THE RIEMANN PROBLEM WITH DELTA INITIAL DATA FOR THE ONE-DIMENSIONAL TRANSPORT EQUATIONS*

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Abstract. In this paper, the Riemann problem with delta initial data for the one dimensional transport equations is studied. The global existence of generalized solutions are obtained constructively by using generalized Rankine-Hugoniot conditions and the entropy condition. Moreover, a new kind of nonclassical wave appears, namely a delta contact discontinuity, which is a Dirac delta function supported on a contact discontinuity. Furthermore, it can be found that the generalized solutions are stable by making use of the perturbation of the initial data.

1. Introduction:

The 1-D transport equations can be expressed as

$$\begin{cases} \rho_t + (\rho u)_x = 0, \\ (\rho u)_t + (\rho u^2)_x = 0, \end{cases} \tag{1.1}$$

where ρ and u, respectively, stand for the density and the velocity of the gas. We also call the system (1.1) zero-pressure gas dynamics. It can be derived from zero-pressure isentropic gas dynamics [1]. The system (1.1) is referred to as the adhesion particle dynamics system to describe the motion process of free particles sticking under collision in the low temperature and the information of large-scale structure in the universe [16, 17]. We can also obtain the system from the flux-splitting numerical schemes [2, 11].

For the Riemann problem of (1.1), it is not difficult to see that the delta shock and vacuum do occur, see [22]. For more study of transport equations, we refer to [10, 26, 27, 28, 30]. In order to deal with the interaction of delta shock with the vacuum states, we adopt the idea proposed by Liu and Smoller [12] when they considered the vacuum problem for isentropic gas dynamic equations, and then approximate them by a set of small non-admissible shocks like as the front tracking algorithm [3]. The same method has been used in [19, 24]. As for delta shock waves, we refer readers to [5, 6, 7, 8, 9, 10, 13, 14, 15, 18, 20, 21, 22, 25] and the references cited therein for more details.

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In this paper, our main purpose is to investigate the Riemann problem with delta initial data and possible interactions of delta shock waves and contact vacuum state for (1.1). It is well recognized nowadays that the interactions of elementary waves play an strategic role of building blocks for all fields of theory, numerics and applications. Thus, the perturbation of Riemann initial data

$$(u,\rho)(x,0)=\begin{cases}(u_-,\rho_-), & x<-\epsilon,\\ (u_0,\omega_0/\epsilon), & -\epsilon<x<\epsilon,\\ (u_+,\rho_+), & x>\epsilon\end{cases}\tag{1.2}$$

where $\epsilon>0$ is sufficiently small, is firstly considered here. We constructively obtain the solutions, global structure and large time-asymptotic behaviors of solutions for problem (1.1)-(1.2). Moreover, a new kind of nonclassical wave is obtained, namely a delta contact discontinuity, which is a Dirac delta function supported on a contact discontinuity and has already appeared in the interaction process for the magnetohydrodynamics equations [15]. Let $\epsilon\to 0$, under the stability theory of weak solutions, we obtain solutions for system (1.1) with delta initial data

$$(u,\rho)(x,0)=\begin{cases}(u_-,\rho_-), & x<0,\\ (u_0,\omega_0\delta), & x=0,\\ (u_+,\rho_+), & x>0,\end{cases}\tag{1.3}$$

where δ is the standard Dirac delta function. The global existence of generalized solutions for problem (1.1) and (1.3) are obtained constructively by using generalized Rankine-Hugoniot conditions and the entropy condition. Moreover, a new kind of nonclassical wave, namely a delta contact discontinuity, appears too. Furthermore, if we let $u_0=0$ and $\omega_0=0$, the solution of (1.1) and (1.3) corresponds to the solutions of Riemann problem for (1.1). This method have been used by Wang and Zhang [29] to study the Riemann problem with delta initial data for the 1-D Chaplygin gas equations. Recently, there have some results on initial data contain Dirac delta function. See [16, 31].

This paper is organized as follows. In section 2, we present some preliminary knowledge for the system (1.1) and display the Riemann solutions of (1.1) with constant initial data. In section 3, we firstly construct the solutions of (1.1) and (1.2) case by case. Then letting $\epsilon\to 0$, we obtain the generalized solutions of (1.1) and (1.3).

2. Preliminaries

In this section, we review the Riemann solutions of transport equations (1.1) with initial data

$$(u(x,0),\rho(x,0))=(u_\pm,\rho_\pm),\pm x>0,\tag{2.1}$$

where $\rho_\pm>0$, the detailed study of which can be found in [22]. For more details about the Riemann problem for hyperbolic conservation laws, see [4, 23].

The transport equations (1.1) have a double eigenvalue $\lambda=u$ with only one corresponding right eigenvector $r=(1,0)^\top$. By simple calculation, we obtain $\nabla\lambda\cdot r=0$. Thus the system (1.1) is linearly degenerate.

Given two constant states $(u_\pm, \rho_\pm)$, we can constructively obtain the Riemann solutions of (1.1) and (2.1) containing contact discontinuities, vacuum or delta shock wave.

For the case $u_- < u_+$, the solutions can be expressed as

$$(u,\rho)(x,t) = \begin{cases} (u_-,\rho_-), & x \le u_- t, \\ (\xi, 0), & u_- t \le x \le u_+ t, \\ (u_+,\rho_+), & x \ge u_+ t. \end{cases} \tag{2.2}$$

For the case $u_- = u_+$, we connect the constant $(u_\pm, \rho_\pm)$ by contact discontinuity.

For the case $u_- > u_+$, a solution containing a weighted δ-measure supported on a line will be constructed to connect the constant $(u_\pm, \rho_\pm)$. Consider a piecewise smooth solution of (1.1) in the form

$$(u,\rho)(t,x) = \begin{cases} (u_-,\rho_-), & x < u_\delta t, \\ (u_\delta, \omega(t)\delta(x - u_\delta t)), & x = u_\delta t, \\ (u_+,\rho_+), & x > u_\delta t, \end{cases} \tag{2.3}$$

where $\omega(t)$ and u_δ are weight and velocity of Dirac delta wave respectively, satisfying the generalized Rankine-Hugoniot conditions

$$\begin{cases} \dfrac{\mathrm{d}\omega(t)}{\mathrm{d}t} = u_\delta[\rho] - [\rho u], \\ \dfrac{\mathrm{d}\omega(t) u_\delta}{\mathrm{d}t} = u_\delta[\rho u] - [\rho u^2], \end{cases} \tag{2.4}$$

with initial data $\omega(0) = 0$, where $[\rho] = \rho_+ - \rho_-$. By simple calculation, we obtain

$$\omega(t) = \sqrt{\rho_-\rho_+}(u_- - u_+)\ t, \tag{2.5}$$

$$u_\delta = \frac{\sqrt{\rho_+}u_+ + \sqrt{\rho_-}u_-}{\sqrt{\rho_+} + \sqrt{\rho_-}}. \tag{2.6}$$

We also can justify the delta shock wave satisfies the entropy condition:

$$u_+ < u_\delta < u_-, \tag{2.7}$$

which means that all the characteristics on both sides of the delta shock are incoming.

Thus, we have obtained the solutions of the 1-D Riemann problem for transport equations.

3. Interactions of delta shock waves and contact discontinuity

In this section, we first consider the three pieces constant initial data (1.1)-(1.2) and obtain the solutions constructively. Then letting $\epsilon \to 0$, we get the solution of (1.1) and (1.3) under the stability theory of weak solutions. In order to cover all the cases completely, our discussion should be divided into four cases. According to the different combinations from $(-\epsilon, 0)$ and $(\epsilon, 0)$, the four cases can be expressed as

$$(1)\ u_- < u_0 < u_+,\ (2)\ u_0 < u_- < u_+ (u_- < u_+ < u_0),$$

$$(3)\ u_0 < u_+ < u_- (u_+ < u_0 < u_-),\ (4)\ u_+ < u_0 < u_-.$$

Case 1: $u_- < u_0 < u_+$.

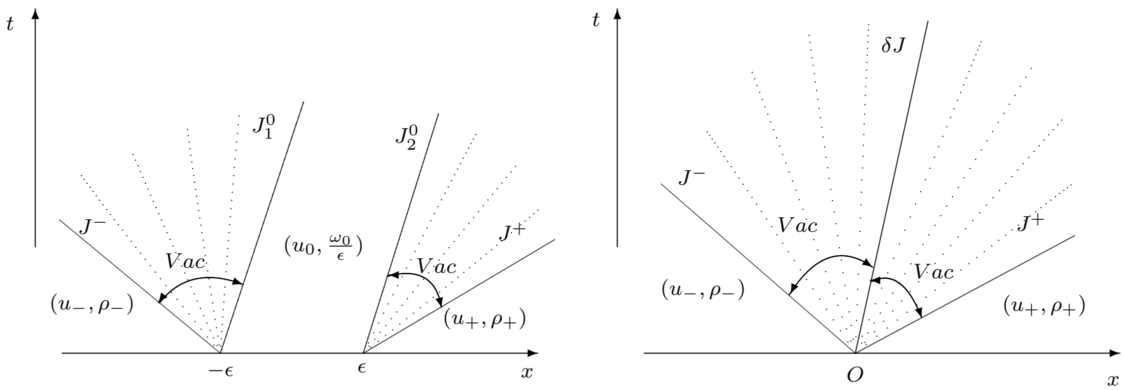

Fig. 3.1.

In this case, when t is small enough, the solution of the initial value problem (1.1) and (1.2) can be expressed briefly as follows (see Fig. 3.1.):

$$(u_-, \rho_-) + J^- + Vac + J_1^0 + (u_0, \frac{\omega_0}{\epsilon}) + J_2^0 + Vac + J^+ + (u_+, \rho_+),$$

where"+" means "followed by". The propagation speed of the J_1^0 and J_2^0 are u_0. Thus we see that the J_1^0 can not overtake J_2^0 at a finite time. So far, the solutions of (1.1) and (1.2) have been constructed completely. Letting $\epsilon \to 0$, we obtain a solution of (1.1) and (1.3) as follows, see Fig. 3.1.,

$$(u_-, \rho_-) + J^- + Vac + \delta J + Vac + J^+ + (u_+, \rho_+),$$

where the propagation speed of the δJ is u_0. Note that a new kind of nonclassical wave δJ, namely a delta contact discontinuity, which is a Dirac delta function supported on a contact discontinuity appears. So we introduce the delta contact discontinuity as in [15, 20], as follows.

Definition 3.1. Let Ω be a region where u is a continuous function and the curve Γ in Ω of slope $\lambda = u$. A pair of distributions $(u, \rho) \in C(\Omega) \times D'(\Omega)$ is called a delta contact discontinuity, if u is a weak solution of the equations (1.1) and ρ is a sum of locally integrable function on Ω and a delta function on Γ which solves the equations (1.1) in the sense of distribution.

If $u_0 = 0$ and $\omega_0 = 0$, it is easy to see that the solution of (1.1) and (1.3) is consistent with the Riemann solutions of (1.1) and (2.1), which implies that the solution constructed here is stable under some perturbations.

Case 2: $u_0 < u_- < u_+$ (If $u_- < u_+ < u_0$, then the structure of the solution is similar.)

In this case, when t is small enough, the solution of the initial value problem (1.1) and (1.2) can be expressed briefly as follows (see Fig. 3.2.):

$$(u_-, \rho_-) + \delta S_1 + (u_0, \frac{\omega_0}{\epsilon}) + J^0 + Vac + J^+ + (u_+, \rho_+).$$

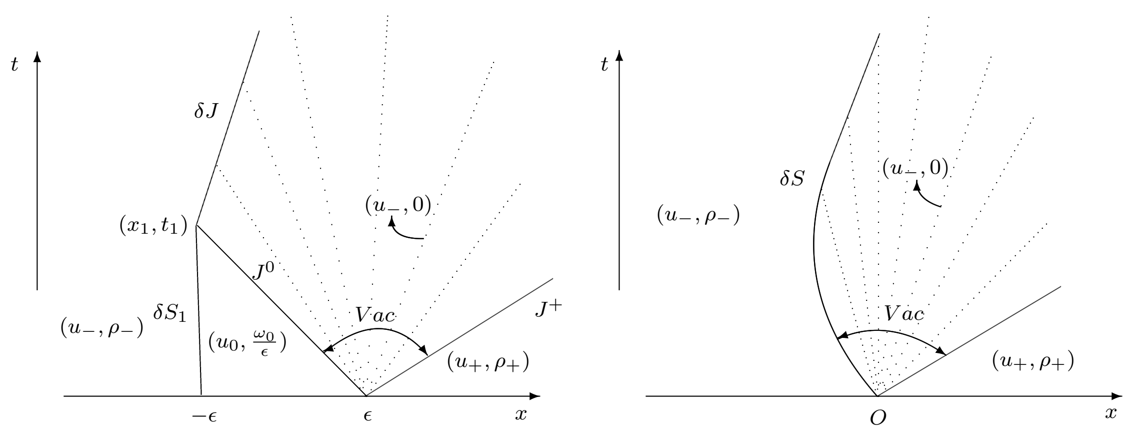

Fig. 3.2.

The propagating speed of δS_1 is $u_{\delta 1}$ which satisfies

$$u_{\delta 1} = \frac{\sqrt{\frac{\omega_0}{\epsilon}}u_0 + \sqrt{\rho_-}u_-}{\sqrt{\frac{\omega_0}{\epsilon}} + \sqrt{\rho_-}},\ u_0 < u_{\delta 1} < u_-.$$

The propagating speed of J^0 is u_0. Thus, it is easy to see that δS_1 and J^0 will meet at a finite time. The interaction point (x_1, t_1) is determined by

$$\begin{cases} x_1 + \epsilon = u_{\delta 1} t_1, \\ x_1 - \epsilon = u_0 t_1, \end{cases} \tag{3.1}$$

which gives

$$(x_1, t_1) = (\frac{2u_0\epsilon}{u_{\delta 1} - u_0} + \epsilon, \frac{2\epsilon}{u_{\delta 1} - u_0}) \tag{3.2}$$

Now at the time $t = t_1$, we again have a Riemann problem with initial data

$$(u, \rho)(x, t_1) = \begin{cases} (u_-, \rho_-), & x < x_1 \\ (u, 0), & x > x_1, \end{cases} \tag{3.3}$$

where u satisfies $u_0 \le u \le u_+$, which is resolved by delta contact discontinuity δJ. Notice that the right-side hand of δJ is vacuum state, we now turn our attention to the interaction of delta contact discontinuity and vacuum. The generalized Rankine-Hugoniot conditions can be detailed by

$$\begin{cases} \dfrac{\mathrm{d}x(t)}{\mathrm{d}t} = u_{\delta 2}, \\ \dfrac{\mathrm{d}\omega(t)}{\mathrm{d}t} = -u_{\delta 2}\rho_- + \rho_- u_-, \\ \dfrac{\mathrm{d}\omega(t)u_{\delta 2}}{\mathrm{d}t} = -u_{\delta 2}\rho_- u_- + \rho_- u_-^2. \end{cases} \tag{3.4}$$

From (3.4), we have $u_{\delta 2} = u_-$. When $t > t_1$, the wave δJ begins to penetrate the vacuum with speed of propagation u_-. The process of penetration are determined by

$$\begin{cases} \dfrac{dx}{dt} = u_-, \\ x - \epsilon = ut, \end{cases} \tag{3.5}$$

where u satisfies $u_0 \leq u \leq u_+$. Differentiate the second equation of (3.5) with respect to t, we obtain

$$\frac{dx}{dt} = \frac{du}{dt}t + u. \tag{3.6}$$

From (3.5) and (3.6), we obtain

$$\frac{du}{dt}t + u = u_-. \tag{3.7}$$

Integrate (3.7), we have

$$\frac{u_- - u_0}{u - u_-} = \frac{t}{t_1}. \tag{3.8}$$

From (3.8), it is clear that $t \to \infty$ as $u \to u_-$. Thus, the delta shock wave does not penetrate vacuum completely. In brief, when $t \to \infty$, the solutions can be expressed as

$$(u_-, \rho_-) + \delta J + Vac + J^+ + (u_+, \rho_+).$$

So far, the solutions of (1.1) and (1.2) have been constructed completely. Letting $\epsilon \to 0$, from (3.2), we know that (x_1, t_1) tends to $(0, 0)$. So we obtain a solutions of (1.1) and (1.3) as follows, see Fig. 3.2.,

$$(u_-, \rho_-) + \delta S + Vac + J^+ + (u_+, \rho_+).$$

Now, we prove that the solution is reasonable. Integrate (3.4) from 0 to t with initial data

$$(x, \omega, u_{\delta 2}) = (0, \omega_0, u_0), \tag{3.9}$$

we obtain

$$\begin{cases} \omega - \omega_0 = -\rho_- x + \rho_- u_- t, \\ \omega u_\delta = -\rho_- u_- x + \rho_- u_-^2 t. \end{cases} \tag{3.10}$$

From (3.10), we get

$$\omega_0 u_0 - \omega_0 u_\delta = -\rho_- u_\delta x + \rho_- u_- u_\delta t + \rho_- u_- x - \rho_- u_- t, \tag{3.11}$$

i.e.,

$$\frac{d}{dt}\{\frac{1}{2}\rho_- \mathrm{x}^2 - \rho_- \mathrm{u}_- \mathrm{xt} - \omega_0 \mathrm{x} + \frac{1}{2}\rho_- \mathrm{u}_-^2 \mathrm{t}^2 + \omega_0 \mathrm{u}_0 \mathrm{t}\} = 0. \tag{3.12}$$

Integrate (3.12) from 0 to t, we have

$$\frac{1}{2}\rho_- x^2 - \rho_- u_- xt - \omega_0 x + \frac{1}{2}\rho_- u_-^2 t^2 + \omega_0 u_0 t = 0. \tag{3.13}$$

Solving (3.13), we obtain

$$x(t) = \frac{\omega_0 + \rho_- u_- - \omega(t)}{\rho_-}, \tag{3.14}$$

and

$$x(t) = \frac{\omega_0 + \rho_- u_- + \omega(t)}{\rho_-}, \tag{3.15}$$

where

$$w(t) = \sqrt{\omega_0^2 + 2\omega_0 \rho_- u_- t - 2\omega_0 \rho_- u_0 t}. \tag{3.16}$$

Differentiate (3.14) and (3.15) with respect to t, we obtain

$$u_\delta(t) = \frac{\rho_- u_- - \frac{dw}{dt}}{\rho_-}, \tag{3.17}$$

and

$$u_\delta(t) = \frac{\rho_- u_- + \frac{dw}{dt}}{\rho_-}, \tag{3.18}$$

where

$$\frac{dw}{dt} = \frac{\omega_0 \rho_- (u_- - u_0)}{\sqrt{\omega_0 + 2\omega_0 \rho_- u_- t - 2\omega_0 \rho_- u_0 t}}. \tag{3.19}$$

It is easy to conclude that the solution of (3.17) satisfies entropy condition. We give up the solutions of (3.15) and (3.18) for (3.18) does not satisfy the entropy condition.

From (3.17) and (3.19), we know that

$$\lim_{t\to\infty} u_\delta = u_-. \tag{3.20}$$

So the delta shock does not penetrate vacuum completely. It also means that δS converts to δJ when $t \to \infty$.

Remark 1. From (3.14) and (3.16), if $u_0 = 0$, $\omega_0 = 0$, then $(x, \omega, u) = (u_- t, 0, u_-)$.

This is consistent with the results to the Riemann problem (1.1) and (2.1). It implies that the solution constructed here is stable under some perturbations.

Case 3: $u_0 < u_+ < u_-$ (If $u_+ < u_0 < u_-$, then the structure of the solution is similar.)

In this case, when t is small enough, the solution of the initial value problem (1.1) and (1.2) can be expressed briefly as follows (see Fig.3.3.):

$$(u_-, \rho_-) + \delta S_1 + (u_0, \frac{\omega_0}{\epsilon}) + J^0 + Vac + J^+ + (u_+, \rho_+).$$

It is easy to see that the interaction point (x_1, t_1) satisfies (3.2). The interaction point (x_2, t_2) is determined by

$$\begin{cases} x_2 - x_1 = u_- t_2, \\ x_2 - \epsilon = u_+ t_2, \end{cases} \tag{3.21}$$

which means that

$$(x_2, t_2) = (\frac{(\epsilon - x_1)u_+}{u_- - u_+} + \epsilon, \frac{\epsilon - x_1}{u_- - u_+}), \tag{3.22}$$

where x_1 satisfies (3.2).

When $t > t_2$, the solution can be expressed by

$$(u_-, \rho_-) + \delta S_3 + (u_+, \rho_+).$$

So far, the solutions of (1.1) and (1.2) have been constructed completely. Letting $\epsilon \to 0$, we obtain a solution of (1.1) and (1.3), see Fig. 3.3. When t is small,the solution can be expressed by

$$(u_-, \rho_-) + \delta S + Vac + J^+ + (u_+, \rho_+).$$

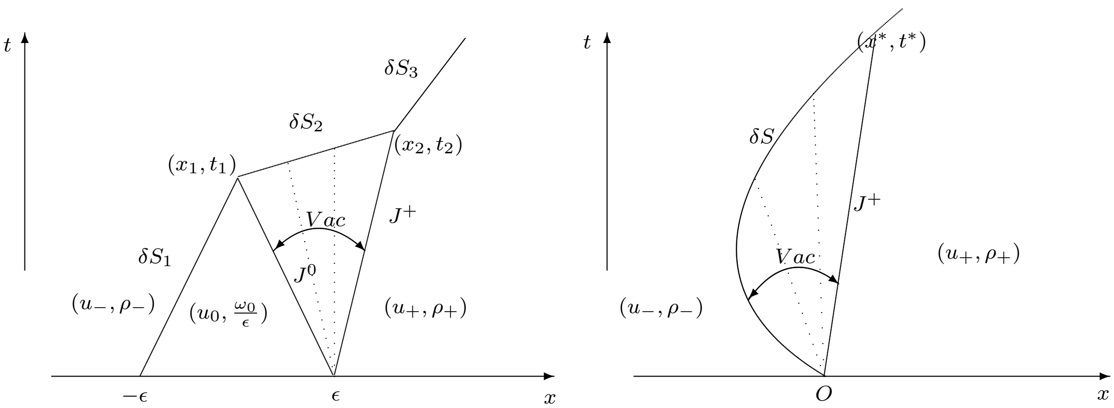

Fig. 3.3.

Notice that the right hand of δJ is vacuum state, we turn our attention to the interaction of delta shock and vacuum. The generalized Rankine-Hugoniot conditions can be detailed by

$$\begin{cases} \dfrac{\mathrm{d}x(t)}{\mathrm{d}t} = u_\delta, \\ \dfrac{\mathrm{d}\omega(t)}{\mathrm{d}t} = -u_\delta\rho_- + \rho_- u_-, \\ \dfrac{\mathrm{d}\omega(t)u_\delta}{\mathrm{d}t} = -u_\delta\rho_- u_- + \rho_- u_-^2. \end{cases} \tag{3.23}$$

From (3.23), like the case 2, we can obtain (3.17) and (3.19). It is easy to see that the delta shock will interact J^+ at a finite time t^*. From (3.17), it is clear that δS will meet J^+ at a finite time $t = t^*$. Now at the time $t = t^*$, we have a Riemann problem with initial data

$$(u, \rho)(x, t_1) = \begin{cases} (u_-, \rho_-), & x < x^* \\ (u_+, \rho_+), & x > x^*. \end{cases} \tag{3.24}$$

when $t > t^*$, the solution can be expressed

$$(u_-, \rho_-) + \delta S + (u_+, \rho_+).$$

If $u_0 = 0$, $\omega_0 = 0$, the solution of this case is consistent with the results to the Riemann problem (1.1) and (2.1). It implies that the solution constructed here is stable under some perturbations.

Case 4: $u_+ < u_0 < u_-$.

In this case, when t is small enough, the solution of the initial value problem (1.1) and (1.3) can be expressed briefly as follows (see Fig. 3.4.):

$$(u_-, \rho_-) + \delta S_1 + (u_0, \frac{\omega_0}{\epsilon}) + \delta S_2 + (u_+, \rho_+),$$

The propagation speed of the wave δS_1 is $u_{\delta 1}$, satisfying $u_0 < u_{\delta 1} < u_-$. The propagation speed of the wave δS_2 is $u_{\delta 2}$, satisfying $u_+ < u_{\delta 2} < u_0$. Thus we see that the δS_1 will overtake

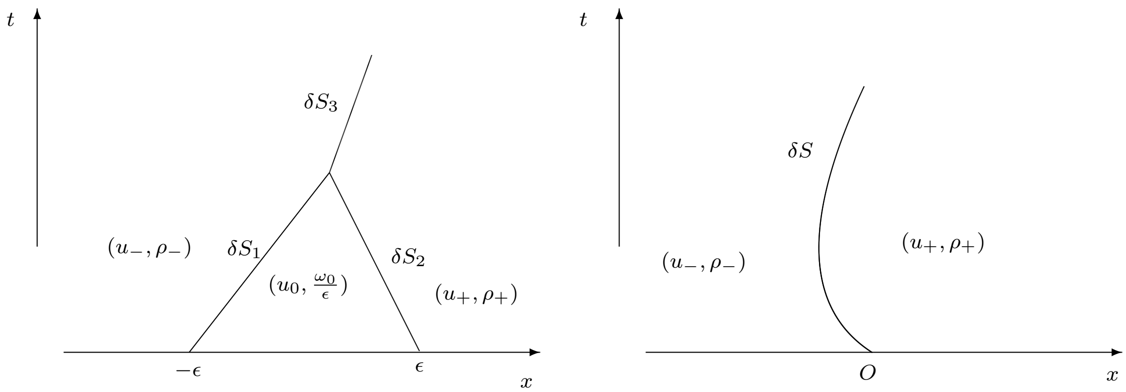

Fig. 3.4.

δS_2 at a finite time. The interaction point (x_1, t_1) is determined by

$$\begin{cases} x_1 + \epsilon = u_{\delta 1} t_1, \\ x_1 - \epsilon = u_{\delta 2} t_1, \end{cases} \tag{3.25}$$

which means that

$$(x_1, t_1) = (\frac{2u_{\delta 1}\epsilon}{u_{\delta 1} - u_{\delta 2}} - \epsilon, \frac{2\epsilon}{u_{\delta 1} - u_{\delta 2}}). \tag{3.26}$$

At the time $t = t_1$, we again have a Riemann problem with initial data

$$(u, \rho)(x, t_1) = \begin{cases} (u_-, \rho_-), & x < x_1, \\ (u_+, \rho_+), & x > x_1, \end{cases} \tag{3.27}$$

which is resolved by a new delta shock wave δS_3 (See Fig. 3.4.). Letting $\epsilon \to 0$, we obtain a solutions of (1.1) and (1.3), see Fig. 3.4.

Now, we prove that solution is reasonable. The solution

$$(u, \rho)(x, t_1) = \begin{cases} (u_-, \rho_-), & x < x(t), \\ (u_\delta, \omega(t)\delta(x - x(t))), & x = x(t), \\ (u_+, \rho_+), & x > x(t), \end{cases} \tag{3.28}$$

satisfies the generalized Rankine-Hugoniot conditions:

$$\begin{cases} \frac{\mathrm{d}x(t)}{\mathrm{d}t} = u_\delta, \\ \frac{\mathrm{d}\omega(t)}{\mathrm{d}t} = u_\delta[\rho] - [\rho u], \\ \frac{\mathrm{d}\omega(t)u_\delta}{\mathrm{d}t} = u_\delta[\rho u] - [\rho u], \end{cases} \tag{3.29}$$

where $[\rho] = \rho_+ - \rho_-$, with initial data (3.9). Like case 2, we obtain

$$\begin{cases} x(t) = \frac{1}{2}(u_- + u_+)t, & [\rho] = 0, \\ x(t) = \frac{-\omega_0 + [\rho u]t + \omega(t)}{[\rho]} & [\rho] \neq 0, \end{cases} \tag{3.30}$$

where

$$w(t)=\sqrt{(\omega_0-[\rho u]t)^2-[\rho]([\rho u^2]t^2-\omega_0 u_0 t)}. \tag{3.31}$$

From (3.30), we know that

$$\lim_{t\to\infty} u_\delta=\begin{cases}\frac{1}{2}(u_-+u_+), & [\rho]=0,\\ \frac{\sqrt{\rho_+}u_++\sqrt{\rho_-}u_-}{\sqrt{\rho_+}+\sqrt{\rho_-}}, & [\rho]\neq 0.\end{cases} \tag{3.32}$$

So we have

$$u_+<\lim_{t\to\infty} u_\delta<u_-.$$

Remark 2. From (3.30) and (3.31), if $u_0=0$, $\omega_0=0$, then

$$(x,u_\delta,\omega)(t)=\begin{cases}(\frac{1}{2}(u_-+u_+)t, \frac{1}{2}(u_-+u_+), (\rho_-u_--\rho_+u_+)t), & [\rho]=0,\\ (\frac{\sqrt{\rho_+}u_++\sqrt{\rho_-}u_-}{\sqrt{\rho_+}+\sqrt{\rho_-}}t, \frac{\sqrt{\rho_+}u_++\sqrt{\rho_-}u_-}{\sqrt{\rho_+}+\sqrt{\rho_-}}, \sqrt{\rho_+\rho_-}(u_--u_+)t), & [\rho]\neq 0.\end{cases}$$

This is consistent with the results to the Riemann problem (1.1) and (2.1). It implies that the solution constructed here is stable under some perturbations.

So far, we have finished the discussion for all kinds of interactions. The global solutions for the perturbed initial value problem (1.1)-(1.2) have been constructed completely. Moreover we obtain the solutions of (1.1) and (1.3) by letting $\epsilon\to 0$ for the solutions of (1.1)-(1.2). We summarize our results in the following.

Theorem 3.2. *The solutions of (1.1) and (1.3) can be obtained by letting $\epsilon\to 0$ for the solutions of (1.1)-(1.2). If letting $(\omega_0,u_0)=(0,0)$, the solutions of (1.1) and (1.3) are exactly corresponding to the solutions of Riemann problem (1.1) and (2.1). It implies that the solution constructed here is stable under some perturbations.*

References

1. F. Bouchut, *On zero-pressure gas dynamics//Advance in Kinetic Theory and Computing*, *Ser. Adv. Math. Appl. Sci. 22. River Edge, NJ: World Scientific*, (1994), pp: 171-190.
2. Y. Brenier, E. Grenier, *Sticky particles and scalar conservation laws*, *SIAM J. Numer Anal.*, **35**(1998), pp: 2317-2328.
3. A. Bressan, *Hyperbolic systems of Conservation Laws: The one-Dimensional Cauchy Problem*, *Oxford lecture Ser. Math. Appl.*, vol. 20, Oxford Univ. Press, Oxford, 2000.
4. T. Chang, L. Hsiao, *The Riemann problem an Interaction of Waves in Gas Dynamics*, *Pitman Monographs and Survey in Pure and Applied Mathematics*, **Vol. 41**, Longman Scientific and Technical, Essex, Englang, 1989.
5. G. Q. Chen, H. Liu, *Formation of δ-shocks and vacuum states in the vanishing pressure limit of solutions to the Euler equations for isentropic fluids*, *SIAM. J. Math. Anal.*, **34**(2003), pp: 925-938.
6. G. Q. Chen, H. Liu, *Concentration and cavition in the vanishing pressure limit of solutions to the Euler equations for nonisentropic fluids*, *Physica D*, **189**(2004), pp: 141-165.
7. V. G. Danilov, V. M. Shelkovich, *Delta-shock waves type solution of hyperbolic systems of conservation laws*, *Quart. Appl. Math.*, **63**(2005), pp: 401-427.
8. L.H.Guo,W.C.Sheng,T.Zhang, *The two-dimensional Riemann problem for isentropic Chaplygin gas dynamic system*, *Communications on Pure and Applied Analysis*, **9**(2010), pp: 431-458.
9. D. J. Korchinski, *Solutions of a Riemann problem for a system of conservation laws possessing no classical weak solution*, *Thesis, Adelphi University*, 1977.

10. J.Q. Li, T. Zhang, S. Yang, *The two-dimensional Riemann problem in gas dynamics*, in: Pitman Monographs and Surveys in Pure and Applied Mathematics 98, Longman, Scientific and Technical, Harlow, 1998.
11. Y. Li, Y. Cao, *Large partial difference method with second accuracy in gas dynamics*, Sci Sinica A, **28**(1985), pp: 1024-1035.
12. T. Liu, J. Smoller, *on the vacuum state for isentropic gas dynamic equations*, Adv. in Appl. Math., **1**(1980), pp: 345-359.
13. M. Nedeljkov, *Singular shock waves in interactions*, Quart. Appl. Math.,**66**(2008), pp: 281-302.
14. M. Nedeljkov, *Delta and singular delta locus for one dimensional systems of conservation laws*, Math. Methods Appl. Sci.,**27**(2004), pp: 931 - 955.
15. M. Nedeljkov, M. Oberguggenberger, *Interactions of delta shock waves in a strictly hyperbolic system of conservation laws*, J. Math. Anal. Appl., **344**(2008), pp: 1143-1157.
16. E. W. Rykov, G. Yu, Ya G. Sinai, *Generalized varinational principled, global weak solutions and behavior with random initial data for system of conservation laws arising in adhesion particle dynamics*, Comm. Math. Phys., **177**(1996), pp: 349-380.
17. S. F. Shandarin, Y. B. Zeldovich, *The large-scale structure of the universe: turbulence, intermittency, structure in a self-gravitatuing medium*, Rev. Mod. Phys., **61**(1989), pp: 185-220.
18. V. M. Shelkovich, *$\delta-$ and $\delta'-$ shock wave types of singular solutions of systems of conservation laws and transport and concentration processes,*, Russian Math. Surv., **63**(2008), pp: 473-546.
19. C. Shen, M. Sun, *Interactions of delta shock waves for the transport equations with split delta functions*, J. Math. Anal. Appl., **351**(2009), pp: 747-755.
20. C. Shen, M. Sun, Stability of the Riemann solutons for a nonstrictly hyperbolic system of conservation laws, *Nonlinear Analysis*, **73**(2010), pp: 3284-3294.
21. C. Shen, M. Sun, *Formation of delta shocks and vacuum states in the vanishing pressure limit of Riemann solutions to the perturbed Aw-Rascle model*, J. Differential Equations, **249**(2010), pp: 3024-3051.
22. W. C. Sheng, T. Zhang, *The Riemann Problem for the Transportation Equations in Gas Dynamics*, Mem. Amer. Math. Soc., **137**(1999), 654.
23. J. Smoller, *Shock Waves and Reaction-Diffusion Equation*, Springer-Verlag, New York, 1994.
24. M. Sun, *Interactions of elementary waves for the Aw-Rascle model*, SIAM J. Appl. Math., **69**(2009), No. 6, pp: 1542-1558.
25. D. Tan, T. Zhang, Y. Zheng, *Delta shock waves as limits of vanishing viscosity for hyperbolic system of conservation laws*, J. Differential Equations,**112**(1994), pp: 1-32.
26. Z. Wang, X. Ding, *Uniqueness of generalized solution for the cauchy problem of transportation equations*, Acta Mathematica Scientia, **1**73(1997), pp: 341-352.
27. Z. Wang, X. Ding, *On the cauchy problem of transportation equations*, Acta Mathematica Scientia, **1**32(1997), pp: 113-122.
28. Z. Wang, Q. L. Zhang, *Spiral solutions to the two-dimensional transport equations*, Acta Math. Appl. Scientia, **3**06(2010), pp: 2110-2128.
29. Z. Wang, Q. L. Zhang, *The Riemann problem with delta initial data for the one-diemensional Chaplygin gas equations*, Acta Mathematica Scientia, **3**(2012), pp: 825-841.
30. H. Yang *generalized plane delta-shock waves for n-dimensional zero-pressure gas dynamics*, J. Math. Anal. Appl.,**260**(2001), pp: 18-35.
31. H. Yang, W. Sun, *The Riemann problem with initial data for a class of coupled hyperbolic system of comservation laws*, Nonlinear Analysis: TMA,**67**(2007), pp: 3041-3049.