On characterizations of gamma distribution

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Abstract. Let X and Y be independent identically distributed nondegenerate and positive random variables with common absolutely continuous distribution function $F(x)$. Put $Z = max(X,Y)$ and $W = min(X,Y)$. In this paper, it is proved that $\frac{Z}{Z+W}$ $\left(or\ \frac{W}{Z+W}\right)$ and $Z+W$ are independent if and only if X and Y have gamma distributions. Also, we obtain that $\left(\frac{Z}{Z+W}\right)^2$ $\left(or\ \left(\frac{W}{Z+W}\right)^2\right)$ and $Z+W$ are independent if and only if X and Y have gamma distributions.

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1. Introduction

Let X and Y be two independent identically distributed (i.i.d.) nondegenerate and positive random variables with common absolutely continuous distribution function $F(x)$ and the corresponding density function $f(x)$.

It is known that X/Y and $X+Y$ are independently distributed if and only if both X and Y have gamma distributions with the same scale parameter [see Lukacs(1955)].

The current investigation was induced by a characterization of exponentiality by Kotz and Steutel(1988). Now, set $Z = max(X,Y)$ and $W = min(X,Y)$. Namely, if we write $U = \frac{Z}{(Z+W)}$, one can ask whether the independence of U and $Z+W = X+Y$ characterizes exponentiality. Since $U(X+Y) = Z$ is distributed as X or Y in the case of the exponential distribution, it would give a characterization similar to the result of Kotz and Steutel(1988), without the additional assumptions of U being uniform and $U(X+Y)$ having the same distribution as X. The answer in paper is that we do get a characterization, however, not the exponential distribution but the larger family of gamma variables. In other words, we can see the need for additional assumptions if we want the exponential distribution but not the full set of conditions of Kotz and Steutel(1988). The details are as follows.

Note that $\frac{Z}{Z+W}$ is a scale invariant statistic. So, by Lukacs and Laha (1964, P. 73), $\frac{Z}{Z+W}$ is independent of $Z+W = X+Y$ for gamma distributions. However, it is generally not true that the independence of a scale invariant statistic and $X+Y$ characterizes the gamma family.

In this paper, we show that $\frac{Z}{Z+W}$ and $Z+W$ are independent if and only if X and Y have gamma distributions. Also, we characterize that $\left(\frac{Z}{Z+W}\right)^2 \left(or \left(\frac{W}{Z+W}\right)^2\right)$ and $Z+W$ are independent if and only if X and Y have gamma distributions.

2. Main Results

Theorem 2.1. Let X and Y be i.i.d. nondegenerate and positive random variables with common absolutely continuous distribution function $F(x)$ and pdf $f(x)$ and $E(X^2) < \infty$. Then $\frac{Z}{Z+W} \left(or\ \frac{W}{Z+W}\right)$ and $(Z+W)$ are independent if and only if X and Y have gamma distributions.

Theorem 2.2. Let X and Y be i.i.d. nondegenerate and positive random variables with common absolutely continuous distribution function $F(x)$ and pdf $f(x)$ and $E(X^2) < \infty$. Then $\left(\frac{Z}{Z+W}\right)^2 \left(or \left(\frac{W}{Z+W}\right)^2\right)$ and $(Z+W)$ are independent if and only if X and Y have gamma distributions.

3. Proofs

Proof of Theorem 2.1. Write $Z = max(X,Y)$ and $W = min(X,Y)$. Since $\frac{Z}{Z+W}$ and $\frac{W}{Z+W}$ are scale invariant statistics, by Lukacs and Laha (1964), $\frac{Z}{Z+W} \left(or\ \frac{W}{Z+W}\right)$ is independent of $Z+W = X+Y$ for gamma variables. So, we have to prove the converse.

Denote the characteristic functions of $\frac{Z}{Z+W}$, $Z+W$ and $\left(\frac{Z}{Z+W}, Z+W\right)$ by $\phi_1(t)$, $\phi_2(t)$ and $\phi(t,s)$ respectively.

The independence of $\frac{Z}{Z+W}$ and $Z+W$ is equivalent to

$$\phi(t,s) = \phi_1(t)\phi_2(t). \tag{3.1}$$

By the lemma of Lukacs and Laha (1964),

$$\begin{aligned}\phi(t,s) &= \int\int_{0<x\le y<\infty} e^{it\frac{y}{x+y}+is(x+y)} f(x)f(y)dxdy \\ &+ \int\int_{0<y<x<\infty} e^{it\frac{x}{x+y}+is(x+y)} f(x)f(y)dxdy.\end{aligned} \tag{3.2}$$

By interchanging x and y in the first integral, we get

$$\phi(t,s) = 2\int_0^\infty \int_y^\infty e^{it\frac{x}{x+y}+is(x+y)} f(x)f(y)dxdy.$$

By the same method with regard to $\phi_1(t)$ and $\phi_2(t)$, (3.1) becomes

$$\begin{aligned}&\int_0^\infty \int_y^\infty e^{it\frac{x}{x+y}+is(x+y)} f(x)f(y)dxdy \\ &= \left(\int_0^\infty \int_y^\infty e^{it\frac{x}{x+y}} f(x)f(y)dxdy\right)\left(\int_0^\infty \int_0^\infty e^{is(x+y)} f(x)f(y)dxdy\right).\end{aligned} \tag{3.3}$$

Since $\dfrac{W}{Z+W} = 1 - \dfrac{Z}{Z+W}$, $\dfrac{W}{Z+W}$ and $Z+W$ are also independent.

So an equation similar to (3.1) holds for $\dfrac{W}{Z+W}$ and $Z+W$. That is, we obtain the following equation

$$\begin{aligned}&\int_0^\infty \int_0^y e^{it\frac{x}{x+y}+is(x+y)} f(x)f(y)dxdy \\ &= \left(\int_0^\infty \int_0^y e^{it\frac{x}{x+y}} f(x)f(y)dxdy\right)\left(\int_0^\infty \int_0^\infty e^{is(x+y)} f(x)f(y)dxdy\right).\end{aligned} \tag{3.4}$$

Adding equations (3.3) and (3.4), we get

$$\begin{aligned}&\int_0^\infty \int_0^\infty e^{it\frac{x}{x+y}+is(x+y)} f(x)f(y)dxdy \\ &= \left(\int_0^\infty \int_0^\infty e^{it\frac{x}{x+y}} f(x)f(y)dxdy\right)\left(\int_0^\infty \int_0^\infty e^{is(x+y)} f(x)f(y)dxdy\right).\end{aligned} \tag{3.5}$$

The integrals in (3.5) exist not only for reals t and s but also for complex values $t = u + iv$, $s = u^* + iv^*$, where u and u^* are reals, for which $v = Im(t) \ge 0$, $v^* = Im(s) \ge 0$ and they are analytic for all t, s for $v = Im(t) > 0$, $v^* = Im(s) > 0$ [see Lukacs (1955)].

Differentiating (3.5) twice, first with respect to t and then with respect to s, and setting $t = 0$, we get

$$\int_0^\infty \int_0^\infty x^2 e^{is(x+y)} f(x)f(y)dxdy = \theta \int_0^\infty \int_0^\infty (x+y)^2 e^{is(x+y)} f(x)f(y)dxdy \tag{3.6}$$

where $\theta = E\left[\left(\frac{X}{X+Y}\right)^2\right]$, $0 < \theta < 1$.

Denote the characteristic function of X by

$$\varphi(s) = \int_0^\infty e^{isx} f(x)dx. \tag{3.7}$$

Then we know that

$$\varphi^{'}(s) = i\int_0^\infty xe^{isx} f(x)dx \quad \text{and} \quad \varphi^{''}(s) = -\int_0^\infty x^2 e^{isx} f(x)dx. \tag{3.8}$$

By using (3.7) and (3.8), we can express (3.6) as a differential equation

$$-\varphi^{''}(s)\varphi(s) = \theta\left[-\varphi^{''}(s)\varphi(s) - 2\varphi^{'}(s)\varphi^{'}(s) - \varphi^{''}(s)\varphi(s)\right]$$

that is,

$$\frac{\varphi^{''}(s)}{\varphi^{'}(s)} = \frac{2\theta}{1-2\theta}\frac{\varphi^{'}(s)}{\varphi(s)},\ 0 < \theta < 1.$$

After integrating and taking the initial conditions $\varphi(0) = 1$, $\varphi^{'}(0) = iE(X)$, we obtain

$$\varphi^{'}(s) = iE(X)\varphi(s)^{\frac{2\theta}{1-2\theta}}. \tag{3.9}$$

Note that $\theta = E\left[\left(\frac{X}{X+Y}\right)^2\right] = E\left[\left(\frac{Y}{X+Y}\right)^2\right]$ for i.i.d. random variables X and Y. Then,

$$2\theta = E\left[\frac{X^2+Y^2}{(X+Y)^2}\right] = E\left[\frac{1}{1+\frac{2XY}{X^2+Y^2}}\right]. \tag{3.10}$$

Note that, for $x > 0$, $y > 0$, $0 < 2xy \le x^2 + y^2$, and the equality on the right hand side occurs only if $x = y$. By the assumed continuity of $F(x)$, $P(x = y) = 0$, so $0 < \frac{2xy}{x^2+y^2} < 1$, that is, by (3.10),

$$\frac{1}{4} < \theta < \frac{1}{2}. \tag{3.11}$$

Hence, from (3.9) and (3.11), by uniqueness theorem of the differential equation for $\frac{2\theta}{1-2\theta} > 1$, there exists a unique solution

$$\varphi(s) = \left(1 - \frac{iE(X)}{\lambda}s\right)^{-\lambda}$$

where $\lambda = \frac{1-2\theta}{4\theta-1} > 0$.

This completes the proof.

Proof of Theorem 2.2. Since $\left(\frac{Z}{Z+W}\right)^2$ and $\left(\frac{W}{Z+W}\right)^2$ are scale invariant statistics, by Lukacs and Laha (1964), $\left(\frac{Z}{Z+W}\right)^2$ $\left(or\ \left(\frac{W}{Z+W}\right)^2\right)$ is independent of $Z+W = X+Y$ for gamma variables. So, we have to prove the converse. Denote the characteristic functions of $\left(\frac{Z}{Z+W}\right)^2$, $Z+W$ and $\left(\left(\frac{Z}{Z+W}\right)^2, Z+W\right)$ by $\phi_1(t)$, $\phi_2(t)$ and $\phi(t,s)$ respectively.

The independence of $\left(\frac{Z}{Z+W}\right)^2$ and $Z+W$ is equivalent to

$$\phi(t,s) = \phi_1(t)\phi_2(t). \tag{3.12}$$

Then (3.12) gives

$$\begin{aligned} &\int_0^\infty \int_y^\infty e^{it(\frac{x}{x+y})^2+is(x+y)} f(x)f(y)dxdy \\ &= \left(\int_0^\infty \int_y^\infty e^{it(\frac{x}{x+y})^2} f(x)f(y)dxdy\right)\left(\int_0^\infty \int_0^\infty e^{is(x+y)} f(x)f(y)dxdy\right). \end{aligned} \tag{3.13}$$

Since $\left(\frac{W}{Z+W}\right)^2 = \left(1-\frac{Z}{Z+W}\right)^2$, $\left(\frac{W}{Z+W}\right)^2$ and $Z+W$ are also independent. So an equation similar to (3.12) holds for $\left(\frac{W}{Z+W}\right)^2$ and $Z+W$.

Then we have

$$\begin{aligned} &\int_0^\infty \int_0^y e^{it(\frac{x}{x+y})^2+is(x+y)} f(x)f(y)dxdy \\ &= \left(\int_0^\infty \int_0^y e^{it(\frac{x}{x+y})^2} f(x)f(y)dxdy\right)\left(\int_0^\infty \int_0^\infty e^{is(x+y)} f(x)f(y)dxdy\right). \end{aligned} \tag{3.14}$$

Adding equations (3.13) and (3.14), we have

$$\begin{aligned} &\int_0^\infty \int_0^\infty e^{it(\frac{x}{x+y})^2+is(x+y)} f(x)f(y)dxdy \\ &= \left(\int_0^\infty \int_0^\infty e^{it(\frac{x}{x+y})^2} f(x)f(y)dxdy\right)\left(\int_0^\infty \int_0^\infty e^{is(x+y)} f(x)f(y)dxdy\right). \end{aligned} \tag{3.15}$$

The integrals in (3.15) exist not only for reals t and s but also for complex values $t = u+iv$, $s = u^*+iv^*$, where u and u^* are reals, for which $v = Im(t) \geq 0$, $v^* = Im(s) \geq 0$ and they are analytic for all t, s for $v = Im(t) > 0$, $v^* = Im(s) > 0$ [see Lukacs (1955)].

Differentiating (3.15) one time with respect to t and then two times with respect to s, and setting $t = 0$, we get

$$\int_0^\infty \int_0^\infty x^2 e^{is(x+y)} f(x)f(y)dxdy \quad = \theta \int_0^\infty \int_0^\infty (x+y)^2 e^{is(x+y)} f(x)f(y)dxdy \quad (3.16)$$

where $\theta = E\left[\left(\frac{X}{X+Y}\right)^2\right]$, $0 < \theta < 1$.

By the same method of proof of Theorem 2.1, there exists a unique solution

$$\varphi(s) = \left(1 - \frac{iE(X)}{\lambda} s\right)^{-\lambda}$$

where $\lambda = \frac{1-2\theta}{4\theta - 1} > 0$.

This completes the proof.

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References

[1] J. Aczel, *Lectures on functional equations and their applications*, Academic Press, NY, 1966.

[2] S. Kotz and F.W. Steutel, *Note on a characterization of exponential distribution*, Statist. Probab. Lett. 6 (1988), 201-203.

[3] E. Lukacs, *A characterization of the gamma distribution* Ann. Math. Statist. 26 (1955), 319-324.

[4] E. Lukacs and R.G. Laha, *Applications of Characteristic Functions*, Charles Griffin, London, 1964.