|v\|_{q(\cdot)} \le \rho_k, \forall (u,v) \in S_{\rho_k}^{(k)}.$$

It follows from the compactness of $S_{\rho_k}^{(k)}$ and the definition of the norm $\|\cdot\|$ that there exists constant $\theta_k^{\#} > 0$ such that

$$\int_{\Omega_0} \sigma |u|^{\epsilon_1(x)} dx \ge \theta_k^{\#}, \int_{\Omega_0} \sigma |v|^{\epsilon_2(x)} dx \ge \theta_k^{\#}, \forall (u,v) \in S_{\rho_k}^{(k)}.$$

Without loss of generality, we may assume that $\max_{x\in\overline{\Omega}_0} \epsilon_1(x) < \min_{x\in\overline{\Omega}_0} p(x), \max_{x\in\overline{\Omega}_0} \epsilon_2(x) < \min_{x\in\overline{\Omega}_0} q(x)$. For any $(u,v) \in S_{\rho_k}^{(k)}$ and $t \in (0,1)$, combining the definition of $E_k$ and condition (ii), we have

$$\begin{aligned} J(tu,tv) &\le \Phi(tu,tv) - \int_{\Omega_0} \sigma(t^{\epsilon_1(x)} |u|^{\epsilon_1(x)} + t^{\epsilon_2(x)} |v|^{\epsilon_2(x)}) dx \\ &\le -\frac{1}{2} \int_{\Omega_0} \sigma(t^{\epsilon_1(x)} |u|^{\epsilon_1(x)} + t^{\epsilon_2(x)} |v|^{\epsilon_2(x)}) dx \\ &\le -\frac{1}{2} t^{p^+ + q^+} \theta_k^{\#} \text{ as } t \to 0^+. \end{aligned}$$

We can find $t_k \in (0,1)$ and $\varepsilon_k > 0$ such that

$$J(t_k u, t_k v) \le -\varepsilon_k < 0, \ \forall (u,v) \in S_{\rho_k}^{(k)},$$

that is

$$J(u,v) \le -\varepsilon_k < 0, \ \forall (u,v) \in S_{t_k\rho_k}^{(k)}.$$

Obviously, $\gamma(S_{t_k\rho_k}^{(k)}) = k$, so $b_k \le -\varepsilon_k < 0$.

By the genus theory (see [9], page 219 Theorem 3.3), each $b_k$ is a critical value of $J$, hence there is a sequence of solutions $\{\pm(u_k, v_k) \mid k = 1, 2, \cdots\}$ such that $J(\pm(u_k, v_k)) < 0$.

It only remains to prove $b_k \to 0$ as $k \to \infty$.

Since $J$ is coercive, there exists a constant $R > 1$ such that $J(u,v) > 0$ when $\|(u,v)\| \geq R$. Taking arbitrarily $A \in \Re_k$, then $\gamma(A) \geq k$. Let $Y_k$ and $Z_k$ be the subspaces of $X$ as mentioned in (3), according to the properties of genus we know that $A \cap Z_k \neq \varnothing$. Let

$$\phi_k = \sup\{|\Psi(u,v)|| (u,v) \in Z_k, \|(u,v)\| \leq R\}.$$

By Lemma 3.6, we have $\phi_k \to 0$ as $k \to \infty$. Thus

$$J(u,v) = \Phi(u,v) - \Psi(u,v) \geq -\Psi(u,v) \geq -\phi_k, \text{ when } (u,v) \in Z_k \text{ and } \|(u,v)\| \leq R.$$

Hence $\sup\limits_{(u,v)\in A} J(u,v) \geq -\phi_k$, and then $b_k \geq -\phi_k$, this concludes $b_k \to 0$ as $k \to \infty$. □

**Theorem 19** *If $F$ satisfies **(A)**, for every $i = 1, \cdots, m$, $G_i(x,s,t) = o(|s|^{p(x)+\tau} + |t|^{q(x)+\tau})$ for $x \in \overline{\Omega}$ uniformly, as $(s,t) \to (0,0)$, where $\tau$ is a positive constant, and $G_i$ satisfies one of the following conditions*

*(i) $\lambda_i a_i > 0$ and **($B_2$)** or **($B_2'$)** is satisfied;*

*(ii) $\lambda_i a_i \leq 0$ and **($B_3$)** is satisfied;*

*and $\Lambda_1 = \{i \in \Lambda\ |(i)$ is satisfied$\}$ is nonempty, and there exist some $i_1, i_2 \in \Lambda_1$ such that $|a_{i_1}(\cdot)|^{-p(\cdot)/(\theta_1(\cdot)-p(\cdot))}, |a_{i_2}(\cdot)|^{-q(\cdot)/(\theta_2(\cdot)-q(\cdot))} \in L^1(\Omega)$, then (1) has a nontrivial solution.*

**Proof** We will prove $J$ satisfies the conditions of Mountain Pass lemma (see [44]).

Since $F$ satisfies **(A)**, then $F(x,u(x),v(x))$ is integrable on $\Omega$ for any $(u,v) \in X$. According to Lemma 11, $J$ satisfies $(PS)$ condition.

Denote $\Lambda = \{i = 1, \cdots, m\}$, $\Lambda_1 = \{i \in \Lambda\ |\text{(i) is satisfied}\}$, $\Lambda_2 = \{i \in \Lambda\ |\text{(ii) is satisfied}\}$, then $\Lambda_1 \cup \Lambda_2 = \Lambda$.

We divided $\Omega$ into small disjoint measurable subsets $\Omega_1, \cdots, \Omega_{n_0}$, such that

$$\min_{x\in\Omega_j} p(x) + \tau > \max_{x\in\Omega_j} p(x), \min_{x\in\Omega_j} q(x) + \tau > \max_{x\in\Omega_j} q(x), j = 1, \cdots, n_0,$$

and

$$\alpha_{i,j}^- := \min_{x\in\overline{\Omega_j}} \alpha_i(x) > p_j^+ := \max_{x\in\overline{\Omega_j}} p(x), \text{ and } \beta_{i,j}^- := \min_{x\in\overline{\Omega_j}} \beta_i(x) > q_j^+ := \max_{x\in\overline{\Omega_j}} q_i(x), i \in \Lambda_1, j = 1, \cdots, n_0.$$

In the following, for any $f \in C(\overline{\Omega})$, we denote

$$f_j^- = \min_{x\in\overline{\Omega_j}} f(x), f_j^+ = \max_{x\in\overline{\Omega_j}} f(x), j = 1, \cdots, n_0,$$

and

$$\Phi_{\Omega_j}(u,v)=\int_{\Omega_j}\frac{1}{p(x)}(|\nabla u|^{p(x)}+a(x)\,|u|^{p(x)})dx+\int_{\Omega_j}\frac{1}{q(x)}(|\nabla v|^{q(x)}+b(x)\,|v|^{q(x)})dx.$$

Hence, when $\|(u,v)\|=\delta$ is small enough, we have

$$\begin{aligned}J(u,v)\ &\geq\ \sum_{j=1}^{n_0}\{\Phi_{\Omega_j}(u,v)-\sum_{i\in\Lambda_1}\int_{\Omega_j}|\lambda_i a_i(x)|\,[\varepsilon(|u|^{p_j^-+\tau}+|v|^{q_j^-+\tau})+C(\varepsilon)(|u|^{\alpha_i(x)}+|v|^{\beta_i(x)})]dx\\&\quad-\sum_{i\in\Lambda_2}\int_{\Omega_j}|\lambda_i a_i(x)|\,[\varepsilon(|u|^{p_j^-+\tau}+|v|^{q_j^-+\tau})+C(\varepsilon)(|u|^{\theta_{1,j}^+}+|v|^{\theta_{2,j}^+})]dx\\&\geq\ \sum_{j=1}^{n_0}\{\Phi_{\Omega_j}(u,v)-C(\|u_j\|_{p(\cdot)}^{p_j^-+\tau}+\|u_j\|_{p(\cdot)}^{\alpha_{i,j}^-}+\|u_j\|_{p(\cdot)}^{\theta_{1,j}^+}+\|v_j\|_{q(\cdot)}^{q_j^-+\tau}+\|v_j\|_{q(\cdot)}^{\beta_{i,j}^-}+\|v_j\|_{q(\cdot)}^{\theta_{2,j}^+})\},\end{aligned}$$

where $u_j=u|_{\Omega_j}$, $v_j=v|_{\Omega_j}$.

Since $p_j^-+\tau,\alpha_{i,j}^-,\theta_{1,j}^+>p_j^+$, and $q_j^-+\tau,\beta_{i,j}^-,\theta_{2,j}^+>q_j^+$, we can get

$$J(u,v)\geq\sum_{j=1}^{n}\frac{1}{4}\Phi_{\Omega_j}(u,v)=\frac{1}{4}\Phi(u,v),\forall(u,v)\in X\text{ with }\|(u,v)\|=\delta\text{ is small enough.}$$

Let $\delta>0$ is small enough, then $J(u,v)\geq c>0$ for any $(u,v)\in X$ with $\|(u,v)\|=\delta$.

From (i) and Proposition 5, we have

$$\begin{aligned}G_i(x,s,t)\ &\geq\ c_1[(|s|^{\theta_1}+|t|^{\theta_2})-1],\forall(x,s,t)\in\overline{\Omega}\times\mathbb{R}^2,\forall i\in\Lambda_1,\\G_i(x,s,t)\ &\leq\ c_2[(|s|^{\frac{\theta_1}{1+\delta}}+|t|^{\frac{\theta_2}{1+\delta}})+1],\forall(x,s,t)\in\overline{\Omega}\times\mathbb{R}^2,\forall i\in\Lambda_2.\end{aligned}$$

For fixed $(u_0,v_0)\in X\backslash\{0\}$ with $suppu_0,suppv_0\subset\overline{\Omega_1}$ and $t>1$, we have

$$\begin{aligned}J(tu_0,tv_0)\ &\leq\ \Phi_{\Omega_1}(tu_0,tv_0)-\sum_{i\in\Lambda_2}\int_{\Omega_1}F_i(x,tu_0,tv_0)dx\\&\quad-C_1\sum_{i\in\Lambda_1}\int_{\Omega_1}|\lambda_i a_i(x)|\,(t^{\theta_1}|u_0|^{\theta_1}+t^{\theta_2}|u_0|^{\theta_2})dx+C_2\\&\leq\ \Phi_1(tu_0)+\Phi_2(tv_0)+\sum_{i\in\Lambda_2}\int_{\Omega}|\lambda_i a_i(x)|\,(|t|^{\frac{\theta_1}{1+\delta}}|u_0|^{\frac{\theta_1}{1+\delta}}+|t|^{\frac{\theta_2}{1+\delta}}|v_0|^{\frac{\theta_2}{1+\delta}})dx\\&\quad-C_1\sum_{i\in\Lambda_1}\int_{\Omega}|\lambda_i a_i(x)|\,(t^{\theta_1}|u_0|^{\theta_1}+t^{\theta_2}|v_0|^{\theta_2})dx+C_2.\end{aligned}$$

Without loss of generality, we may assume that $p_1^+,\theta_{1,1}^+/(1+\delta)<\theta_{1,1}^-$ and $q_1^+,\theta_{2,1}^+/(1+\delta)<\theta_{2,1}^-$. Since $p_1^+,\theta_{1,1}^+/(1+\delta)<\theta_{1,1}^-$ and $q_1^+,\theta_{2,1}^+/(1+\delta)<\theta_{2,1}^-$ on $\overline{\Omega}_1$, we have $J(tu_0,tv_0)\to-\infty$ $(t\to+\infty)$. Obviously, $J(0,0)=0$, then $J$ satisfies the conditions of Mountain Pass Lemma. So $J$ admits at least one nontrivial critical point. $\square$

**Theorem 20** *If $F$ satisfies **(A)**, and $G_i(x,u)$ satisfy one of the following conditions*

*(i) $\lambda_i a_i > 0$ and **($B_2$)** or **($B_2'$)** is satisfied, $G_i(x,s,t) = o(|s|^{p(x)+\tau} + |t|^{q(x)+\tau})$ for $x \in \overline{\Omega}$ uniformly as $(s,t) \to (0,0)$,*

*(ii) $\lambda_i a_i \leq 0$ and **($B_3$)** is satisfied, $G_i(x,s,t) = o(|s|^{p(x)+\tau} + |t|^{q(x)+\tau})$ for $x \in \overline{\Omega}$ uniformly as $(s,t) \to (0,0)$,*

*(iii) $|\lambda_i|$ is small enough, and **($B_1$)** is satisfied,*

*and $\Lambda_1 = \{i \in \Lambda \mid (i) \text{ is satisfied}\} \neq \varnothing$, and there exist some $i_1, i_2 \in \Lambda_2$ such that $|a_{i_1}(\cdot)|^{-p(\cdot)/(\theta_1(\cdot)-p(\cdot))}, |a_{i_2}(\cdot)|^{-q(\cdot)/(\theta_2(\cdot)-q(\cdot))} \in L^1(\Omega)$, then (1) has a nontrivial solution.*

**Proof** We will prove $J$ satisfies the conditions of Mountain Pass lemma (see [44]). Since $F$ satisfies **(A)**, $F(x,u(x),v(x))$ is integrable on $\Omega$ for any $u \in X$.

According to Lemma 11, $J$ satisfies $(PS)$ condition. Denote $\Lambda = \{1, \cdots, m\}$, $\Lambda_1 = \{i \mid \text{(i) is satisfied}\}$, $\Lambda_2 = \{i \in \Lambda \mid \text{(ii) is satisfied}\}$, $\Lambda_3 = \Lambda \backslash (\Lambda_1 \cup \Lambda_2)$. When $\|(u,v)\| \leq 1$, we have

$$\begin{aligned}
J(u,v) \geq\ & \Phi(u,v) - \sum_{i\in\Lambda_3} \int_\Omega F_i(x,u,v)dx \\
& - \sum_{i\in\Lambda_1} \int_\Omega |\lambda_i a_i(x)| \left[\varepsilon(|u|^{p(x)+\tau} + |v|^{q(x)+\tau}) + C(\varepsilon)(|u|^{\alpha_i} + |v|^{\beta_i})\right]dx \\
& - \sum_{i\in\Lambda_2} \int_\Omega |\lambda_i a_i(x)| \left[\varepsilon(|u|^{p(x)+\tau} + |v|^{q(x)+\tau}) + C(\varepsilon)(|u|^{\theta_1} + |v|^{\theta_2})\right]dx.
\end{aligned}$$

Similar to the proof of Theorem 19, there exists an positive constant $\delta < 1$, such that

$$\begin{aligned}
& \Phi(u,v) - \sum_{i\in\Lambda_1} \int_\Omega \lambda_i a_i(x)[\varepsilon(|u|^{p(x)+\tau} + |v|^{q(x)+\tau}) + C(\varepsilon)(|u|^{\alpha_i} + |v|^{\beta_i})]dx \\
& - \sum_{i\in\Lambda_2} \int_\Omega |\lambda_i a_i(x)| \left[\varepsilon(|u|^{p(x)+\tau} + |v|^{q(x)+\tau}) + C(\varepsilon)(|u|^{\theta_1} + |v|^{\theta_2})\right]dx \\
\geq\ & \frac{1}{2p^+} \|u\|_{p(\cdot)}^{p^+} + \frac{1}{2q^+} \|v\|_{q(\cdot)}^{q^+}, \text{ when } \|(u,v)\| = \delta.
\end{aligned}$$

Let

$$|\lambda_i| \leq [\frac{1}{4p^+} \|u\|_{p(\cdot)}^{p^+} + \frac{1}{4q^+} \|v\|_{q(\cdot)}^{q^+}] \frac{1}{\sum_{i\in\Lambda_3} \max\limits_{\|(u,v)\|\leq 1} \int_\Omega |a_i(x)G_i(x,u,v)|\, dx + 1},$$

then we have

$$J(u,v) \geq \frac{1}{4p^+} \|u\|_{p(\cdot)}^{p^+} + \frac{1}{4q^+} \|v\|_{q(\cdot)}^{q^+} > C > 0, \text{ when } \|(u,v)\| = \delta.$$

Similar to the proof Theorem 19, we get the existence of solutions for (1). □

**Theorem 21** *If $F$ satisfies **(A)**, $F(x,-s,-t) = F(x,s,t)$, and we assume for each $i = 1,\cdots,m$, $F_i$ satisfy one of the following*

*($1^0$) **($B_1$)** is satisfied,*

*($2^0$) $\lambda_i a_i > 0$ and **($B_2$)** or **($B_2'$)** is satisfied,*

*($3^0$) $\lambda_i a_i \le 0$ and **($B_3$)** is satisfied,*

*and $\Lambda_2 = \{i \mid (2^0)$ is satisfied$\} \neq \varnothing$, and there exist some $i_1, i_2 \in \Lambda_2$ such that $|a_{i_1}(\cdot)|^{-p(\cdot)/(\theta_1(\cdot)-p(\cdot))}$, $|a_{i_2}(x)|^{-q(\cdot)/(\theta_2(\cdot)-q(\cdot))} \in L^1(\Omega)$, then problem (1) has solutions $\{\pm(u_k,v_k) \mid k = 1,2,\cdots\}$ such that $J(\pm(u_k,v_k)) \to +\infty$ as $k \to +\infty$.*

**Proof** It is easy to see that $(PS)$ condition is satisfied.

Let

$$\phi_{i,k}(R) = \sup\left\{\int_\Omega |\lambda_i a_i(x)|\,|G_i(x,u,v)|dx \middle| (u,v) \in Z_k, \|(u,v)\| \le R\right\}.$$

If $G_i$ doesn't satisfy ($\mathbf{B}_2'$), from Theorem 8 and Lemma 12, we have $\lim\limits_{k\to\infty} \phi_{i,k}(R) = 0$.

If $G_i$ satisfies ($\mathbf{B}_2'$), since $G_i \in C^1(\overline{\Omega}\times\mathbb{R}^2)$, we have

$$|G_i(x,u,v)| \le C_\varepsilon(|u|+|v|) + \frac{\varepsilon}{2(C_{p^*}^* + C_{q^*}^*)}(|u|^{p^*(x)} + |v|^{q^*(x)}), \forall\varepsilon > 0,$$

from Lemma 12 we can see that $\phi_{i,k}(R) \le \varepsilon$ when $k$ is large enough. Therefore $\lim\limits_{k\to\infty} \phi_{i,k}(R) = 0$. Thus, for any $i = 1,\cdots,m$, we have $\lim\limits_{k\to\infty} \phi_{i,k}(R) = 0$. Denote $\phi_k(R) = \max\limits_{1\le i\le m} \phi_{i,k}(R)$, then $\lim\limits_{k\to\infty} \phi_k(R) = 0$.

For fixed $R \ge 1$, and $(u,v) \in Z_k$ with $\|(u,v)\| = R$, we have

$$\begin{aligned} J(u,v) &\ge \Phi(u,v) - \sum_{i\in\Lambda} \int_\Omega |\lambda_i a_i(x)|\,|G_i(x,u,v)|dx \\ &\ge \min(\frac{1}{p^+}, \frac{1}{q^+})R - m\phi_k(R) \\ &\ge \frac{1}{2}\min(\frac{1}{p^+}, \frac{1}{q^+})R, \text{ as } k \to +\infty. \end{aligned}$$

Let $R_1 = 1$, then there exists a constant $k_1$ is large enough such that

$$J(u,v) \ge \frac{1}{2}\min(\frac{1}{p^+}, \frac{1}{q^+})R_1, \forall (u,v) \in Z_k \text{ with } \|(u,v)\| = R_1 \text{ and } k \ge k_1.$$

Let $R_n = 2^n$, then there exists a constant $k_n > k_{n-1}$ is large enough such that

$$J(u,v) \ge \frac{1}{2}\min(\frac{1}{p^+}, \frac{1}{q^+})R_n, \forall (u,v) \in Z_k \text{ with } \|(u,v)\| = R_n \text{ and } k \ge k_n.$$

For fixed $(u_0, v_0) \in X\backslash\{0\}$ with $suppu_0, suppv_0 \subset \overline{\Omega_1}$ and $t > 1$, where $\overline{\Omega_1}$ is defined in Theorem 19, we have

$$\begin{aligned} J(tu_0, tv_0) &\leq \Phi_{\Omega_1}(tu_0, tv_0) + \sum_{i\in\Lambda_1} \int_{\Omega_1} |\lambda_i a_i(x)| \, (t^{\alpha_i(x)} |u_0|^{\alpha_i(x)} + t^{\beta_i(x)} |u_0|^{\beta_i(x)}) dx \\ &+ \sum_{i\in\Lambda_3} \int_{\Omega_1} |\lambda_i a_i(x)| \, (t^{\frac{\theta_1}{1+\delta}} |u_0|^{\frac{\theta_1}{1+\delta}} + t^{\frac{\theta_2}{1+\delta}} |u_0|^{\frac{\theta_2}{1+\delta}}) dx \\ &- \sum_{i\in\Lambda_2} \int_{\Omega_1} |\lambda_i a_i(x)| \, (t^{\theta_1} |u_0|^{\theta_1} + t^{\theta_2} |v_0|^{\theta_2}) dx + C_6, \end{aligned}$$

where $\Phi_{\Omega_1}(tu_0, tv_0)$ is defined in Theorem 19. Without loss of generality, we may assume that

$$\max\{\max_{x\in\overline{\Omega_1}} p(x), \max_{x\in\overline{\Omega_1}} \frac{\theta_1(x)}{1+\delta}\} < \min_{x\in\overline{\Omega_1}} \theta_1(x), \ \max\{\max_{x\in\overline{\Omega_1}} q(x), \max_{x\in\overline{\Omega_1}} \frac{\theta_2(x)}{1+\delta}\} < \min_{x\in\overline{\Omega_1}} \theta_2(x),$$

then the definitions of $\overline{\Omega_1}$ implies $J(tu_0, tv_0) \to -\infty$ $(t \to +\infty)$. Since $J(0, 0) = 0$, $J$ satisfies the conditions of Fountain theorem. $\square$

**Theorem 22** *If $F(x, -s, -t) = F(x, s, t)$, and for each $i = 1, \cdots, m$, $G_i$ satisfies one of the following*

*($1^0$) $\lambda_i a_i(x) > 0$, $G_i(x, u, v) = |u|^{\alpha_i(x)} + |v|^{\beta_i(x)}$, and **($B_1$)** is satisfied,*

*($2^0$) $\lambda_i a_i > 0$ and **($B_2$)** or **($B_2'$)** is satisfied, $G_i(x, s, t) = o(|s|^{p(x)+\tau} + |t|^{q(x)+\tau})$ for $x \in \overline{\Omega}$ uniformly as $(s, t) \to (0, 0)$, and $|a_i(\cdot)|^{-p(\cdot)/(\theta_1 - p(\cdot))}, |a_i(\cdot)|^{-q(\cdot)/(\theta_2 - q(\cdot))} \in L^1(\Omega)$, where $\theta_1$ and $\theta_2$ are constants,*

*($3^0$) $\lambda_i a_i \leq 0$ and **($B_3$)** is satisfied, $G_i(x, s, t) = o(|s|^{p(x)+\tau} + |t|^{q(x)+\tau})$ for $x \in \overline{\Omega}$ uniformly as $(s, t) \to (0, 0)$,*

*and $\Lambda_1 = \{i \mid (1^0) \text{ is satisfied}\} \neq \varnothing$, then problem (1) has solutions $\{\pm(u_k, v_k) \mid k = 1, 2, \cdots\}$ such that $J(\pm(u_k, v_k)) < 0$ and $J(\pm(u_k, v_k)) \to 0$ as $k \to \infty$.*

**Proof** Let's verify the conditions of Proposition 15 item by item.

Let

$$\phi_k(\gamma) = \sup \{\Psi(u, v) \mid \|(u, v)\| \leq \gamma, (u, v) \in Z_k\}.$$

Similar to the proof of Theorem 21, we have $\phi_k(\gamma) \to 0$ as $k \to \infty$. Thus there exists a positive integer $k_0$ such that $\phi_k(1) \leq \frac{1}{2} \min\{\frac{1}{p^+}, \frac{1}{q^+}\}$ for all $k \geq k_0$. Setting $\rho_k = 1$, then for $k \geq k_0$ and $u \in Z_k \cap S_1$, we have

$$J(u, v) \geq \min\{\frac{1}{p^+}, \frac{1}{q^+}\} - \phi_k(1) \geq \frac{1}{2} \min\{\frac{1}{p^+}, \frac{1}{q^+}\},$$

which shows that the condition $(D_1)$ of Proposition 15 is satisfied.

Denote $\Lambda_2 = \{i \mid (2^0) \text{ is satisfied}\}$, $\Lambda_3 = \{i \mid (3^0) \text{ is satisfied}\}$. We may choose $\{Y_k \mid k = 1, 2, \cdots\}$, a sequence of finite dimensional vector subspaces of $X$ defined by (3). For each $Y_k$, because all the norms on $Y_k$ are equivalent, there exists $\epsilon \in (0,1)$ such that for every $(u,v) \in Y_k \cap B_\epsilon$, $\|(u,v)\|$ and $|u|_{(\alpha_i(\cdot),|a_i(\cdot)|)} + |v|_{(\beta_i(\cdot),|a_i(\cdot)|)}$ are small enough. For every $(u,v) \in Y_k \cap B_\epsilon$, similar to the proof of Theorem 19, we have

$$
\begin{aligned}
J(u,v) &\leq \sum_{j=1}^{n_0}\{\Phi_{\Omega_j}(u,v) + c_1 \sum_{i\in\Lambda_2} \int_{\Omega_j} |\lambda_i a_i(x)| \left(|u|^{p(x)+\tau} + |v|^{q(x)+\tau} + |u|^{\alpha_i(x)} + |v|^{\beta_i(x)}\right) dx \\
&\quad + c_2 \sum_{i\in\Lambda_3} \int_{\Omega_j} |\lambda_i a_i(x)| \left(|u|^{p(x)+\tau} + |v|^{q(x)+\tau} + |u|^{\theta_1} + |v|^{\theta_2}\right) dx \\
&\quad - \sum_{i\in\Lambda_1} \int_{\Omega_j} |\lambda_i a_i(x)| \left(|u|^{\alpha_i(x)} + |v|^{\beta_i(x)}\right) dx\} \\
&\leq -\sum_{j=1}^{n_0}\{\frac{1}{2} \sum_{i\in\Lambda_1} \int_{\Omega_j} |\lambda_i a_i(x)| \left(|u|^{\alpha_i(x)} + |v|^{\beta_i(x)}\right) dx\} \\
&= -\frac{1}{2} \sum_{i\in\Lambda_1} \int_{\Omega} |\lambda_i a_i(x)| \left(|u|^{\alpha_i(x)} + |v|^{\beta_i(x)}\right) dx.
\end{aligned}
$$

According to the definition of $\Lambda_1, \Lambda_2$ and $\Lambda_3$, there exists $\gamma_k \in (0,\epsilon)$ which is small enough, such that

$$\zeta_k := \max\{J(u,v) | \, (u,v) \in Y_k, \|(u,v)\| = \gamma_k\} < 0.$$

Thus the condition $(D_2)$ of Proposition 15 is satisfied.

Because $Y_k \cap Z_k \neq \varnothing$ and $\gamma_k < \rho_k$, we have $\eta_k \leq \zeta_k < 0$.

On the other hand, for any $(u,v) \in Z_k$ with $\|(u,v)\| \leq 1 = \rho_k$, we have $J(u,v) = \Phi(u,v) - \Psi(u,v) \geq -\Psi(u,v) \geq -\phi_k(1)$. Noting that $\phi_k(1) \to 0$ as $k \to \infty$, we obtain $\eta_k \to 0$, *i.e.*, $(D_3)$ of Proposition 15 is satisfied.

At last, let's prove that $J$ satisfies $(PS)_c^*$ condition for every $c \in \mathbb{R}$ on $X$.

Suppose that $\{(u_{n_j}, v_{n_j})\} \subset X$ such that $n_j \to \infty$, $(u_{n_j}, v_{n_j}) \in Y_{n_j}$ , $J(u_{n_j}, v_{n_j}) \to c$ and $(J|_{Y_{n_j}})'(u_{n_j}, v_{n_j}) \to 0$. Similar to the process of verifying the $(PS)$ condition in the proof of Lemma 11, we can get the boundedness of $\{(u_{n_j}, v_{n_j})\}$. Going if necessary to a subsequence, we can assume that $(u_{n_j}, v_{n_j}) \rightharpoonup (u,v)$ in $X$. As $X = \overline{\bigcup_{n_j} Y_{n_j}}$, we can choose $(\widetilde{u}_{n_j}, \widetilde{v}_{n_j}) \in Y_{n_j}$ such that $(\widetilde{u}_{n_j}, \widetilde{v}_{n_j}) \to (u,v)$. Since $\{(u_{n_j}, v_{n_j})\}$ is bounded and $J$ is $C^1$ and bounded on $X$, we have

$$\lim_{n_j\to\infty} J'(u_{n_j}, v_{n_j})(u_{n_j} - u, v_{n_j} - v)$$

$$= \lim_{n_j\to\infty} J'(u_{n_j}, v_{n_j})(u_{n_j} - \widetilde{u}_{n_j}, v_{n_j} - \widetilde{v}_{n_j}) + \lim_{n_j\to\infty} J'(u_{n_j}, v_{n_j})(\widetilde{u}_{n_j} - u, \widetilde{v}_{n_j} - v)$$
$$= \lim_{n_j\to\infty} (J|_{Y_{n_j}})'(u_{n_j}, v_{n_j})(u_{n_j} - \widetilde{u}_{n_j}, v_{n_j} - \widetilde{v}_{n_j}) = 0.$$

As $J$ is of $(S_+)$ type, we can conclude $(u_{n_j}, v_{n_j}) \to (u, v)$ in $X$. Furthermore, we have $J'(u_{n_j}, v_{n_j}) \to J'(u, v)$. It only remains to prove $J'(u, v) = 0$. Taking arbitrarily $(u_k^{\#}, v_k^{\#}) \in Y_k$, notice that when $n_j \geq k$ we have

$$\begin{aligned} J'(u,v)(u_k^{\#}, v_k^{\#}) &= (J'(u,v) - J'(u_{n_j}, v_{n_j}))(u_k^{\#}, v_k^{\#}) + J'(u_{n_j}, v_{n_j})(u_k^{\#}, v_k^{\#}) \\ &= (J'(u,v) - J'(u_{n_j}, v_{n_j}))(u_k^{\#}, v_k^{\#}) + (J|_{Y_{n_j}})'(u_{n_j}, v_{n_j})(u_k^{\#}, v_k^{\#}). \end{aligned}$$

Going to limit in the right side of above equation, yields

$$J'(u,v)(u_k^{\#}, v_k^{\#}) = 0, \forall (u_k^{\#}, v_k^{\#}) \in Y_k.$$

So $J'(u,v) = 0$, which shows that $J$ satisfies the $(PS)_c^*$ condition for every $c \in \mathbb{R}$. Then $(D_4)$ of Proposition 15 is satisfied. □

In the following, we will consider the existence of solutions for (1), when $F(x, \cdot, v)$ satisfies sub-$p(x)$ growth condition, and $F(x, u, \cdot)$ satisfies super-$q(x)$ growth condition.

**Theorem 23** *If $F$ satisfies **(A)**, and $F$ satisfies the following condition*

*(i) $a_1(x) > 0$ and $|a_1(\cdot)|^{-p(\cdot)/(\theta_1(\cdot)-p(\cdot))} \in L^1(\Omega)$,*

*(ii) for $i = 1, \cdots, m$, $G_i(x, s, t) = o(|s|^{p(x)+\tau} + |t|^{q(x)+\tau})$ for $x \in \overline{\Omega}$ uniformly as $(s,t) \to (0,0)$, where $\tau$ is a positive constant, and $F_i(x,u,v)$ satisfies one of the following*

*($1^0$) **($B_1$)** is satisfied,*

*($2^0$) $\alpha_i(x) < p(x)$, $q(x) < \beta_i(x) \leq q^*(x)(r_i(x) - 1)/r_i(x)$, $r_i(\cdot) \in C_+(\overline{\Omega})$, and $F_i(x, u, \cdot)$ satisfies*

$$0 \leq F_i(x,u,v) \leq \frac{v}{\theta_2(x)} \frac{\partial}{\partial v} F_i(x,u,v), \forall (x,u) \in \overline{\Omega} \times \mathbb{R}, |v| \geq M,$$

*where $q(x) < \theta_2(x) < \beta_i(x)$, and $F_i(x,u,v) > 0$ when $|u| \geq M, |v| \geq M, \forall x \in \overline{\Omega}$,*

*then (1) has a nontrivial solution.*

**Proof** Without loss of generality, we may assume that $\theta_2(x) \leq \beta_1(x)$. Let $\{(u_n, v_n)\}$ be a $(PS)$ sequence. Similar to the proof of the Lemma 11, we have

$$c + 1 + \|v_n\|_{q(\cdot)} \geq J(u_n, v_n) - J'(u_n, v_n)(0, \frac{1+\delta}{\theta_2(x)} v_n) \geq \frac{l_2}{3} \Phi(u_n, v_n) \text{ as } n \to \infty,$$

where $l_2$ is defined in (8).

Thus $J$ satisfies $(PS)$ condition. Without loss of generality, we may assume that

$$p(x)+2\tau<\frac{r_i(x)-1}{r_i(x)}p^*(x), \text{ and } q(x)+2\tau<\frac{r_i(x)-1}{r_i(x)}q^*(x), \forall x\in\overline{\Omega}.$$

Denote

$$\beta_i^{\#}(x)=\max\{q(x)+2\tau,\beta_i(x)\}, \forall x\in\overline{\Omega}, i=1,\cdots,m.$$

We have

$$|F_i(x,s,t)|\leq\varepsilon(|s|^{p(x)+\tau}+|t|^{q(x)+\tau})+C(\varepsilon)(|s|^{p(x)+2\tau}+|t|^{\beta_i^{\#}(x)}), i=1,\cdots,m.$$

Similar to the proof of Theorem 19, when $\|(u,v)\|=\delta$ is small enough, we can get

$$J(u,v)\geq\sum_{j=1}^{n}\frac{1}{4}\Phi_{\Omega_j}(u,v)=\frac{1}{4}\Phi(u,v).$$

Let $\delta>0$ be small enough, then $J(u,v)\geq c>0$ for any $(u,v)\in X$ with $\|(u,v)\|=\delta$.

For $(M,t)\in X$ and $t>1$, we have

$$\begin{aligned} J(M,t) &= \int_\Omega\frac{M^{p(x)}}{p(x)}dx+\int_\Omega\frac{t^{q(x)}}{q(x)}dx-\int_\Omega F(x,M,t)dx \\ &= \int_\Omega\frac{M^{p(x)}}{p(x)}dx+\int_\Omega\frac{t^{q(x)}}{q(x)}dx-\sum_{1\leq i\leq m}\int_\Omega F_i(x,M,t)dx-\int_\Omega|\lambda_i a_i(x)|\,t^{\beta_1(x)}dx \\ &\leq \sum_{j=1}^{n_0}\{\int_{\Omega_j}\frac{t^{q(x)}}{q(x)}dx+\sum_{1\leq i\leq m}C\int_{\Omega_j}|\lambda_i a_i(x)|\,t^{q(x)}dx-\int_{\Omega_j}|a_1(x)|\,t^{\beta_1(x)}dx\}+C, \end{aligned}$$

where $\Omega_j, j=1,\cdots,n_0$, are defined in Theorem 19.

Thus $J(M,t)\to-\infty$ $(t\to+\infty)$. Obviously, $J(0,0)=0$, then $J$ satisfies the conditions of Mountain Pass lemma (see [44]). So $J$ admits at least one nontrivial critical point. □

**Theorem 24** *If $F$ satisfies **(A)**, $F(x,-s,-t)=F(x,s,t)$, and $F$ satisfies the following condition*

*(i) $a_1(x)>0$ and $|a_1(\cdot)|^{-p(\cdot)/(\theta_1(\cdot)-p(\cdot))}\in L^1(\Omega)$,*

*(ii) for $i=1,\cdots,m$, $F_i(x,u,v)$ satisfies one of the following*

*($1^0$) **($B_1$)** is satisfied,*

*($2^0$) $\alpha_i(x)<p(x)$ and $r_i(x)\geq(p(x)/\alpha_i(x))^0$, $q(x)<\beta_i(x)\leq q^*(x)(r_i(x)-1)/r_i(x)$, $r_i(x)\in C(\overline{\Omega})$, and $F_i(x,u,\cdot)$ satisfies*

$$0\leq F_i(x,u,v)\leq\frac{v}{\theta_2(x)}\frac{\partial}{\partial v}F_i(x,u,v), \forall(x,u)\in\overline{\Omega}\times\mathbb{R}, |v|\geq M,$$

*where* $q(x) < \theta_2(x) < \beta_i(x)$, *and* $F_i(x,u,v) > 0$ *when* $|u| \geq M, |v| \geq M, \forall x \in \overline{\Omega}$,

*then (1) has a sequence of solutions.*

**Proof** Denote $\Lambda_1 = \{i \geq 1 \,|(1^0)$ is satisfied$\}$, $\Lambda_2 = \{i \geq 1 \,|(2^0)$ is satisfied$\}$. Let $\{(u_n, v_n)\}$ be a $(PS)$ sequence. Similar to the proof of the Lemma 11, we have

$$c + 1 + \|v_n\|_{q(\cdot)} \geq J(u_n, v_n) - J'(u_n, v_n)(0, \frac{1+\delta}{\theta_2(x)} v_n) \geq \frac{l_2}{3}\Phi(u_n, v_n) \text{ as } n \to \infty,$$

where $l_2$ is defined in (8).

Thus $\{(u_n, v_n)\}$ is bounded, and then $J$ satisfies $(PS)$ condition. Let $V_k^+ = Z_k$, it is a closed linear subspace of $X$ and $V_k^+ \oplus Y_{k-1} = X$.

Let $h_i \in C_0^\infty(\Omega)$ satisfy

$$supph_i \cap supph_j = \varnothing, \, \forall i \neq j.$$

Set $V_k^- = span\{(0, h_1), \cdots, (0, h_k)\}$. Similar to the proof of Theorem 21, it is easy to see that for every pair of $V_k^+$ and $V_k^-$, $J$ satisfies the conditions of Proposition 16 and the corresponding critical value $\varpi_k := \inf_{g \in \Gamma} \sup_{(u,v) \in V_k^-} J(g(u,v)) \to +\infty$ when $k \to +\infty$. $\square$

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