Moore Systems and Moore Convergence Classes of Families of Nets

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Abstract. This paper presents a new cryptomorphic mathematical structure of the Moore system, called Moore convergence class of families of nets. It is proved that, for a given set X, appropriate order relation $\leq$ can be defined on $\mathbb{MCCFN}(X)$ (the set of all Moore convergence classes of families of nets on X) and a one-to-one correspondence $\tau : (\mathbb{MCCFN}(X), \leq) \longrightarrow (\mathbb{MS}(X), \supseteq)$ (the set of all Moore systems on the set X) can be defined such that $(\mathbb{MCCFN}(X), \leq)$ is a complete lattice and τ is a complete lattice isomorphism. This means that Moore systems and Moore convergence classes of families of nets are cryptomorphic mathematical structures. It is also proved (by considering the restrictions of τ) that pre-cotopologies and pre-convergence classes of families of nets are cryptomorphic mathematical structures and that cotopologies and convergence classes of nets are cryptomorphic mathematical structures (a classical result). This naturally gives a new approach to a Moore system and a pre-cotopology.

1. Introduction and preliminaries

Let X be a set. By a cotopology on X we mean a family $\mathscr{F}$ of subsets (called closed sets) of X with $\mathscr{F}' = \{V \mid X - V \in \mathscr{F}\}$ a topology on X. By a topological closure operator (or Kuratowski closure operator) on X we mean a mapping $c : 2^X \longrightarrow 2^X$ (the set of all subsets of X) which satisfies the following conditions (CO1)–(CO4):

(CO1) $c(\emptyset) = \emptyset$.

(CO2) $A \subseteq c(A) \quad (\forall A \in 2^X)$.

(CO3) $c(A \cup B) = c(A) \cup c(B) \quad (\forall A, B \in 2^X)$.

(CO4) $c(c(A)) = c(A) \quad (\forall A \in 2^X)$.

Denote the set of all cotopologies on X by $\mathbb{CT}(X)$, and the set of all topological closure operators on X by $\mathbb{TCL}(X)$. Define a binary relation $\leq$ on $\mathbb{TCL}(X)$ by putting $c_1 \leq c_2$ iff $c_1(A) \subseteq c_2(A) \quad (\forall A \in 2^X)$. Then both $(\mathbb{CT}(X), \supseteq)$ and $(\mathbb{TCL}(X), \leq)$ are complete lattices, and there exists a one-to-one correspondence $\phi_0 : (\mathbb{TCL}(X), \leq) \longrightarrow (\mathbb{CT}(X), \supseteq)$ such that both ϕ_0 and its inverse mapping ϕ_0^{-1}

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preserve orders (and thus both preserve infimum and supremum, [**9**]), i.e. ϕ_0 is a complete lattice isomorphism. In this case, we say that cotopologies and topological closure operators are cryptomorphic mathematical structures (or there is a cryptomorphic relationship between cotopologies and topological closure operators). Finding a cryptomorphic version of a mathematical structure $\mathscr{A}$ or establishing a cryptomorphic relationship between $\mathscr{A}$ and another mathematical structure $\mathscr{B}$ is of much importance. For example, tools and results developed in $\mathscr{A}$ can be translated to $\mathscr{B}$, and vice versa. There is a cryptomorphic relationship between cotopologies and interior operators (resp., exterior operators, boundary operators, derived operators, difference derived operators, neighborhood operators, remote neighborhood operators and convergence class operators) ([**9, 12, 23, 27, 30**]). There is also a cryptomorphic relationship between systems of matroid independent sets and systems of matroid minimal circuits (resp., matroid bases, matroid rank functions, matroid closure operators, systems of matroid closed sets, matroid derived operators and matroid difference derived operators) ([**26**]). Encouraged by applications of Moore systems in some non-mathematics areas ([**1, 21, 22**]), this paper will study cryptomorphic mathematical structures of Moore systems (see the following Definition 1.1, which is a kind generalization of cotopologies).

Definition 1.1. [**4, 11, 20**] Let X be a set. If $\mathscr{F} \subseteq 2^X$ is closed under the operations of arbitrary intersection (here we make an agreement that $\bigwedge \emptyset = X$), then we call $\mathscr{F}$ a Moore system on X, and $(X, \mathscr{F})$ a Moore space. Any element in $\mathscr{F}$ is called a closed set. A Moore space $(X, \mathscr{F})$ satisfying $\emptyset \in \mathscr{F}$ is called a pre-cotopological space (in this case, $\mathscr{F}$ is called a pre-cotopology). The set of all Moore systems (resp., all pre-cotopologies) on X is denoted by $\mathbb{MS}(X)$ (resp., $\mathbb{PCT}(X)$).

Definition 1.2. [**3, 6**] A Moore closure operator on a given set X is a mapping $c : 2^X \longrightarrow 2^X$ which satisfies the following conditions:
(C1) $A \subseteq B$ implies $c(A) \subseteq c(B)$ $(\forall A, B \in 2^X)$.
(C2) $A \subseteq c(A)$ $(\forall A \in 2^X)$.
(C3) $c(c(A)) = c(A)$ $(\forall A \in 2^X)$.
A Moore closure operator c on X satisfying $c(\emptyset) = \emptyset$ is called a pre-closure operator. The set of all Moore closure operators (resp., all pre-closure operators) on X is denoted by $\mathbb{MCL}(X)$ (resp., $\mathbb{PCL}(X)$).

Moore closure operator is a cryptomorphic mathematical structure (see [**25**]) of Moore system, and both occur in a quantity of domains: algebra, topology, geometry (see [**6, 24**]), lattice theory, logic (see [**2, 3, 17**]), combinatorics, computer science (see [**4**]), relational data bases (see [**5**]), data analysis (see [**8, 10**]), knowledge structures (see [**7**]), mathematical social sciences (see [**13, 14, 18**]), artificial life (see [**1**]), evolutionary theory (see [**22**]), combinatorial chemistry (see [**22**]), evolutionary biology (see [**21**]), etc. This paper will present a new cryptomorphic version of Moore systems, called Moore convergence classes of families of nets. We also prove that pre-cotopologies and pre-convergence classes of families of nets are cryptomorphic mathematical structures, and this generalizes the main result in [**29**] from a finite set to an arbitrary set.

Now we introduce some preliminary notions and results needed in this paper. It is well-known that net is a very useful tool for the convergence theory in topology [**12**]. But it is not in a Moore system and in a pre-cotopology since the intersection of two closed sets may be not a closed set. Thus we give a new notion — a family of nets in place of net.

Definition 1.3. [**29**] Let X be a set. A family $\overline{\mathscr{S}} = \{\overline{S}_i \mid i \in T\}$ of nets (i.e. $\overline{S}_i$ is a net for each $i \in T$) in X is said to be a family of subnets of another family $\mathscr{S} = \{S_i \mid i \in T\}$ of nets in X if $\overline{S}_i$ is a subnet of S_i for each $i \in T$.

Remark 1.4. The empty family of nets will be regarded as a family of subnets of any family of nets.

Definition 1.5. Suppose that $\mathscr{F}$ is a Moore system on X, $x \in X$, and $\mathscr{S}$ is a family of nets in X.

(1) $U \in 2^X$ is called a closed remote neighborhood of x related to $\mathscr{F}$ iff it satisfies $U \in \mathscr{F}$ and $x \notin U$. The set of all closed remote neighborhoods of x related to $\mathscr{F}$ is denoted by $\mathscr{F}(x)$.

(2) $\mathscr{S}$ is said to converge to x with respect to $\mathscr{F}$ iff there exists a net $S \in \mathscr{S}$ which is eventually not in U for all $U \in \mathscr{F}(x)$ (that is, there exists $m \in D$ such that $S(n) \notin U$ when $m \leq n$). Clearly, any family $\mathscr{S}$ of nets (particularly, empty family of nets) converges to x if $\mathscr{F}(x) = \emptyset$ and empty family of nets does not converge to x whenever $\mathscr{F}(x) \neq \emptyset$.

Surprisingly, empty family of nets (which is written as $\emptyset$) is of much importance in a Moore system, that is why we give an emphasis on empty family of nets in Remark 1.4 and Definition 1.5.

Convergence is an important and commonly used definition in general topology and fuzzy topology (see [**12, 15**]). A convergence theory, which is different from the others, is introduced in the following.

Definition 1.6. A space of Moore convergence class of families of nets is a pair $(X, (\Phi, F))$, where X is a set, $F \subseteq X$ such that the following conditions are satisfied (the pair (Φ, F) is called a Moore convergence class of families of nets):

(S1) $\{(\mathscr{S}, x) \mid \mathscr{S}$ is a family of nets in X$, x \in F\} \subseteq \Phi$.

(S2) For each $x \in X - F$, $(\emptyset, x) \notin \Phi$.

(S3) If $\mathscr{S}$ is a family of nets in X which contains a net S taking a constant value $x \in X$, then $(\mathscr{S}, x) \in \Phi$.

(S4) If $(\mathscr{S}, x) \in \Phi$ and $\overline{\mathscr{S}}$ is a nonempty family of subnets of $\mathscr{S}$, then $(\overline{\mathscr{S}}, x) \in \Phi$.

(S5) If $\mathscr{S}$ is a nonempty family of nets and $(\mathscr{S}, x) \notin \Phi$, then there exists a nonempty family $\overline{\mathscr{S}}$ of subnets of $\mathscr{S}$ such that $(\overline{\overline{\mathscr{S}}}, x) \notin \Phi$ for any family $\overline{\overline{\mathscr{S}}}$ of nets in $\mathrm{Img}(\overline{\mathscr{S}}) = \bigcup\{\mathrm{Img}(S) \mid S \in \overline{\mathscr{S}})\}$, where $\mathrm{Img}(S)$ denotes the range of S $(S \in \overline{\mathscr{S}})$.

(S6) Let $(\mathscr{S}, x) \in \Phi$. If, for each member $S = \{S(m) \mid m \in D\}$ of $\mathscr{S}$ and each $m \in D$, there exists a nonempty family $\mathscr{S}_{(S,m)} = \{\{S(m,n) \mid n \in E\} \mid E \in \mathscr{T}_{S(m)}\}$ of nets in X (here $\mathscr{T}_{S(m)}$ is a family of directed sets) such that $(\mathscr{S}_{(S,m)}, S(m)) \in \Phi$, then $(\mathscr{S}^*, x) \in \Phi$ for some $I_S \in \prod_{m \in D} \mathscr{T}_{S(m)}$, where

$$\mathscr{S}^* = \{\{S \circ R_{I_S}(m, f) \mid (m, f) \in D \times \prod_{m \in D} I_S(m)\} \mid S = \{S(m) \mid m \in D\}, S \in \mathscr{S}\},$$

$R_{I_S}(m, f) = (m, f(m))$ and $D \times \prod_{m \in D} I_S(m)$ is a directed set with point-wise order.

A Moore convergence class (Φ, F) of families of nets satisfying $F = \emptyset$ is called a pre-convergence class of families of nets. The set of all Moore convergence classes of families of nets (resp., all pre-convergence classes of families of nets) on X is denoted by $\mathbb{MCCFN}(X)$ (resp., $\mathbb{PCCFN}(X)$). In this paper, we will make no distinction between a special family $\mathscr{S} = \{S\}$ of nets and its unique member S. In this way, we may verify that the notions defined in Definitions 1.5 and 1.6 is a generalization of the corresponding notions defined for nets. We will use $\mathbb{CCN}(X)$ to denote the set of all convergence classes of nets on X.

Remark 1.7. For each $c \in \mathbb{MCL}(X)$, let $F(c) = \mathscr{F}_c = \{A \in 2^X \mid c(A) = A\}$. Then we obtain a mapping $\phi : \mathbb{MCL}(X) \longrightarrow \mathbb{MS}(X)$. It can be shown that $\phi|\mathbb{TCL}(X) = \phi_0$, $\phi|\mathbb{PCL}(X) : \mathbb{PCL}(X) \longrightarrow \mathbb{PCT}(X)$ and ϕ are one-to-one correspondences, and its inverse mapping is defined as $c(A) = \{x \in X \mid (X - V) \cap A \neq \emptyset$ for each $V \in \mathscr{F}(x)\} \cup c(\emptyset)$ (see [**25**]). One can also define a one-to-one correspondence $\varphi_0 : \mathbb{CCN}(X) \longrightarrow \mathbb{TCL}(X)$ (cf. [**12**]). In this paper we will define an order relation $\leq$ on $\mathbb{MCCFN}(X)$ such that $(\mathbb{MCCFN}(X), \leq)$ is a complete lattice, and define a complete lattice isomorphism $\varphi : (\mathbb{MCCFN}(X), \leq) \longrightarrow (\mathbb{MCL}(X), \leq)$ such that $\phi \circ \varphi|\mathbb{CCN}(X) = \phi_0 \circ \varphi_0 : (\mathbb{CCN}(X), \leq) \longrightarrow (\mathbb{CT}(X), \supseteq)$, $\phi \circ \varphi|\mathbb{PCCFN}(X) : (\mathbb{PCCFN}(X), \leq) \longrightarrow (\mathbb{PCT}(X), \supseteq)$, and $\phi \circ \varphi : (\mathbb{MCCFN}(X), \leq) \longrightarrow (\mathbb{MS}(X), \supseteq)$ are all complete lattice isomorphisms.

Lemma 1.8. [**4**] Let X be a complete lattice, and Y a poset. If there is an order-isomorphism $f : X \longrightarrow Y$ (i.e. f is a one-to-one correspondence, and both f and its inverse mapping f^{-1} are order-preserving), then Y is a complete lattice too.

2. Main results

For a given set X, let $\leq$ be a relation on $\mathbb{MCCFN}(X)$ defined by $(\Phi_1, F_1) \leq (\Phi_2, F_2)$ if and only if $F_1 \subseteq F_2$ and $\Phi_1 \subseteq \Phi_2$, and define the relation $\leq$ on $\mathbb{MCL}(X)$ by $c_1 \leq c_2$ if and only if $c_1(A) \subseteq c_2(A)$ $(\forall A \in 2^X)$.

Theorem 2.1. *For any set X, $(\mathbb{MCCFN}(X), \leq)$ and $(\mathbb{MCL}(X), \leq)$ are complete lattices and they are isomorphic.*

Proof. **Step 1** For each $(\Phi, F) \in \mathbb{MCCFN}(X)$, let $\varphi(\Phi, F)(A) = \{x \in X \mid (\mathscr{S}, x) \in \Phi$ for some family of nets $\mathscr{S}$ in $A\}$ $(\forall A \in 2^X)$. Then $\varphi(\Phi, F)$ is a Moore closure operator on X, and thus $\varphi : (\mathbb{MCCFN}(X), \leq) \longrightarrow (\mathbb{MCL}(X), \leq)$ is a mapping. Obviously, $\varphi(\Phi, F)$ satisfies (C1). For each $A \in X$, we have $A \subseteq \varphi(\Phi, F)(A)$ by (S3), thus (C2) holds. It remains to show $\varphi(\Phi, F)(A) = \varphi(\Phi, F)(\varphi(\Phi, F)(A))$. By (C2), it suffices to show $\varphi(\Phi, F)(\varphi(\Phi, F)(A)) \subseteq \varphi(\Phi, F)(A)$. For each $x \in \varphi(\Phi, F)(\varphi(\Phi, F)(A))$, if $x \in F$, then, by (S1), there exists a family of nets $\mathscr{S}$ in A such that $(\mathscr{S}, x) \in \Phi$, and thus $x \in \varphi(\Phi, F)(A)$. If $x \notin F$, then, by (S2), there exists a nonempty family of nets $\mathscr{S}$ in $\varphi(\Phi, F)(A)$ such that $(\mathscr{S}, x) \in \Phi$. That is, for each member $S = \{S(m) \mid m \in D\}$ of $\mathscr{S}$, $S(m) \in \varphi(\Phi, F)(A)$ $(\forall m \in D)$. It follows from definition of $\varphi(\Phi, F)(A)$ that there exists a family of nets $\mathscr{S}_{(S,m)}$ in A such that $(\mathscr{S}_{(S,m)}, S(m)) \in \Phi$. If $\mathscr{S}_{(S,m)}$ is an empty family of nets for some $m \in D$, then $S(m) \in F$ by (S2). Replace $\mathscr{S}_{(S,m)}$ by any nonempty family of nets $\mathscr{T}_{(S,m)}$ in A, then, by (S1), $(\mathscr{T}_{(S,m)}, S(m)) \in \Phi$. In this process, we obtain a nonempty family of nets $\mathscr{S}_{(S,m)}$ in A such that $(\mathscr{S}_{(S,m)}, S(m)) \in \Phi$ for each S and $m \in D$. By (S6), there exists a family of nets $\mathscr{S}^*$ in A such that $(\mathscr{S}^*, x) \in \Phi$, which means $x \in \varphi(\Phi, F)(A)$. Therefore, $\varphi(\Phi, F)(\varphi(\Phi, F)(A)) \subseteq \varphi(\Phi, F)(A)$.

Step 2 For each $c \in \mathbb{MCL}(X)$, let $\psi(c) = (\Phi, c(\emptyset))$, where $\Phi = \{(\mathscr{S}, x) \mid x \in X$, $\mathscr{S}$ is a family of nets in X which converges to x with respect to $\mathscr{F}_c$ $\}$, $\mathscr{F}_c$ is the Moore system on X induced by c. Then $\psi(c) = (\Phi, c(\emptyset))$ is a Moore convergence class of families of nets on X, and thus $\psi : (\mathbb{MCL}(X), \leq) \longrightarrow (\mathbb{MCCFN}(X), \leq)$ is a mapping. Let $F = c(\emptyset)$. Obviously, Φ satisfies (S1)–(S4). It remains to verify that Φ satisfies (S5) and (S6).

Suppose that $\mathscr{S}$ is a nonempty family of nets and $(\mathscr{S}, x) \notin \Phi$. Then $x \notin F$, and thus $\mathscr{F}_c(x) \neq \emptyset$. Since $\mathscr{S}$ does not converge to x with respect to $\mathscr{F}_c$ (i.e. there exists a closed remote neighborhood U of x related to $\mathscr{F}_c$ such that each member S of $\mathscr{S}$ has a subnet $\overline{S}$ which is in U), we obtain a family $\overline{\mathscr{S}} = \{\overline{S} \mid S \in \mathscr{S}\}$ of subnets of $\mathscr{S}$, where $\overline{S} = \{\overline{S}(m) \mid m \in \overline{D}\}$ $(\overline{S} \in \overline{\mathscr{S}})$. Let $A_{\overline{\mathscr{S}}} = \bigcup_{\overline{S} \in \overline{\mathscr{S}}} \{\overline{S}(m) \mid m \in \overline{D}\}$, and $\overline{\overline{\mathscr{S}}}$ be a family of nets in $A_{\overline{\mathscr{S}}}$. If $\overline{\overline{\mathscr{S}}}$ is a nonempty family of nets, then $\overline{\overline{\mathscr{S}}}$ is in U, and thus does not converge to x with respect to $\mathscr{F}_c$, that is $(\overline{\overline{\mathscr{S}}}, x) \notin \Phi$. If $\overline{\overline{\mathscr{S}}}$ is an empty family of nets, then $(\overline{\overline{\mathscr{S}}}, x) \notin \Phi$ by (S2) and the fact $x \notin F$. Therefore Φ satisfies (S5).

We now prove that Φ satisfies (S6). Let $\mathscr{S}$ be a nonempty family of nets and $(\mathscr{S}, x) \in \Phi$. If $x \in F$, then any family of nets converges to x with respect to $\mathscr{F}_c$. Suppose $x \notin F$ and for each member $S = \{S(m) \mid m \in D\}$ of $\mathscr{S}$ and each $m \in D$, there exists a nonempty family $\mathscr{S}_{(S,m)} = \{\{S(m,n) \mid n \in E\} \mid E \in \mathscr{T}_{S(m)}\}$ of nets in X such that $(\mathscr{S}_{(S,m)}, S(m)) \in \Phi$. For each member $S = \{S(m) \mid m \in D\}$ of $\mathscr{S}$, we need to find an $I_S \in \prod_{m \in D} \mathscr{T}_{S(m)}$ such that $(\mathscr{S}^*, x) \in \Phi$, where

$$\mathscr{S}^* = \{\{S \circ R_{I_S}(m,f) \mid (m,f) \in D \times \prod_{m \in D} I_S(m)\} \mid S = \{S(m) \mid m \in D\}, S \in \mathscr{S}\}.$$

Since $x \notin F$, $F \in \mathscr{F}_c(x)$, which implies $\mathscr{F}_c(x) \neq \emptyset$. Take an arbitrary member U of $\mathscr{F}_c(x)$. As $(\mathscr{S}, x) \in \Phi$, there exists a member $S = \{S(m) \mid m \in D\}$ of $\mathscr{S}$ which is eventually not in U (i.e. there exists an $m_U \in D$ such that $S(m) \notin U$ whenever $m \geq m_U$). Consequently, $U \in \mathscr{F}_c(S(m))$ $(m \geq m_U)$. For such a member S of $\mathscr{S}$ and each $m \geq m_U$, as the corresponding nonempty family $\mathscr{S}_{(S,m)}$ of nets satisfies $(\mathscr{S}_{(S,m)}, S(m)) \in \Phi$, there exists a member $S_{(S,m,U)} = \{S(m,n) \mid n \in E_U^m\}$ of $\mathscr{S}_{(S,m)}$ which is eventually not in U (i.e. there exists an $n_U \in E_U^m$ such that $S(m,n) \notin U$ whenever $n \geq n_U$). For such a net S, first fix $E^m \in \mathscr{T}_{S(m)}$ for each $m \in D$ satisfying $m \ngeq m_U$, and then define an $I_S \in \prod_{m \in D} \mathscr{T}_{S(m)}$ as follows:

$$I_S(m) = \begin{cases} E_U^m, & m \geq m_U, \\ E^m, & m \ngeq m_U. \end{cases}$$

We will show that I_S is the required one. First fix $n_m \in E^m$ for each $m \in D$ satisfying $m \ngeq m_U$, and then define an $f_S \in \prod_{m \in D} I_S(m)$ as follows:

$$f_S(m) = \begin{cases} n_U, & m \geq m_U, \\ n_m, & m \ngeq m_U. \end{cases}$$

We obtain a member $S \circ R_{I_S}$ of $\mathscr{S}^*$ which is eventually not in U because $S \circ R_{I_S}(m,f) = S(m, f(m)) \notin U$ whenever $(m,f) \geq (m_U, f_S)$. As $U \in \mathscr{F}_c(x)$ is arbitrary, $\mathscr{S}^*$ converges to x with respect to $\mathscr{F}_c$, i.e. $(\mathscr{S}^*, x) \in \Phi$.

Step 3 $\psi \circ \varphi(\Phi, F) = (\Phi, F)$ $(\forall (\Phi, F) \in \mathbb{MCCFN}(X))$, i.e. $F_1 = F$ and $\Phi_1 = \Phi$ hold, where $F_1 = \varphi(\Phi, F)(\emptyset)$ and $\Phi_1 = \{(\mathscr{S}, x) \mid \mathscr{S}$ is a family of nets which converges to x with respect to $\mathscr{F}_{\varphi(\Phi,F)}\}$. It can be easily seen from (S1), (S2) and definition of $\varphi(\Phi, F)(\emptyset)$ that $\varphi(\Phi, F)(\emptyset) = F$. Therefore we only need to show $\Phi_1 = \Phi$.

First, we show $\Phi_1 \subseteq \Phi$. For each $(\mathscr{S}, x) \in \Phi_1$, if $x \in F$, then, by (S1), $(\mathscr{S}, x) \in \Phi$. If $x \notin F = \varphi(\Phi, F)(\emptyset)$, then $\mathscr{F}_{\varphi(\Phi,F)}(x) \neq \emptyset$, and thus $\mathscr{S}$ is a nonempty family of nets. Assume that $(\mathscr{S}, x) \notin \Phi$. By (S5), there exists a $\overline{\mathscr{S}}$ (a nonempty family of subnets of $\mathscr{S}$) such that $(\overline{\overline{\mathscr{S}}}, x) \notin \Phi$ for any family $\overline{\overline{\mathscr{S}}}$ of nets in $\mathrm{Img}(\overline{\mathscr{S}})$. On the other hand, $\overline{\mathscr{S}}$ converges to x with respect to $\mathscr{F}_{\varphi(\Phi,F)}$ because $\mathscr{S}$ does. Since $(\mathscr{S}, x) \notin \Phi$, $x \notin F$. Then $\mathscr{F}_{\varphi(\Phi,F)}(x) \neq \emptyset$. By Definition 1.5, $(X - U) \cap \mathrm{Img}(\overline{\mathscr{S}}) \neq \emptyset$ for each closed remote neighborhood U of x related to $\mathscr{F}_{\varphi(\Phi,F)}$. By Remark 1.7, $x \in \varphi(\Phi, F)(\mathrm{Img}(\overline{\mathscr{S}}))$. It follows form definition $\varphi(\Phi, F)(\mathrm{Img}(\overline{\mathscr{S}}))$ that $(\overline{\overline{\mathscr{S}}}, x) \in \Phi$ for some family $\overline{\overline{\mathscr{S}}}$ in $\mathrm{Img}(\overline{\mathscr{S}})$ of nets. This contradicts (S5).

Next, we show $\Phi \subseteq \Phi_1$. For each $(\mathscr{S}, x) \in \Phi$, if $x \in F$, then $x \in \varphi(\Phi, F)(\emptyset)$, and thus $\mathscr{S}$ converges to x with respect to $\mathscr{F}_{\varphi(\Phi,F)}$, which means $(\mathscr{S}, x) \in \Phi_1$. Otherwise, $x \notin F$ and $(\mathscr{S}, x) \in \Phi$. By (S2), $\mathscr{S}$ is a nonempty family of nets. Suppose that $(\mathscr{S}, x) \notin \Phi_1$ (i.e. $\mathscr{S}$ does not converge to x with respect to $\mathscr{F}_{\varphi(\Phi,F)}$). Then no net in $\mathscr{S}$ is eventually in $X - U$ for some closed remote neighborhood U of x related to $\mathscr{F}_{\varphi(\Phi,F)}$ (notice that $\mathscr{F}_{\varphi(\Phi,F)}(x) \neq \emptyset$). In other words, for each member $S = \{S(m) \mid m \in D\}$ of $\mathscr{S}$, there exists a subnet $\overline{S} = \{S(m) \mid m \in \overline{D}\}$ of S which is in U. We obtain a family of subnets $\overline{\mathscr{S}} = \{\overline{S} \mid S \in \mathscr{S}\}$ of $\mathscr{S}$ in U. By (S4), $(\overline{\mathscr{S}}, x) \in \Phi$. By definition of φ, $x \in \varphi(\Phi, F)(U) = U$. This is a contradiction because $x \notin U$.

Step 4 $\varphi \circ \psi(c) = c$ for each $c \in \mathbb{MCL}(X)$, i.e. $\varphi \circ \psi(c)(A) = c(A)$ for each $c \in \mathbb{MCL}(X)$ and each $A \in 2^X$. First, we show $\varphi \circ \psi(c)(A) \subseteq c(A)$, where $\varphi \circ \psi(c)(A) = \{x \in X \mid$ there exists a family $\mathscr{S}$ of nets in A such that $\mathscr{S}$ converges to x with respect to $\mathscr{F}_c\}$. For each $x \in \varphi \circ \psi(c)(A)$, if $x \in c(\emptyset)$, then $x \in c(A)$ by (C1). If $x \notin c(\emptyset)$, there exists a family $\mathscr{S}$ of nets in A such that $\mathscr{S}$ converges to x with respect to $\mathscr{F}_c$. Clearly $\mathscr{S}$ is a nonempty family of nets and $\mathscr{F}_c(x) \neq \emptyset$. For each $U \in \mathscr{F}_c(x)$, there exists a member S of $\mathscr{S}$ such that S is eventually in $X - U$, thus $A \cap (X - U) \neq \emptyset$. By remark 1.7, $x \in c(A)$. Therefore $\varphi \circ \psi(c)(A) \subseteq c(A)$.

Next, we show $c(A) \subseteq \varphi \circ \psi(c)(A)$. It can be easily seen from definition of $\varphi \circ \psi(c)(A)$ that $c(\emptyset) \subseteq \varphi \circ \psi(c)(A)$. Assume that $x \in c(A) - c(\emptyset)$ (i.e. $\mathscr{F}_c(x) \neq \emptyset$). Then for each $U \in \mathscr{F}_c(x)$, there exists an $a^U \in X$ such that $a^U \in A \cap (X - U)$. Let S_U be the sequence taking a constant value a^U. Then the nonempty family $\mathscr{S}=\{S_U \mid U \in \mathscr{F}(x)\}$ of nets in A converges to x with respect to $\mathscr{F}_c$, which means $x \in \varphi \circ \psi(c)(A)$. Therefore $c(A) \subseteq \varphi \circ \psi(c)(A)$.

Step 5 From the above, φ is a one-to-one correspondence whose inverse mapping is ψ. We will prove that both φ and ψ are order-preserving mappings.

First, we show that $\varphi(\Phi_1, F_1)(A) \subseteq \varphi(\Phi_2, F_2)(A)$ for any $A \in 2^X$, any (Φ_1, F_1), and any $(\Phi_2, F_2) \in \mathbb{MCCFN}(X)$ satisfying $(\Phi_1, F_1) \leq (\Phi_2, F_2)$ (which means φ is an order-preserving mapping). Suppose $x \in \varphi(\Phi_1, F_1)(A)$, then there exists a family of nets $\mathscr{S}$ in A such that $(\mathscr{S}, x) \in \Phi_1$. Since $\Phi_1 \subseteq \Phi_1$, $(\mathscr{S}, x) \in \Phi_2$, which means $x \in \varphi(\Phi_2, F_2)(A)$.

Next, assume that c_1, c_2 are Moore closure operators and $c_1 \leq c_2$, we prove that $F_1 \subseteq F_2$ and $\Phi_1 \subseteq \Phi_2$ (which means ψ is an order-preserving mapping), where $F_i = c_i(\emptyset)$, $\Phi_i = \{(\mathscr{S}, x) \mid x \in X,\ \mathscr{S}$ is a family of nets in X which converges to x with respect to $\mathscr{F}_{c_i}\}$, and $\mathscr{F}_{c_i}$ is a Moore system on X induced by c_i $(i = 1, 2)$.

Obviously, $F_1 = c_1(\emptyset) \subseteq c_2(\emptyset) = F_2$ since $c_1 \le c_2$. Suppose that $(\mathscr{S}, x) \in \Phi_1$. If $x \in F_2$, then $(\mathscr{S}, x) \in \Phi_2$. If $x \notin F_2$, then $\mathscr{F}_{c_2}(x) \neq \emptyset$. As $c_1 \le c_2$ and Remark 1.7, we have $\mathscr{F}_{c_1} \supseteq \mathscr{F}_{c_2}$, and thus $\mathscr{F}_{c_1}(x) \supseteq \mathscr{F}_{c_2}(x)$. Since $(\mathscr{S}, x) \in \Phi_1$, for each $U \in \mathscr{F}_{c_2}(x) \subseteq \mathscr{F}_{c_1}(x)$, there exists a member S of $\mathscr{S}$ such that S is eventually not in U, which means that $\mathscr{S}$ converges to x with respect to $\mathscr{F}_{c_2}$. Hence $(\mathscr{S}, x) \in \Phi_2$.

Step 6 As in the case of topological spaces, there exists an order-isomorphism $\phi : (\mathbb{MCL}(X), \le) \longrightarrow (\mathbb{MS}(X)(X), \supseteq)$ (cf. [**25**]). By Step 5, $\phi \circ \varphi : (\mathbb{MCCFN}(X), \le) \longrightarrow (\mathbb{MS}(X), \supseteq)$ is an order-isomorphism. Clearly $(\mathbb{MS}(X), \supseteq)$ is a complete lattice, and so is $(\mathbb{MCCFN}(X), \le)$ by Lemma 1.8. Therefore $(\mathbb{MCCFN}(X), \le)$ is isomorphic to $(\mathbb{MS}(X), \supseteq)$. □

Corollary 2.2. *$\varphi(\mathbb{PCCFN}(X)) = \mathbb{PCL}(X)$ and $\varphi(\mathbb{CCN}(X)) = \mathbb{TCL}(X)$, thus $\mathbb{PCCFN}(X)$ is isomorphic to $\mathbb{PCL}(X)$ and $\mathbb{CCN}(X)$ is isomorphic to $\mathbb{TCL}(X)$.*

Proof. We only show $\varphi(\mathbb{PCCFN}(X)) = \mathbb{PCL}(X)$. Suppose that $c \in \mathbb{PCL}(X)$. By Step 2 above, $\psi(c) = (\Phi, c(\emptyset))$ is a Moore convergence class of families of nets. Since c is a pre-closure operator, $c(\emptyset) = \emptyset$, and thus $\psi(c) \in \mathbb{PCCFN}(X)$. It follows that $c = \varphi \circ \psi(c) \in \varphi(\mathbb{PCCFN}(X))$. Therefore $\mathbb{PCL}(X) \subseteq \varphi(\mathbb{PCCFN}(X)$. Conversely, assume that $(\Phi, F) \in \mathbb{PCCFN}(X)$, then $\varphi(\Phi, F)$ is a Moore closure operator by Step 1 above. Since (Φ, F) is a pre-convergence class of families of nets, $F = \emptyset$. By (S2), $(\emptyset, x) \notin \Phi$ for any x in X. Hence $\varphi(\Phi, F)(\emptyset) = \emptyset$, which means $\varphi(\Phi, F)$ is a pre-closure operator. Therefore $\varphi(\mathbb{PCCFN}(X)) \subseteq \mathbb{PCL}(X)$ is also true. □

Remark 2.3. (1) Corollary 2.2 improves the main result in [**29**] which says $\varphi(\mathbb{PCCFN}(X)) = \mathbb{PCL}(X)$ when X is a finite set.

(2) For a given set X, let $\mathbb{CT}(X)$ be the set of all cotopologies on X. If we use $J \sqsubseteq L$ to denote that J is a complete sublattice of the complete lattice L, then we may show $\mathbb{CCN}(X) \sqsubseteq \mathbb{PCCFN}(X) \sqsubseteq \mathbb{MCCFN}(X)$, $\mathbb{TCL}(X) \sqsubseteq \mathbb{PCL}(X) \sqsubseteq \mathbb{MCL}(X)$, and $\mathbb{CT}(X) \sqsubseteq \mathbb{PCT}(X) \sqsubseteq \mathbb{MS}(X)$. By Theorem 2.1, $\phi \circ \varphi : (\mathbb{MCCFN}(X), \le) \longrightarrow (\mathbb{MS}(X), \supseteq)$ is an isomorphism; by Corollary 2.2, $\phi \circ \varphi | \mathbb{PCCFN}(X) : (\mathbb{PCCFN}(X), \le) \longrightarrow (\mathbb{PCT}(X), \supseteq)$ and $\phi \circ \varphi | \mathbb{CCN}(X) = \phi_0 \circ \varphi_0 : (\mathbb{CCN}(X), \le) \longrightarrow (\mathbb{CT}(X), \supseteq)$ are also isomorphisms.

3. Concluding remarks

Moore system is a useful mathematical structure which occurs in a quantity of domains, including algebra, topology, geometry, lattice theory, logic, combinatorics, computer science, relational data base, data analysis, knowledge structure, mathematical social science, etc. Finding a cryptomorphic version of a Moore system is of much importance. It is well known that Moore closure operator is a cryptomorphic mathematical structure of Moore system. In this paper we construct a new cryptomorphic mathematical structure of Moore system, called Moore convergence class of families of nets, which allows us to study Moore system and related or analogous system in a different way (we will demonstrate this point in the appendix section). In addition, there may be some connections between the notion of Moore convergence class of families of nets and multi-agent or multi-source approximation space (or its generalization, called dynamic space, see [**19**]). We will discuss this topic in the subsequent paper.

4. Appendix

The notion of continuity is an important concept in mathematics (see [**16, 28**]). In this appendix we will show an application of Moore convergence classes of families of nets by using this notion to characterize continuous mappings in Moore spaces.

Theorem 4.1. *For a mapping f from one Moore space $(X, \mathscr{F}_X)$ to another Moore space $(Y, \mathscr{F}_Y)$, the following statements are equivalent:*

(1) f is continuous, i.e. $f^{-1}(B) \in \mathscr{F}_X \quad (\forall B \in \mathscr{F}_Y)$.

(2) If $\mathscr{S}$ converges to x, then $f \circ \mathscr{S}$ converges to $f(x)$, where $f \circ \mathscr{S} = \{f \circ S \mid S \in \mathscr{S}\}$.

Proof. (1) $\Longrightarrow$ (2). Suppose that $\mathscr{S}$ is a family of nets which converges to x. If $f(x) \in F_Y$ (F_Y is the smallest element in $\mathscr{F}_Y$), then $f \circ \mathscr{S}$ converges to $f(x)$. If $f(x) \notin F_Y$, then $\mathscr{F}_Y(f(x)) \neq \emptyset$. For each $U \in \mathscr{F}_Y(f(x))$, we have $f^{-1}(U) \in \mathscr{F}_X(x)$. Since $\mathscr{S}$ converges to x, there exists a member S of $\mathscr{S}$ such that S is eventually not in $f^{-1}(U)$, and thus $f \circ S$ is eventually not in U. Hence $f \circ \mathscr{S}$ converges to $f(x)$.

(2) $\Longrightarrow$ (1). Assume that $B \in \mathscr{F}_Y$ and $\mathscr{S}$ is a family of nets in $f^{-1}(B)$ which converges to x. We show that $x \in f^{-1}(B)$, so that $f^{-1}(B) \in \mathscr{F}_X$. Since $\mathscr{S}$ is a family of nets in $f^{-1}(B)$, $f \circ \mathscr{S}$ is a family of nets in B. Further, it follows that $f \circ \mathscr{S}$ converges to $f(x)$. Thus $f(x) \in B$, which implies $x \in f^{-1}(B)$. □

Remark 4.2. The notion of Moore convergence class of families of nets can also be used to define or characterize compact subsets of a Moore space and to study dynamical systems on a Moore space.

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