HARDY AND LITTLEWOOD INEQUALITIES ON TIME SCALES

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Abstract. In this paper, we will prove some new dynamic inequalities of Hardy and Littlewood type on time scales. The results as special cases contain the integral inequalities due to Hardy and the discrete inequalities due to Hardy and Littlewood. The main results will be proved by using some algebraic inequalities, the Hölder inequality and a simple consequence of Keller's chain rule on time scales.

1. Introduction

The classical Hardy inequality states that for $f \geq 0$ and integrable over any finite interval $(0, x)$ and f^p is integrable and convergent over $(0, \infty)$ and $p > 1$, then

$$\int_0^\infty \left(\frac{1}{x} \int_0^x f(t)dt \right)^p dx \leq \left(\frac{p}{p-1} \right)^p \int_0^\infty f^p(x)dx. \tag{1.1}$$

The constant $(p/(p-1))^p$ is the best possible. This inequality was proved by Hardy in 1925, and it is the continuous version of a discrete inequality from his work in 1920 ([7]). The discrete version of the inequality (1.1) is given by the inequality

$$\sum_{n=1}^\infty \left(\frac{1}{n} \sum_{k=1}^n a_k \right)^p \leq \left(\frac{p}{p-1} \right)^p \sum_{n=1}^\infty a_n^p, \quad p > 1. \tag{1.2}$$

Hardy's inequality (1.1) has been generalized by Hardy himself in [9]. There he showed that, for any integrable function $f(x) > 0$ on $(0, \infty)$, $p > 1$, then

$$\int_0^\infty \frac{1}{x^m} \left(\int_x^\infty f(t)dt \right)^p dx \leq \left(\frac{p}{1-m} \right)^p \int_0^\infty \frac{1}{x^{m-p}} f^p(x)dx, \ m < 1, \tag{1.3}$$

and

$$\int_0^\infty \frac{1}{x^m} \left(\int_0^x f(t)dt \right)^p dx \leq \left(\frac{p}{m-1} \right)^p \int_0^\infty \frac{1}{x^{m-p}} f^p(x)dx, \quad m > 1. \tag{1.4}$$

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Hardy and Littlewood [8] established the discrete versions of (1.3) and (1.4). In particular they proved that if $p > 1$ and a_n is a sequence of positive terms then

$$\sum_{n=1}^{\infty} \frac{1}{n^c} \left(\sum_{k=n}^{\infty} a_k \right)^p \leq M \sum_{n=1}^{\infty} \frac{1}{n^{c-p}} a_n^p, \quad c < 1 \tag{1.5}$$

and

$$\sum_{n=1}^{\infty} \frac{1}{n^c} \left(\sum_{k=1}^{n} a_k \right)^p \leq M \sum_{n=1}^{\infty} \frac{1}{n^{c-p}} a_n^p, \quad c > 1, \tag{1.6}$$

where M is a positive constant (see [17, 28]). For more contributions of Hardy type inequalities, we refer the reader to the books [14, 15, 20] and the papers [4, 12, 13, 16, 18, 19, 22].

Over the last ten years a number of dynamic inequalities of Hardy type have been established in [21, 23, 25, 27] on a time scale $\mathbb{T}$ (which may be an arbitrary closed subset of the real numbers $\mathbb{R}$). The cases when the time scale is equal to the reals or to the integers represent the classical theories of integral and of discrete inequalities. In this paper, without loss of generality, we assume that $\sup \mathbb{T} = \infty$, and define the time scale interval $[t_0, \infty)_{\mathbb{T}}$ by $[t_0, \infty)_{\mathbb{T}} := [t_0, \infty) \cap \mathbb{T}$. The three most popular examples of calculus on time scales are differential calculus, difference calculus, and quantum calculus, i.e, when $\mathbb{T} = \mathbb{R}$, $\mathbb{T} = \mathbb{N}$ and $\mathbb{T} = q^{\mathbb{N}_0} = \{q^t : t \in \mathbb{N}_0\}$ where $q > 1$. For more details of time scale analysis, we refer the reader to the two books by Bohner and Peterson [2], [3] which summarize and organize much of the time scale calculus. For applications of time scale calculus on oscillation of dynamic equations on time scales we refer to the papers [5, 11, 26, 29] and the book [24].

In [23] the author applied the technique used by Elliott [6] to prove the inequality (1.2) and established a time scale version of the Hardy inequality (1.1). In particular he proved that if $p > 1$ and g is a nonnegative and such that the delta integral $\int_a^{\infty} (g(t))^p \, \Delta t$ exits as a finite number, then

$$\int_a^{\infty} \left(\frac{1}{\sigma(x) - a} \int_a^{\sigma(x)} g(t) \Delta t \right)^p \Delta x \leq \left(\frac{p}{p-1} \right)^p \int_a^{\infty} g^p(x) \Delta x. \tag{1.7}$$

If in addition $\mu(t)/t \to 0$ as $t \to \infty$, then the constant is the best possible. In the proof of the inequality (1.7) the author assumed that $\varphi^{\Delta}(t) > 0$ where $\varphi(t) = \int_a^t f(s) \Delta s/(t-a)$.

The paper is organized as follows. In Section 2, we prove some new dynamic inequalities of Hardy and Littlewood type on time scales. The main results will be proved by making use of the chain rule, Hölder's inequality and some algebraic inequalities. These inequalities contain the integral and discrete inequalities (1.3)-(1.5) as special cases when $\mathbb{T} = \mathbb{R}$ and $\mathbb{T} = \mathbb{N}$.

2. Main Results

In this section, we will prove the main results. For completeness, we recall the following concepts related to the notion of time scales. A time scale $\mathbb{T}$ is an arbitrary nonempty closed subset of the real numbers $\mathbb{R}$. We assume throughout that $\mathbb{T}$ has the topology that it inherits from the standard topology on the real numbers $\mathbb{R}$. The forward jump operator and the backward jump operator are defined by:

$$\sigma(t) := \inf\{s \in \mathbb{T} : \ s > t\}, \quad \rho(t) := \sup\{s \in \mathbb{T} : \ s < t\},$$

where $\sup \emptyset = \inf \mathbb{T}$. A point $t \in \mathbb{T}$, is said to be left–dense if $\rho(t) = t$ and $t > \inf \mathbb{T}$, is right–dense if $\sigma(t) = t$, is left–scattered if $\rho(t) < t$ and right–scattered if $\sigma(t) > t$. A function $g : \mathbb{T} \to \mathbb{R}$ is said to be right–dense continuous (rd–continuous) provided g is continuous at right–dense points and at left–dense points in $\mathbb{T}$, left hand limits exist and are finite. The set of all such rd–continuous functions is denoted by $C_{rd}(\mathbb{T})$.

The graininess function μ for a time scale $\mathbb{T}$ is defined by $\mu(t) := \sigma(t) - t$, and for any function $f : \mathbb{T} \to \mathbb{R}$ the notation $f^{\sigma}(t)$ denotes $f(\sigma(t))$. Fix $t \in \mathbb{T}$ and let $x : \mathbb{T} \to \mathbb{R}$. Define $x^{\Delta}(t)$ to be the number (if it exists) with the property that given any $\epsilon > 0$ there is a neighborhood U of t with

$$|[x(\sigma(t)) - x(s)] - x^{\Delta}(t)[\sigma(t) - s]| \leq \epsilon|\sigma(t) - s|, \quad \text{for all } s \in U.$$

In this case, we say $x^{\Delta}(t)$ is the (delta) derivative of x at t and that x is (delta) differentiable at t. We will frequently use the following results due to Hilger [10]. Throughout the paper will assume that $g : \mathbb{T} \to \mathbb{R}$ and let $t \in \mathbb{T}$.
(i) If g is differentiable at t, then g is continuous at t.
(ii) If g is continuous at t and t is right-scattered, then g is differentiable at t with $g^{\Delta}(t) = \frac{g(\sigma(t))-g(t)}{\mu(t)}$.
(iii) If g is differentiable and t is right-dense, then

$$g^{\Delta}(t) = \lim_{s \to t} \frac{g(t) - g(s)}{t - s}.$$

(iv) If g is differentiable at t, then $g(\sigma(t)) = g(t) + \mu(t)g^{\Delta}(t)$.

Note that if $\mathbb{T} = \mathbb{R}$ then

$$\sigma(t) = t, \ \mu(t) = 0, \ f^{\Delta}(t) = f'(t), \ \int_a^b f(t)\Delta t = \int_a^b f(t)dt,$$

if $\mathbb{T} = \mathbb{Z}$, then

$$\sigma(t) = t + 1, \ \mu(t) = 1, \ f^{\Delta}(t) = \Delta f(t), \ \int_a^b f(t)\Delta t = \sum_{t=a}^{b-1} f(t),$$

if $\mathbb{T} = h\mathbb{Z}$, $h > 0$, then $\sigma(t) = t + h$, $\mu(t) = h$, and

$$y^{\Delta}(t) = \Delta_h y(t) := \frac{y(t+h) - y(t)}{h}, \ \int_a^b f(t)\Delta t = \sum_{k=0}^{\frac{b-a-h}{h}} f(a + kh)h,$$

and if $\mathbb{T} = \{t : t = q^k,\ k \in \mathbb{N}_0,\ q > 1\}$, then $\sigma(t) = qt$, $\mu(t) = (q-1)t$,

$$x^{\Delta}(t) = \Delta_q x(t) = \frac{(x(q\,t) - x(t))}{(q-1)t}, \quad \int_{t_0}^{\infty} f(t)\Delta t = \sum_{k=n_0}^{\infty} f(q^k)\mu(q^k),$$

where $t_0 = q^{n_0}$, and if $\mathbb{T} = \mathbb{N}_0^2 := \{n^2 : n \in \mathbb{N}_0\}$, then $\sigma(t) = (\sqrt{t}+1)^2$,

$$\mu(t) = 1 + 2\sqrt{t},\ \ \Delta_N y(t) = \frac{y((\sqrt{t}+1)^2) - y(t)}{1+2\sqrt{t}}.$$

In this paper we will refer to the (delta) integral which we can define as follows. If $G^{\Delta}(t) = g(t)$, then the Cauchy (delta) integral of g is defined by $\int_a^t g(s)\Delta s := G(t) - G(a)$. It can be shown (see [2]) that if $g \in C_{rd}(\mathbb{T})$, then the Cauchy integral $G(t) := \int_{t_0}^t g(s)\Delta s$ exists, $t_0 \in \mathbb{T}$, and satisfies $G^{\Delta}(t) = g(t)$, $t \in \mathbb{T}$. An infinite integral is defined as $\int_a^{\infty} f(t)\Delta t = \lim_{b\to\infty} \int_a^b f(t)\Delta t$. We will make use of the following product and quotient rules for the derivative of the product fg and the quotient f/g (where $gg^{\sigma} \neq 0$, here $g^{\sigma} = g \circ \sigma$) of two differentiable function f and g

$$(fg)^{\Delta} = f^{\Delta}g + f^{\sigma}g^{\Delta} = fg^{\Delta} + f^{\Delta}g^{\sigma}, \text{ and } \left(\frac{f}{g}\right)^{\Delta} = \frac{f^{\Delta}g - fg^{\Delta}}{gg^{\sigma}}. \tag{2.1}$$

We say that a function $p : \mathbb{T} \to \mathbb{R}$ is regressive provided $1 + \mu(t)p(t) \neq 0$, $t \in \mathbb{T}$. The chain rule formula that we will use in this paper is

$$(x^{\gamma}(t))^{\Delta} = \gamma \int_0^1 [hx^{\sigma} + (1-h)x]^{\gamma-1}\, dhx^{\Delta}(t), \tag{2.2}$$

which is a simple consequence of Keller's chain rule [2, Theorem 1.90]. Using the fact that $g^{\sigma}(t) = g(t) + \mu(t)g^{\Delta}(t)$, we obtain

$$(x^{\gamma}(t))^{\Delta} = \gamma \int_0^1 \left[x + h\mu(t)x^{\Delta}(t)\right]^{\gamma-1} dhx^{\Delta}(t). \tag{2.3}$$

The integration by parts formula is given by

$$\int_a^b u(t)v^{\Delta}(t)\Delta t = [u(t)v(t)]_a^b - \int_a^b u^{\Delta}(t)v^{\sigma}(t)\Delta t. \tag{2.4}$$

To prove the main results, we will use the following Hölder inequality [2, Theorem 6.13]. Let a, $b \in \mathbb{T}$. For u, $v \in C_{rd}(\mathbb{T}, \mathbb{R})$, we have

$$\int_a^b |u(t)v(t)|\,\Delta t \leq \left[\int_a^b |u(t)|^q\,\Delta t\right]^{\frac{1}{q}} \left[\int_a^b |v(t)|^p\,\Delta t\right]^{\frac{1}{p}}, \tag{2.5}$$

where $p > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$.

Throughout the paper, we will assume that the functions in the statements of the theorems are nonnegative and the integrals considered are assumed to exist.

Now, we are ready to state and prove the main results in this paper. Our first two results, Theorem 2.1 (respectively Theorem 2.2) are the time scale version of (1.3) (respectively (1.4)).

Theorem 2.1. *Let $\mathbb{T}$ be a time scale with $a \in (0,\infty)_{\mathbb{T}}$ and $p, q > 0$ such that* $p/q > 1$ and $\gamma < 1$. *Furthermore assume that g is a nonnegative and such that the delta integral $\int_a^\infty (\sigma(t))^{\frac{p}{q}-\gamma} g^{p/q}(t)\Delta t$ exists. Let*

$$\Omega(t) := \int_t^\infty g(s)\Delta s, \qquad \textit{for any} \quad t \in [a,\infty)_{\mathbb{T}}. \tag{2.6}$$

Then

$$\int_a^\infty \frac{(\Omega(t))^{\frac{p}{q}}}{\sigma^\gamma(t)}\Delta t \leq \left(\frac{p}{q\,(1-\gamma)}\right)^{p/q} \int_a^\infty \frac{g^{\frac{p}{q}}(t)}{(\sigma(t))^{\gamma-\frac{p}{q}}}\Delta t. \tag{2.7}$$

Proof. Integrating the left hand side of (2.7) by the parts formula (2.4) with $v^\Delta(t) = 1/\sigma^\gamma(t)$, and $u(t) = (\Omega(t))^{p/q}$, we obtain

$$\int_a^\infty \frac{(\Omega(t))^{\frac{p}{q}}}{\sigma^\gamma(t)}\Delta t = v(t)\Omega^{\frac{p}{q}}(t)\Big|_a^\infty + \int_a^\infty (v^\sigma(t))\left(-\Omega^{p/q}(t)\right)^\Delta \Delta t, \tag{2.8}$$

where $v(t) = \int_a^t (1/\sigma^\gamma(s))\,\Delta s$. Using the chain rule (2.2) and the fact that $\sigma(s) \geq s$ we have

$$\begin{aligned}
\left(s^{1-\gamma}\right)^\Delta &= (1-\gamma)\int_0^1 [h\sigma(s)+(1-h)s]^{-\gamma}dh \\
&= (1-\gamma)\int_0^1 \frac{dh}{[h\sigma(s)+(1-h)s]^\gamma} \\
&\geq (1-\gamma)\int_0^1 \frac{dh}{[h\sigma(s)+(1-h)\sigma(s)]^\gamma} = \frac{(1-\gamma)}{\sigma^\gamma(s)}.
\end{aligned}$$

This implies that

$$\begin{aligned}
v^\sigma(t) &= \int_a^{\sigma(t)} \frac{1}{\sigma^\gamma(s)}\Delta s \leq \frac{1}{1-\gamma}\int_a^{\sigma(t)}\left(\frac{1}{s^{\gamma-1}}\right)^\Delta \Delta t \\
&= \frac{1}{1-\gamma}\frac{1}{(\sigma(t))^{\gamma-1}} - \frac{1}{1-\gamma}\frac{1}{a^{\gamma-1}} \leq \frac{1}{1-\gamma}(\sigma(t))^{1-\gamma}.
\end{aligned} \tag{2.9}$$

Combining (2.8), (2.9) and using the facts that $\Omega(\infty) = 0$ and $v(a) = 0$, we get that

$$\int_a^\infty \frac{(\Omega(t))^{p/q}}{\sigma^\gamma(t)}\Delta t \leq \frac{p}{q(1-\gamma)}\int_a^\infty \frac{g(t)}{(\sigma(t))^{\gamma-1}}\left(-(\Omega^{p/q}(t)\right)^\Delta)\Delta t. \tag{2.10}$$

Applying the chain rule $f^\Delta(g(t)) = f^{'}(g(c))g^\Delta(t)$, where $c \in [t,\sigma(t)]$, we see that there exists $c \in [t,\sigma(t)]$ such that

$$-\left(\Omega^{p/q}(t)\right)^\Delta = -(\frac{p}{q})\Omega^{\frac{p}{q}-1}(c)(\Omega^\Delta(t)). \tag{2.11}$$

Since $\Omega^{\Delta}(t)=-g(t)\leq 0$ and $c\geq t$, we have

$$-\left(\Omega^{p/q}(t)\right)^{\Delta}\leq (\frac{p}{q})(\Omega(t))^{\frac{p}{q}-1}g(t). \tag{2.12}$$

Substituting (2.12) into (2.10), we have

$$\int_a^{\infty}\frac{(\Omega(t))^{p/q}}{\sigma^{\gamma}(t)}\Delta t\leq \frac{p}{q\,(1-\gamma)}\int_a^{\infty}\frac{(\Omega(t))^{\frac{p}{q}-1}}{(\sigma(t))^{\gamma-1}}g(t)\Delta t.$$

This implies

$$\int_a^{\infty}\frac{(\Omega(t))^{p/q}}{\sigma^{\gamma}(t)}\Delta t\leq \frac{p}{q\,(1-\gamma)}\int_a^{\infty}\frac{(\sigma^{\gamma}(t))^{(p-q)/p}g(t)}{(\sigma(t))^{\gamma-1}}\frac{(\Omega(t))^{\frac{p}{q}-1}}{(\sigma^{\gamma}(t))^{\frac{(p-q)}{p}}}\Delta t. \tag{2.13}$$

Applying the Hölder inequality (2.5) on the term

$$\int_a^{\infty}\left[\frac{(\sigma^{\gamma}(t))^{(p-q)/p}}{(\sigma(t))^{\gamma-1}}g(t)\right]\left[(\sigma^{\gamma}(t))^{-\frac{(p-q)}{p}}(\Omega(t))^{\frac{p}{q}-1}\right]\Delta t,$$

with indices p/q and $p/(p-q)$, we see that

$$\int_a^{\infty}\left[\frac{(\sigma^{\gamma}(t))^{(p-q)/p}}{(\sigma(t))^{\gamma-1}}g(t)\right](\sigma^{\gamma}(t))^{-\frac{(p-q)}{p}}(\Omega(t))^{\frac{p-q}{q}}\Delta t$$

$$\leq \left[\int_a^{\infty}\left[\frac{(\sigma^{\gamma}(t))^{(p-q)/p}}{(\sigma(t))^{\gamma-1}}g(t)\right]^{p/q}\Delta t\right]^{\frac{q}{p}}\left[\int_a^{\infty}\frac{(\Omega(t))^{p/q}}{\sigma^{\gamma}(t)}\Delta t\right]^{\frac{p-q}{p}}. \tag{2.14}$$

Substituting (2.14) into (2.13), we have

$$\int_a^{\infty}\frac{(\Omega(t))^{p/q}}{(\sigma(t))^{\gamma}}\Delta t\leq \left(\frac{p}{q(1-\gamma)}\right)^{p/q}\int_a^{\infty}(\sigma(t))^{\frac{p}{q}-\gamma}g^{\frac{p}{q}}(t)\Delta t. \tag{2.15}$$

The proof is complete.

Remark 1. *At the end of the proof above if in addition we use the fact that* $\Omega^{\Delta}(t)\leq 0$, *we see that*

$$\int_a^{\infty}\frac{1}{(\sigma(t))^{\gamma}}(\Omega^{\sigma}(t))^{p/q}\Delta t\leq \int_a^{\infty}\frac{1}{(\sigma(t))^{\gamma}}(\Omega(t))^{p/q}\Delta t,$$

and this and (2.15) implies that

$$\int_a^{\infty}\frac{1}{(\sigma(t))^{\gamma}}(\Omega^{\sigma}(t))^{p/q}\Delta t\leq \left(\frac{p}{q\,(1-\gamma)}\right)^{p/q}\int_a^{\infty}\frac{1}{(\sigma(t))^{\gamma-\frac{p}{q}}}g^{\frac{p}{q}}(t)\Delta t.$$

Remark 2. *As a special case of Theorem 2.1 when* $\mathbb{T}=\mathbb{R}$ *and* $p/q=\lambda>1$ *and* $\gamma<1$, *we have the following Hardy inequality*

$$\int_a^{\infty}\frac{1}{t^{\gamma}}\left(\int_t^{\infty}g(s)ds\right)^{\lambda}dt\leq \left(\frac{\lambda}{1-\gamma}\right)^{\lambda}\int_a^{\infty}\frac{1}{t^{\gamma-\lambda}}g^{\lambda}(t)dt\,.$$

Let $G(t)=\int_t^\infty g(s)ds$. *Thus, we have (note that* $G(\infty)=0$*)*

$$\int_a^\infty \frac{1}{t^\gamma}\left(G(t)\right)^\lambda dt \le \left(\frac{\lambda}{1-\gamma}\right)^\lambda \int_a^\infty \frac{1}{t^{\gamma-\lambda}}(G^{'}(t))^\lambda dt,$$

which can be considered as a generalization of Wirtinger's inequality (see [1]*).*

Remark 3. *Assume that* $\mathbb{T}=\mathbb{N}$ *in Theorem 2.1,* $p/q=\lambda>1$, $a=1$ *and* $\gamma<1$. *In this case the inequality in Remark 1 becomes the following discrete Hardy and Littlewood inequality*

$$\sum_{n=1}^{\infty}\frac{1}{(n+1)^\gamma}\left(\sum_{k=n+1}^{\infty}g(k)\right)^\lambda \le \left(\frac{\lambda}{1-\gamma}\right)^\lambda \sum_{n=1}^{\infty}\frac{1}{(n+1)^{\gamma-\lambda}}g^\lambda(n). \tag{2.16}$$

For the rest of the paper, we will assume that there exists a constant $K>0$, with

$$\frac{s}{\sigma(s)} \ge \frac{1}{K}, \qquad \text{for } s\ge a. \tag{2.17}$$

Theorem 2.2. *Let* $\mathbb{T}$ *be a time scale with* $a\in(0,\infty)_{\mathbb{T}}$ *and* $p, q>0$ *such that* $p/q>1$ *and* $\gamma>1$. *Furthermore assume that* g *is a nonnegative and such that the delta integral* $\int_a^\infty t^{\frac{p}{q}-\gamma}g^{p/q}(t)\Delta t$ *exists. Let*

$$\Lambda(t):=\int_a^t g(s)\Delta s, \qquad \textit{for any} \quad t\in[a,\infty)_{\mathbb{T}}. \tag{2.18}$$

Then

$$\int_a^\infty \frac{1}{t^\gamma}(\Lambda^\sigma(t))^{p/q}\Delta t \le \left(\frac{pK^\gamma}{q\,(\gamma-1)}\right)^{p/q}\int_a^\infty \frac{1}{t^{\gamma-\frac{p}{q}}}g^{p/q}(t)\Delta t. \tag{2.19}$$

Proof. Integrating the left hand side of (2.19) using the parts formula (2.4) with $u^\Delta(t)=\frac{1}{t^\gamma}$ and $v^\sigma(t)=(\Lambda^\sigma(t))^{p/q}$, we have

$$\int_a^\infty \frac{(\Lambda^\sigma(t))^{p/q}}{t^\gamma}\Delta t = \left[u(t)\Lambda^{p/q}(t)\right]\Big|_a^\infty + \int_a^\infty(-u(t))\left(\Lambda^{p/q}(t)\right)^\Delta \Delta t, \tag{2.20}$$

where

$$u(t)=\int_t^\infty \left(\frac{-1}{s^\gamma}\right)\Delta s. \tag{2.21}$$

Using the chain rule (2.3), we have that

$$\begin{aligned}\left(\frac{-1}{s^{\gamma-1}}\right)^\Delta &= (\gamma-1)\int_0^1\frac{1}{[h\sigma(s)+(1-h)s]^\gamma}dh\\ &\ge (\gamma-1)\int_0^1\frac{1}{[h\sigma(s)+(1-h)\sigma(s)]^\gamma}dh\\ &= \int_0^1\left(\frac{\gamma-1}{\sigma^\gamma(s)}\right)dh=\frac{(\gamma-1)s^\gamma}{\sigma^\gamma(s)s^\gamma}.\end{aligned} \tag{2.22}$$

From (2.17) and (2.22), we see that

$$\left(\frac{-1}{s^{\gamma-1}}\right)^{\Delta} \geq \frac{(\gamma-1)}{K^{\gamma}\, s^{\gamma}}.$$

Then

$$\begin{aligned}\int_t^{\infty} \frac{-1}{s^{\gamma}}\Delta s &\geq \frac{-K^{\gamma}}{(\gamma-1)}\int_t^{\infty}\left(\frac{-1}{s^{\gamma-1}}\right)^{\Delta}\Delta s = \frac{K^{\gamma}}{(\gamma-1)}\left(\frac{1}{s^{\gamma-1}}\right)\Big|_t^{\infty} \\ &= \frac{-K^{\gamma}}{(\gamma-1)}\left(\frac{1}{t^{\gamma-1}}\right). \end{aligned} \tag{2.23}$$

Hence

$$-u(t) = -\int_t^{\infty}\left(\frac{-1}{s^{\gamma}}\right)\Delta s \leq \frac{K^{\gamma}}{\gamma-1}\left(\frac{1}{t^{\gamma-1}}\right). \tag{2.24}$$

From (2.18), (2.20), (2.21) and (2.24), we have (note that $u(\infty)=0$ and $\Lambda(a)=0$) that

$$\int_a^{\infty}\frac{1}{t^{\gamma}}\left(\Lambda^{\sigma}(t)\right)^{p/q}\Delta t \leq \frac{pK^{\gamma}}{q(\gamma-1)}\int_a^{\infty}\frac{g(t)}{t^{\gamma-1}}\left(\Lambda^{p/q}(t)\right)^{\Delta}\Delta t. \tag{2.25}$$

Applying the chain rule ([2, Theorem 1.87])

$$f^{\Delta}(g(t)) = f^{'}(g(c))g^{\Delta}(t), \text{ where } c\in[t,\sigma(t)],$$

we see that there exists $c \in [t,\sigma(t)]$ such that

$$\left(\Lambda^{p/q}(t)\right)^{\Delta} = (\frac{p}{q})\Lambda^{\frac{p}{q}-1}(c)\Lambda^{\Delta}(t). \tag{2.26}$$

Since $\Lambda^{\Delta}(t) \geq 0$ and $\sigma(t) \geq c$, we have

$$\left(\Lambda^{p/q}(t)\right)^{\Delta} \leq (\frac{p}{q})(\Lambda^{\sigma}(t))^{\frac{p}{q}-1}g(t). \tag{2.27}$$

Substituting (2.27) into (2.25), we have

$$\int_a^{\infty}\frac{(\Lambda^{\sigma}(t))^{p/q}}{t^{\gamma}}\Delta t \leq \frac{pK^{\gamma}}{q\,(\gamma-1)}\int_a^{\infty}\frac{(\Lambda^{\sigma}(t))^{\frac{p}{q}-1}}{t^{\gamma-1}}g(t)\Delta t.$$

This implies that

$$\int_a^{\infty}\frac{(\Lambda^{\sigma}(t))^{p/q}}{t^{\gamma}}\Delta t \leq \frac{pK^{\gamma}}{q\,(\gamma-1)}\int_a^{\infty}\left[\frac{t^{\gamma(p-q)/p}}{t^{\gamma-1}}g(t)\right]\left[t^{-\gamma\frac{(p-q)}{p}}(\Lambda^{\sigma}(t))^{\frac{p}{q}-1}\right]\Delta t. \tag{2.28}$$

Applying the Hölder inequality (2.5) on the term

$$\int_a^{\infty}\left[\frac{t^{\gamma(p-q)/p}}{t^{\gamma-1}}f(t)\right]t^{-\gamma\frac{(p-q)}{p}}(\Lambda^{\sigma}(t))^{\frac{p-q}{q}}\Delta t,$$

with indices p/q and $p/(p-q)$, we see that

$$\int_a^\infty \left[\frac{(t^\gamma)^{(p-q)/p}}{t^{\gamma-1}} f(t)\right] (t^\gamma)^{-\frac{(p-q)}{p}} (\Lambda^\sigma(t))^{\frac{p-q}{q}} \Delta t$$

$$\leq \left[\int_a^\infty \left[\frac{(t^\gamma)^{(p-q)/p}}{t^{\gamma-1}} g(t)\right]^{p/q} \Delta t\right]^{\frac{q}{p}} \left[\int_a^\infty \frac{(\Lambda^\sigma(t))^{p/q}}{t^\gamma} \Delta t\right]^{\frac{p-q}{p}}. \tag{2.29}$$

Substituting (2.29) into (2.28), we have

$$\int_a^\infty \frac{1}{t^\gamma} (\Lambda^\sigma(t))^{p/q} \Delta t \leq \frac{pK^\gamma}{q(\gamma-1)} \left[\int_a^\infty \left[\frac{(t^\gamma)^{(p-q)/p}}{t^{\gamma-1}} g(t)\right]^{p/q} \Delta t\right]^{\frac{q}{p}}$$

$$\times \left[\int_a^\infty \frac{(\Lambda^\sigma(t))^{p/q}}{t^\gamma} \Delta t\right]^{\frac{p-q}{p}}.$$

This gives us that

$$\int_a^\infty \frac{1}{t^\gamma} (\Lambda^\sigma(t))^{p/q} \Delta t \leq \left(\frac{pK^\gamma}{q(\gamma-1)}\right)^{p/q} \int_a^\infty t^{\frac{p}{q}-\gamma} g^{p/q}(t) \Delta t,$$

which is the desired inequality (2.19). The proof is complete.

Remark 4. *Let $G(t) = \int_a^t g(s)ds$. Then from (2.19) we have (note $\sigma(t) \geq t$)*

$$\int_a^\infty \frac{1}{\sigma^\gamma(t)} (G^\sigma(t))^{p/q} \Delta t \leq \left(\frac{pK^\gamma}{q(\gamma-1)}\right)^{p/q} \int_a^\infty \frac{1}{t^{\gamma-\frac{p}{q}}} (G^\Delta(t))^{p/q}(t) \Delta t. \tag{2.30}$$

Remark 5. *As a special case of Remark 4, when $\mathbb{T} = \mathbb{R}$ and $p/q = \lambda > 1$ and $\gamma < 1$, we have the following Hardy inequality*

$$\int_a^\infty \frac{1}{t^\gamma} \left(\int_t^\infty g(s)ds\right)^\lambda dt \leq \left(\frac{\lambda}{1-\gamma}\right)^\lambda \int_a^\infty \frac{1}{t^{\gamma-\lambda}} g^\lambda(t) dt\,.$$

Let $G(t) = \int_a^t g(s)ds$. Thus, we have (note that $G(a) = 0$)

$$\int_a^\infty \frac{1}{t^\gamma} (G(t))^\lambda dt \leq \left(\frac{\lambda}{1-\gamma}\right)^\lambda \int_a^\infty \frac{1}{t^{\gamma-\lambda}} (G^{'}(t))^\lambda dt,$$

which can be considered as a generalization of Wirtinger's inequality (see [1]*). When $\gamma = \lambda > 1$, we have the classical Hardy inequality (1.1)*

$$\int_a^\infty \left(\frac{1}{t}\int_a^t g(s)ds\right)^\lambda dt \leq \left(\frac{\lambda}{\lambda-1}\right)^\lambda \int_a^\infty g^\lambda(t) dt, \quad \lambda > 1.$$

In Theorem 2.2 if we replace the left hand side by

$$\int_a^\infty \frac{1}{\sigma^\gamma(t)} (\Lambda^\sigma(t))^{p/q} \Delta t,$$

and proceeding as in the proof of Theorem 2.2, we get the following result (note (2.17) is not assumed).

Corollary 2.1. *Let $\mathbb{T}$ be a time scale with $a \in (0,\infty)_{\mathbb{T}}$ and $p,\ q > 0$ such that $p/q > 1$ and $\gamma > 1$. Furthermore assume that g is a nonnegative and such that the delta integral $\int_a^\infty \sigma^{\frac{p}{q}-\gamma}(t)\left(\frac{\sigma(t)}{t}\right)^{p(\gamma-1)} g^{p/q}(t)\Delta t$ exists. Let $\Lambda(t)$ be as defined in Theorem 2.2. Then*

$$\int_a^\infty \frac{(\Lambda^\sigma(t))^{p/q}}{(\sigma(t))^\gamma}\Delta t \leq \left(\frac{p}{q\,(\gamma-1)}\right)^{p/q}\int_a^\infty \frac{g^{p/q}(t)}{\sigma^{\gamma-\frac{p}{q}}(t)}\left(\frac{\sigma(t)}{t}\right)^{\frac{p}{q}(\gamma-1)}\Delta t.$$

Remark 6. *Assume that $\mathbb{T}=\mathbb{N}$ in Theorem 2.2, $p/q=\lambda>1$, $a=1$ and $\gamma>1$. Furthermore assume that $\sum_{n=1}^\infty g^\lambda(n)$ is convergent. Note $\frac{n}{\sigma(n)}=\frac{n}{n+1}$ so $\frac{1}{2}\leq\frac{n}{n+1}\leq 1$. In this case, we have the following discrete Hardy and Littlewood inequality (with $K=2$) inequality*

$$\sum_{n=1}^{\infty}\frac{1}{n^\gamma}\left(\sum_{k=1}^{n} g(k)\right)^\lambda \leq \left(\frac{2^\lambda\lambda}{\gamma-1}\right)^\lambda \sum_{n=1}^{\infty}\frac{1}{n^{\gamma-\lambda}}g^\lambda(n).$$

In the following, we will apply the chain rules (2.2) and (2.3) to obtain new inequalities of Hardy and Littlewood type on time scales. These inequalities are not as general as the results in Theorems 2.1 and 2.2. However we include these results and proofs since they provide a strategy which can be used in other situations.

Theorem 2.3. *Let $\mathbb{T}$ be a time scale with $a \in (0,\infty)_{\mathbb{T}}$ and $p,\ q>0$ such that $p/q \geq 2$ and $\gamma>1$. Furthermore assume that g is a nonnegative and such that the delta integral $\int_a^\infty t^{\frac{p}{q}-\gamma}g^{p/q}(t)\Delta t$ exists. Let $\Lambda(t)$ be as defined in (2.18). Then*

$$\int_a^\infty \frac{1}{t^\gamma}(\Lambda^\sigma(t))^{p/q}\Delta t \leq \frac{2^{p/q-2}pK^\gamma}{q\,(\gamma-1)}\left[\int_a^\infty \frac{1}{t^{\gamma-\frac{p}{q}}}g^{p/q}(t)\Delta t\right]^{\frac{q}{p}}$$

$$\times\left[\int_a^\infty \frac{(\Lambda^\sigma(t))\Lambda^{p/q}}{t^\gamma}\Delta t\right]^{\frac{p-q}{p}} + \frac{2^{p/q-2}K^\gamma}{(\gamma-1)}\int_a^\infty \frac{\mu^{p/q-1}(t)}{t^{\gamma-1}}(g(t))^{p/q}\Delta t. \tag{2.31}$$

Proof. We proceed as in the proof of Theorem 2.2 to obtain using the chain rule (2.2) that

$$\int_a^\infty \frac{1}{t^\gamma}\left(\Lambda^\sigma(t)\right)^{p/q}\Delta t \leq \frac{pK^\gamma}{q(\gamma-1)}\int_a^\infty \frac{1}{t^{\gamma-1}}(\Lambda^{p/q}(t)^\Delta\Delta t. \tag{2.32}$$

Applying the chain rule (2.3), we see that $(\Lambda^{p/q}(t))^\Delta=(p/q)g(t)\int\limits_0^1\left[\Lambda+\mu hg\right]^{\frac{p}{q}-1}dh$.

Applying the inequality

$$a^\lambda+b^\lambda\leq(a+b)^\lambda\leq 2^{\lambda-1}(a^\lambda+b^\lambda),\text{ if } a,\ b\geq 0,\ \ \lambda\geq 1, \tag{2.33}$$

on the term $[\Lambda + h\mu g]^{(p/q)-1}$, we see for $p/q \geq 2$ that

$$\begin{aligned} (\Lambda^{p/q}(t)^{\Delta} &= (p/q)g(t)\int_0^1 [\Lambda + h\mu g]^{\frac{p}{q}-1}\,dh \\ &\leq (\frac{p}{q})2^{p/q-2}g(t)\left(\Lambda^{\sigma}(t)\right)^{\frac{p}{q}-1} + 2^{p/q-2}g(t)(\mu g)^{\frac{p}{q}-1}. \end{aligned} \tag{2.34}$$

Substituting (2.34) into (2.32), we have

$$\begin{aligned} \int_a^{\infty}\frac{(\Lambda^{\sigma}(t))^{p/q}}{t^{\gamma}}\Delta t &\leq \frac{p\left(2^{p/q-2}\right)K^{\gamma}}{q\,(\gamma-1)}\int_a^{\infty}\frac{\left(\Lambda^{\sigma}(t)\right)^{p/q-1}}{t^{\gamma-1}}g(t)\Delta t \\ &\quad + \frac{2^{p/q-2}K^{\gamma}}{\gamma-1}\int_a^{\infty}\frac{(\mu(t))^{p/q-1}\,(g(t))^{p/q}}{t^{\gamma-1}}\Delta t. \end{aligned}$$

This implies that

$$\begin{aligned} &\int_a^{\infty}\frac{1}{t^{\gamma}}(\Lambda^{\sigma}(t))^{p/q}\Delta t \\ \leq\;& \frac{2^{p/q-2}pK^{\gamma}}{q\,(\gamma-1)}\int_a^{\infty}\left[\frac{(t^{\gamma})^{(p-q)/p}}{t^{\gamma-1}}g(t)\right]\left[(t^{\gamma})^{-(p-q)/p}(\Lambda^{\sigma}(t))^{(p-q/q)}\right]\Delta t \\ &+ \frac{2^{p/q-2}K^{\gamma}}{\gamma-1}\int_a^{\infty}\frac{\mu^{\frac{p}{q}-1}(t)}{t^{\gamma-1}}(f(t))^{p/q}\Delta t. \end{aligned} \tag{2.35}$$

Applying the Hölder inequality (2.5) on the term

$$\int_a^{\infty}\left[\frac{(t^{\gamma})^{(p-q)/p}}{t^{\gamma-1}}f(t)\right]\left[(t^{\gamma})^{-(p-q)/p}\left(\Lambda^{\sigma}(t)\right)^{(p-q/q)}\right]\Delta t,$$

with indices p/q and $p/(p-q)$, we see that

$$\begin{aligned} \int_a^{\infty}\frac{1}{t^{\gamma}}(\Lambda^{\sigma}(t))^{p/q}\Delta t &\leq \frac{2^{p/q-2}pK^{\gamma}}{q\,(\gamma-1)}\left[\int_a^{\infty}\left[\frac{(t^{\gamma})^{(p-q)/p}}{t^{\gamma-1}}g(t)\right]^{p/q}\Delta t\right]^{\frac{q}{p}} \\ &\times\left[\int_a^{\infty}\frac{(\Lambda^{\sigma}(t))^{p/q}}{t^{\gamma}}\Delta t\right]^{\frac{p-q}{p}} \\ &+\frac{2^{p/q-2}K^{\gamma}}{\gamma-1}\int_a^{\infty}\frac{\mu^{p/q-1}(t)}{t^{\gamma-1}}(g(t))^{p/q}\Delta t, \end{aligned}$$

which is the desired inequality (2.31). The proof is complete.

As a special case of Theorem 2.3 when $\mathbb{T} = \mathbb{R}$, we have the following Hardy-type inequality.

Corollary 2.2. *Assume that* $g \geq 0$, *integrable and* $t^{\lambda-\gamma}g^{\lambda}$ *is integrable and convergent over* $(0,\infty)$. *If* $\lambda \geq 2$ *and* $\gamma > 1$, *then*

$$\int_a^{\infty} \frac{1}{t^{\gamma}} \left(\int_a^t g(s)ds \right)^{\lambda} dt \leq \left(\frac{2^{\lambda-2}\lambda}{\gamma-1} \right)^{\lambda} \int_a^{\infty} \frac{1}{t^{\gamma-\lambda}} g^{\lambda}(t)\, dt.$$

Theorem 2.4. *Let* $\mathbb{T}$ *be a time scale with* $a \in (0,\infty)_{\mathbb{T}}$ *and* p, $q > 0$ *such that* $p/q \geq 2$ *and* $\gamma > 1$. *Furthermore assume that* g *is a nonnegative and such that the delta integral* $\int_a^{\infty} t^{\frac{p}{q}-\gamma} g^{p/q}(t)\Delta t$ *exists. Let* $\Lambda(t)$ *be as defined in (2.18). Then*

$$\int_a^{\infty} \frac{1}{t^{\gamma}} (\Lambda^{\sigma}(t))^{p/q} \Delta t \leq \left(\frac{2^{\frac{p}{q}-1} K^{\gamma}}{(\gamma-1)} \right)^{\frac{p}{q}} \int_a^{\infty} \frac{1}{t^{\gamma-\frac{p}{q}}} g^{p/q}(t)\Delta t. \tag{2.36}$$

Proof. Proceeding as in the proof of Theorem 2.2, we get that

$$\int_a^{\infty} \frac{1}{t^{\gamma}} (\Lambda^{\sigma}(t))^{p/q} \Delta t \leq \frac{K^{\gamma}}{(\gamma-1)} \int_a^{\infty} \frac{(\Lambda^{p/q}(t))^{\Delta}}{t^{\gamma-1}} \Delta t. \tag{2.37}$$

From the chain rule (2.3) and (2.33), we see that

$$\begin{aligned} (\Lambda^{p/q}(t))^{\Delta} &\leq 2^{\frac{p}{q}-2} \frac{p}{q} \int_0^1 \left[(h\Lambda^{\sigma})^{\frac{p}{q}-1} + (1-h)^{\frac{p}{q}-1} \Lambda^{\frac{p}{q}-1} \right] dh \Lambda^{\Delta}(t) \\ &= 2^{\frac{p}{q}-2} \left[(\Lambda^{\sigma})^{\frac{p}{q}-1} + \Lambda^{\frac{p}{q}-1} \right] \Lambda^{\Delta}(t) \\ &\leq 2^{\frac{p}{q}-2} \left[(\Lambda^{\sigma})^{\frac{p}{q}-1} + (\Lambda^{\sigma})^{\frac{p}{q}-1} \right] g(t). \end{aligned} \tag{2.38}$$

This implies that

$$(\Lambda^{p/q}(t))^{\Delta} \leq 2^{\frac{p}{q}-1} (\Lambda^{\sigma}(t))^{\frac{p}{q}-1} g(t). \tag{2.39}$$

Hence

$$\int_a^{\infty} \frac{1}{t^{\gamma}} (\Lambda^{\sigma}(t))^{p/q} \Delta t \leq \frac{2^{\frac{p}{q}-1} K^{\gamma}}{(\gamma-1)} \int_a^{\infty} \frac{1}{t^{\gamma-1}} (\Lambda^{\sigma}(t))^{\frac{p}{q}-1} g(t)\Delta t, \tag{2.40}$$

and thus

(2.41)
$$\int_a^{\infty} \frac{1}{t^{\gamma}} (\Lambda^{\sigma}(t))^{p/q} \Delta t \leq \frac{2^{p/q-1} K^{\gamma}}{(\gamma-1)} \int_a^{\infty} \left[\frac{t^{\gamma\frac{(p-q)}{p}}}{t^{\gamma-1}} g(t) \right] \left[(t^{\gamma})^{-\frac{(p-q)}{p}} (\Lambda^{\sigma}(t))^{\frac{p-q}{q}} \right] \Delta t.$$

Applying the Hölder inequality (2.5) on the right hand side with indices p/q and $p/(p-q)$, we see that

$$\begin{aligned} &\int_a^{\infty} \frac{1}{t^{\gamma}} (\Lambda^{\sigma}(t))^{p/q} \Delta t \\ \leq \;& \frac{2^{p/q-1} K^{\gamma}}{\gamma-1} \left[\int_a^{\infty} \left[\frac{t^{\gamma(p-q)/p}}{t^{\gamma-1}} g(t) \right]^{p/q} \Delta t \right]^{\frac{q}{p}} \left[\int_a^{\infty} \frac{(\Lambda^{\sigma})^{p/q)}(t)}{t^{\gamma}} \Delta t \right]^{\frac{p-q}{p}}. \end{aligned}$$

This implies that

$$\left[\int_a^\infty \frac{1}{t^\gamma}(\Lambda^\sigma(t))^{p/q}\Delta t\right]^{1-\frac{p-q}{p}} \leq \frac{2^{p/q-1}K^\gamma}{\gamma-1}\left[\int_a^\infty \left[\frac{t^{\gamma(p-q)/p}}{t^{\gamma-1}}g(t)\right]^{p/q}\Delta t\right]^{\frac{q}{p}}.$$

Then

$$\int_a^\infty \frac{1}{t^\gamma}(\Lambda^\sigma(t))^{p/q}\Delta t \leq \left(\frac{2^{\frac{p}{q}-1}K^\gamma}{\gamma-1}\right)^{\frac{p}{q}}\int_a^\infty \frac{1}{t^{\gamma-\frac{p}{q}}}g^{p/q}(t)\Delta t,$$

which is the desired inequality (2.36). The proof is complete.

In the following, we consider the case when $p/q \leq 2$ and prove new inequalities of Hardy and Littlewood type on time scales. To prove these results, we need the inequality

$$2^{r-1}\left(a^r+b^r\right) \leq (a+b)^r \leq \left(a^r+b^r\right), \text{ where } a,\, b\geq 0 \text{ and } 0\leq r\leq 1. \tag{2.42}$$

Applying this inequality (2.42) when $r = p/q - 1 < 1$, we see that

$$\frac{p}{q}\int_0^1 \left[\Lambda + h\mu\Lambda^\Delta\right]^{(p/q)-1} dh \leq (p/q)\Lambda^{p/q-1} + (\mu g)^{p/q-1}, \;\; p/q \leq 2.$$

Proceeding as in the proof of Theorem 2.3, we have the following result.

Theorem 2.5. *Let $\mathbb{T}$ be a time scale with $a \in (0,\infty)_\mathbb{T}$ and $p,\, q > 0$ such that $p/q \leq 2$ and $\gamma > 1$. Furthermore assume that g is a nonnegative and such that the delta integral $\int_a^\infty t^{\frac{p}{q}-\gamma}g^{p/q}(t)\Delta t$ exists. Assume that g is nonnegative function and let $\Lambda(t)$ be as defined in (2.18). Then*

$$\int_a^\infty \frac{(\Lambda^\sigma(t))^{p/q}}{t^\gamma}\Delta t - K^\gamma\int_a^\infty \frac{\mu^{p/q-1}(t)}{t^{\gamma-1}}(g(t))^{p/q}\Delta t$$
$$\leq \frac{pK^\gamma}{q(\gamma-1)}\left[\int_a^\infty \frac{1}{t^{\gamma-\frac{p}{q}}}g^{p/q}(t)\Delta t\right]^{\frac{q}{p}}\left[\int_a^\infty \frac{(\Lambda^\sigma(t))^{p/q}}{t^\gamma}\Delta t\right]^{\frac{p-q}{p}}.$$

As in the proof of Theorem 2.4, we have the following theorem.

Theorem 2.6. *Let $\mathbb{T}$ be a time scale with $a \in (0,\infty)_\mathbb{T}$ and $p,\, q > 0$ such that $p/q \leq 2$ and $\gamma > 1$. Assume that g is nonnegative function and let $\Lambda(t)$ be as defined in (2.18). Then*

$$\int_a^\infty \frac{1}{t^\gamma}(\Lambda^\sigma(t))^{p/q}\Delta t \leq \left(\frac{2K^\gamma}{\gamma-1}\right)^{\frac{p}{q}}\int_a^\infty \frac{1}{t^{\gamma-\frac{p}{q}}}g^{p/q}(t)\Delta t.$$

In the following we prove a new class of inequalities when $\gamma < 1$ by using the function $\Omega(t)$ defined in (2.6). Applying the inequality (2.42) on the term

$[h\Omega^{\sigma}+(1-h)\Omega)]^{\frac{p}{q}-1}$, when $p/q-1\leq 1$, we see that

$$\begin{aligned} &\frac{p}{q}\int_0^1 [h\Omega^{\sigma}+(1-h)\Omega)]^{\frac{p}{q}-1}\,dh \\ \leq\ &\frac{p}{q}\int_0^1 \left[h^{\frac{p}{q}-1}(\Omega^{\sigma})^{\frac{p}{q}-1}+(1-h)^{\frac{p}{q}-1}\Omega^{\frac{p}{q}-1}\right]dh \\ =\ &\left[(\Omega^{\sigma})^{\frac{p}{q}-1}+\Omega^{\frac{p}{q}-1}\right]\leq 2\Omega^{\frac{p}{q}-1}(t). \end{aligned} \tag{2.43}$$

This implies that

$$\int_a^{\infty}\frac{1}{\sigma^{\gamma}(t)}(\Omega(t))^{p/q}\Delta t\leq\frac{2}{1-\gamma}\int_a^{\infty}\left[\frac{(\sigma^{\gamma}(t))^{(p-q)/p}}{\sigma^{\gamma-1}(t)}g(t)\right]\left[(\sigma^{\gamma}(t))^{-\frac{(p-q)}{p}}(\Omega(t))^{\frac{p}{q}-1}\right]\Delta t.$$

Proceeding as in the proof of Theorem 2.1, we have the following theorem (note (2.17) is not assumed).

Theorem 2.7. *Let $\mathbb{T}$ be a time scale with $a\in(0,\infty)_{\mathbb{T}}$* and p, $q>0$ *such that $p/q\leq 2$ and $\gamma<1$. Furthermore assume that g is a nonnegative and such that the delta integral $\int_a^{\infty}(\sigma(t))^{\frac{p}{q}-\gamma}g^{p/q}(t)\Delta t$ exists. Then*

$$\int_a^{\infty}\frac{1}{\sigma^{\gamma}(t)}\left(\int_t^{\infty}g(s)\Delta s\right)^{p/q}\Delta t\leq\left(\frac{2}{1-\gamma}\right)^{p/q}\int_a^{\infty}\frac{1}{(\sigma(t))^{\gamma-\frac{p}{q}}}\,(g(t))^{p/q}\,\Delta t.$$

Remark 7. *It is worth mentioning here that the above results are valid also if the upper bound of integration ∞ is replaced by any finite number b i.e. if we replace the time scale interval $[a,\infty)_{\mathbb{T}}$ with $[a,b]_{\mathbb{T}}$.*

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