ON STARLIKE FUNCTIONS CONNECTED WITH k-FIBONACCI NUMBERS

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ABSTRACT. We present a new subclass $\mathcal{SL}^k$ of starlike functions which is related to a shell-like curve. The coefficients of such functions are connected with k-Fibonacci numbers $F_{k,n}$ defined recurrently by $F_{k,0} = 0, F_{k,1} = 1$ and $F_{k,n} = kF_{k,n} + F_{k,n-1}$ for $n \geq 1$, where k is a given positive real number. We investigate some basic properties for the class $\mathcal{SL}^k$.

1. Introduction

Let $\mathbb{D} = \{z : |z| < 1\}$ denote the unit disc. Let $\mathcal{A}$ be the class of all analytic functions f in the open unit disc $\mathbb{D}$ with normalization $f(0) = 0$, $f'(0) = 1$ and let $\mathcal{S}$ denote the subset of $\mathcal{A}$ which is composed of univalent functions. We say that f is subordinate to F in $\mathbb{D}$, written as $f \prec F$, if and only if $f(z) = F(\omega(z))$ for some analytic function ω, $\omega(0) = 0$, $|\omega(z)| < 1$, $z \in \mathbb{D}$. The idea of subordination was used for defining many classes of functions studied in geometric function theory. Let us recall

$$\mathcal{S}^*[\varphi] := \left\{ f \in \mathcal{A} : \ \frac{zf'(z)}{f(z)} \prec \varphi(z), \ z \in \mathbb{D} \right\}, \tag{1.1}$$

$$\mathcal{K}[\varphi] := \left\{ f \in \mathcal{A} : \ \left[1 + \frac{zf''(z)}{f'(z)} \right] \prec \varphi(z), \ z \in \mathbb{D} \right\}, \tag{1.2}$$

where φ is analytic in $\mathbb{D}$ with $\varphi(0) = 1$. For $\varphi(z) = (1+z)/(1-z)$ we obtain the well known classes $\mathcal{S}^*$, $\mathcal{K}$ of starlike and convex functions, respectively. A lot of classes of functions have been defined by exchanging the function φ in (1.1) or in (1.2) by other functions giving very often an interesting image of the unit circle. If $\varphi(z) = (1+(1-2\alpha)z)/(1-z)$, $\alpha < 1$, then $\varphi(\mathbb{D})$ is the half plane $\mathfrak{Re}(w) > \alpha$, and the sets (1.1), (1.2) become the classes $\mathcal{S}^*(\alpha)$ of starlike or $\mathcal{K}(\alpha)$ of convex functions of order α, respectively, introduced in [14]. If $\varphi(z) = (1+Az)(1+Bz)$, $-1 < B < A \leq 1$, then $\varphi(\mathbb{D})$ is a disc, and the classes (1.1), (1.2) become the classes considered in [6, 7]. The class of strongly starlike functions of order β, $0 < \beta \leq 1$, see [20], we obtain from (1.1) with $\varphi(z) = ((1+z)/(1-z))^\beta$. Then $\varphi(\mathbb{D})$ is an angle. If

$$\varphi(z) = 1 + \frac{2}{\pi^2}\left(\log \frac{1+\sqrt{z}}{1-\sqrt{z}}\right)^2,$$

then $\varphi(\mathbb{D})$ is a parabolic region, and the set (1.2) is a class of so called uniformly convex function introduced [5, 11, 15]. Close related classes, connected with a hyperbola or with an ellipse were considered in [8, 9, 10]. If $\varphi(z) = \sqrt{1+z}$, where the branch of the square root is chosen in

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order that $\sqrt{1} = 1$, then $\varphi(\mathbb{D})$ is interior of the right loop of the Lemniscate of Bernoulli and the class (1.1) becomes a class considered in [17, 19]. The function

$$\varphi(z) = \left(\frac{1+z}{1+(1-b)/bz}\right)^{1/\alpha}$$

in (1.1) forms a class considered in [13]. In the above and in other not cited here cases the function φ is a convex univalent function. In [12] Ma and Minda proved some general results for classes (1.1) and (1.2), where φ is assumed to be univalent, $\varphi(\mathbb{D})$ is assumed to be symmetric with respect to real axis and starlike with respect to $\varphi(0) = 1$. The problems in the classes defined by (1.1) and by (1.2) become much more difficult if the function φ is not univalent. In [18] the second author defined the class $\mathcal{SL}$ of shell-like functions as the set of functions $f \in \mathcal{A}$ satisfying the condition that

$$\frac{zf'(z)}{f(z)} \prec \widetilde{p}(z), z \in \mathbb{D},$$

where

$$\widetilde{p}(z) = \frac{1+\tau^2 z^2}{1-\tau z-\tau^2 z^2}, \ \tau = \frac{1-\sqrt{5}}{2} \approx -0.618, \ z \in \mathbb{D}.$$

The class $\mathcal{SL}$ is a subclass of the class of starlike functions $\mathcal{S}^{\star}$. The name attributed to the class $\mathcal{SL}$ is motivated by the shape of the curve

$$\mathcal{C} = \left\{\widetilde{p}(e^{it}) : t \in [0, 2\pi) \setminus \{\pi\}\right\},$$

which is a shell-like curve. Furthermore the coefficients of shell-like functions are connected with well-known Fibonacci numbers F_n defined as

$$F_0 = 0, F_1 = 1 \text{ and } F_{n+1} = F_n + F_{n-1} \text{ for } n \geq 1. \tag{1.3}$$

More recently, a lot of new studies have been done about several classes of functions related to shell-like curves connected with Fibonacci numbers (see [1], [2] and [16]).

Motivated by the above studies, we define new subclasses $\mathcal{SL}^k$ of the class $\mathcal{S}^{\star}$ where k is any positive real number. The coefficients of such functions are connected with k-Fibonacci numbers. For $k = 1$, we obtain the class $\mathcal{SL}$ of shell-like functions.

For any positive real number k, the k-Fibonacci numbers $F_{k,n}$ are defined recurrently by

$$F_{k,0} = 0, F_{k,1} = 1 \text{ and } F_{k+1,n} = kF_{k,n} + F_{k,n-1} \text{ for } n \geq 1. \tag{1.4}$$

The Fibonacci numbers defined in (1.3) we obtain from (1.4) for $k = 1$. It is known that the nth k-Fibonacci number is given by

$$F_{k,n} = \frac{(k-\tau_k)^n - \tau_k^n}{\sqrt{k^2+4}}, \tag{1.5}$$

where $\tau_k = \frac{k-\sqrt{k^2+4}}{2}$ (see [3] and [4] for more details about k-Fibonacci numbers).

2. The Class $\mathcal{SL}^k$

Definition 2.1. *Let k be any positive real number. The function $f \in \mathcal{S}$ belongs to the class $\mathcal{SL}^k$ if satisfies the condition that*

$$\frac{zf'(z)}{f(z)} \prec \widetilde{p}_k(z),\ z \in \mathbb{D},$$

where

$$\widetilde{p}_k(z) = \frac{1+\tau_k^2 z^2}{1-k\tau_k z-\tau_k^2 z^2} = \frac{1+\tau_k^2 z^2}{1-(\tau_k^2-1)z-\tau_k^2 z^2},\ \tau_k = \frac{k-\sqrt{k^2+4}}{2},\ z \in \mathbb{D}. \tag{2.1}$$

Theorem 2.1. *The image of the unit circle of the function $\widetilde{p}_k(z)$ defined in* (2.1) *is the curve $\mathcal{C}_k$ with equation*

$$x = \frac{k\sqrt{k^2+4}}{2\left[k^2+2-2\cos\theta\right]},\ y = \frac{(4\cos\theta-k^2)\sin\theta}{2\left[k^2+2-2\cos\theta\right]\left[1+\cos\theta\right]},\ \theta \in [0,2\pi)\backslash\{\pi\}. \tag{2.2}$$

Proof. The proof follows by some straightforward calculations. □

Recall that the curve which is called conchoid of Sluze has the following equation

$$a(x-a)\left(x^2+y^2\right)+\lambda^2 x^2 = 0, \tag{2.3}$$

where $a > 0$ and $\lambda > 0$. For $\lambda = 2a/k$, the conchoid of Sluze (2.3) becomes the curve:

$$x^3+(x-a)y^2+\left(\frac{4-k^2}{k^2}\right)ax^2 = 0. \tag{2.4}$$

For $k = 1$, this curve is the trisectrix of Maclaurin.

We can find the corresponding Cartesian equation of the curve $\mathcal{C}_k$ with equation (2.2) as

$$\left[(8+2k^2)x-k\sqrt{k^2+4}\right]y^2 = \left(\frac{\sqrt{k^2+4}}{k}-2x\right)\left(\sqrt{k^2+4}x-k\right)^2. \tag{2.5}$$

If we rewrite (2.5) in the following form

$$\left(\frac{k\sqrt{k^2+4}}{k^2+4}-x\right)^3+\frac{4-k^2}{k^2}.\frac{k\sqrt{k^2+4}}{2(k^2+4)}\left(\frac{k\sqrt{k^2+4}}{k^2+4}-x\right)^2$$
$$+\left[\left(\frac{k\sqrt{k^2+4}}{k^2+4}-x\right)-\frac{k\sqrt{k^2+4}}{2(k^2+4)}\right]y^2 = 0,$$

then the image of the unit circle under the function $\widetilde{p}_k$ is translated into a curve with equation (2.4) where

$$a = \frac{k\sqrt{k^2+4}}{2(k^2+4)} = \frac{1-\frac{2\tau_k\left(1-k\sqrt{k^2+4}\right)}{k-\sqrt{k^2+4}}}{2(k^2+4)}.$$

Therefore the curve $\mathcal{C}_k$ has a shell-like shape and symmetric with respect to the real axis, see Figure 1.

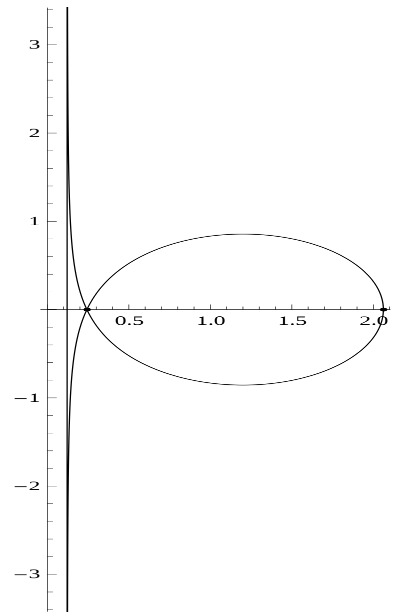

FIGURE 1. The curve $\mathcal{C}_k$ for $k=\frac{1}{2}$.

For $k<2$, note that we have

$$\widetilde{p}\left(e^{\pm i\arccos\left(\frac{k^2}{4}\right)}\right)=\frac{k\sqrt{k^2+4}}{k^2+4},$$

and so the curve $\mathcal{C}_k$ intersects itself on the real axis at the point $\frac{k\sqrt{k^2+4}}{k^2+4}$. Thus $\mathcal{C}_k$ has a loop intersecting the real axis at the points $e=\frac{k\sqrt{k^2+4}}{k^2+4}$ and $f=\frac{\sqrt{k^2+4}}{2k}$. For $k\geq 2$, the curve $\mathcal{C}_k$ has no loops and it is like a conchoid.

Corollary 2.1. *For each $k>0$, $\mathcal{SL}^k\subset\mathcal{S}^*(\alpha_k)$ where $\alpha_k=\frac{k\sqrt{k^2+4}}{2(k^2+4)}=\frac{k(k-2\tau_k)}{2(k^2+4)}$, that is, $f\in\mathcal{SL}^k$ is starlike of order α_k.*

The function $\widetilde{p}_k$ defined in (2.1) is not univalent in $\mathbb{D}$. For example, we have $\widetilde{p}_k(0)=\widetilde{p}(\frac{-k}{2\tau_k})=1$ and $\widetilde{p}(1)=\widetilde{p}(\tau_k^4)=\frac{\sqrt{k^2+4}}{2k}$. We can give the following theorem.

Theorem 2.2. *For each $k>0$ the function $\widetilde{p}_k$ is univalent in the disc $\mathbb{D}_{r_k}=\{z:|z|<r_k\}$, where*

$$r_k=\frac{2-\sqrt{k^2+4}}{k\tau_k}=\frac{k^2-2k+4+(k-2)\sqrt{k^2+4}}{2k} \tag{2.6}$$

and it is not univalent in the disc $\mathbb{D}_{r_k}$ for each $r\geq r_k$.

Proof. Suppose that $\widetilde{p}_k(z)=\widetilde{p}_k(w)$ for some $z,w\in\mathbb{D}$. After some calculations we have

$$\tau_k\,(z-w)\left(w-\frac{2\tau_k z+k}{k\tau_k^2 z-2\tau_k}\right)=0. \tag{2.7}$$

We see that the function

$$g_k(z)=\frac{2\tau_k z+k}{k\tau_k^2 z-2\tau_k}$$

maps a circle $|z| = r < 2/(k\tau_k)$ onto a circle centered at $m = -\frac{2k(1+2\tau_k^2 r^2)}{\tau_k\left(4-k^2\tau_k^2 r^2\right)}$ and of radius $\rho = \frac{r(k^2+4)}{4-k^2\tau_k^2 r^2}$ with the diameter from $g_k(-r)$ to $g_k(r)$. Therefore g_k maps the circle $|z| = r_k$ onto a circle with the diameter from the point $g_k(r_k) = r_k$ to the point $g_k(-r_k)$. We have $g_k(-r_k) > g_k(r_k) = r_k$ for all k because the function $g_k(x)$, $x \in \mathbb{R}$ has negative derivative for all real x. Therefore, if $|w| \le r_k$ and $|z| \le r_k$, then the third factor in (2.7) is equal to 0 for $w = z - r_k$ only. Consequently, we see that (2.7) is not satisfied when $|w| < r_k$ and $|z| < r_k$, which proves that in the disc (2.6) the function $\widetilde{p}_k(z)$ is univalent.

On the other hand, the derivative of the function $\widetilde{p}_k(z)$ is

$$\widetilde{p}_k'(z) = \frac{(z - r_k)\left(z - \frac{2+\sqrt{k^2+4}}{k\tau_k}\right)}{\left(1 - k\tau_k z - \tau_k^2 z^2\right)^2}.$$

The function $\widetilde{p}_k'(z)$ vanishes at the point $z = r_k$ and hence we see that the function $\widetilde{p}_k(z)$ fails to be univalent for $|z| \ge r_k$. □

Theorem 2.3. *Let $(F_{k,n})$ be the sequence of k-Fibonacci numbers defined in* (1.4)*. If*

$$\widetilde{p}_k(z) = \frac{1 + \tau_k^2 z^2}{1 - k\tau_k z - \tau_k^2 z^2} = 1 + \sum_{n=1}^{\infty} p_n z^n,$$

then we have

$$p_n = (F_{k,n-1} + F_{k,n+1})\tau_k^n,\ n = 1, 2, 3, \dots \ . \tag{2.8}$$

Proof. Let us denote $u = \tau_k z$, $|u| < |\tau_k|$. Using the equations $\tau_k (k - \tau_k) = -1$ and $2\tau_k - k = -\sqrt{k^2+4}$, we have

$$\begin{aligned}
\widetilde{p}_k(z) &= \frac{1 + \tau_k^2 z^2}{1 - k\tau_k z - \tau_k^2 z^2} = \frac{1 + u^2}{1 - ku - u^2} = \left(u + \frac{1}{u}\right)\frac{u}{1 - ku - u^2} \\
&= \left(u + \frac{1}{u}\right)\frac{1}{\sqrt{k^2+4}}\left(\frac{1}{1 + \frac{u}{\tau_k}} - \frac{1}{1 + \frac{u}{k-\tau_k}}\right) \\
&= \left(u + \frac{1}{u}\right)\frac{1}{\sqrt{k^2+4}}\sum_{n=1}^{\infty}(-1)^n\left[\left(\frac{u}{\tau_k}\right)^n - \left(\frac{u}{k-\tau_k}\right)^n\right] \\
&= \left(u + \frac{1}{u}\right)\sum_{n=1}^{\infty}\frac{(k-\tau_k)^n - \tau_k^n}{\sqrt{k^2+4}}u^n.
\end{aligned}$$

Now by the equation (1.5), we find

$$\begin{aligned}
\widetilde{p}_k(z) &= \left(u + \frac{1}{u}\right)\sum_{n=1}^{\infty} F_{k,n} u^n \\
&= 1 + \sum_{n=1}^{\infty}\left(F_{k,n-1} + F_{k,n+1}\right)u^n \\
&= 1 + \sum_{n=1}^{\infty}\left(F_{k,n-1} + F_{k,n+1}\right)\tau_k^n z^n,
\end{aligned}$$

and hence we obtain (2.8). □

Theorem 2.4. *A function $f \in \mathcal{S}$ belongs to the class $\mathcal{SL}^k$ if and only if there exists a function q, $q \prec \widetilde{p}_k(z) = \frac{1+\tau_k^2 z^2}{1-k\tau_k z-\tau_k^2 z^2}$ such that*

$$f(z) = z \exp \int_0^z \frac{q(\zeta)-1}{\zeta} d\zeta,\ z \in \mathbb{D}. \tag{2.9}$$

Proof. Let $f \in \mathcal{SL}^k$. Then by definition

$$\frac{zf'(z)}{f(z)} = \widetilde{p}_k(\omega(z)),\ \ |\omega(z)| < 1,\ z \in \mathbb{D}. \tag{2.10}$$

If we take $q(z) = \widetilde{p}(\omega(z))$, we see that the equation (2.10) is equivalent to (2.9). □

For $\widetilde{p}_k(z) = \frac{1+\tau_k^2 z^2}{1-k\tau_k z-\tau_k^2 z^2}$ the formula (2.9) gives $f_0(z) = \frac{z}{1-k\tau_k z-\tau_k^2 z^2}$. Hence the function f_0 belongs to the class $\mathcal{SL}^k$ and it is extremal function for several problems in this class.

Theorem 2.5. *If $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ belongs to the class $\mathcal{SL}^k$, then we have*

$$|a_n| \le |\tau_k|^{n-1} F_{k,n}, \tag{2.11}$$

where $(F_{k,n})$ is the sequence of k-Fibonacci numbers and $\tau_k = \frac{k-\sqrt{k^2+4}}{2}$. Equality holds in (2.11) for the function $f_0(z) = \frac{z}{1-k\tau_k z-\tau_k^2 z^2}$.

Proof. Let $f \in \mathcal{SL}^k$, $f(z) = \sum_{m=0}^{\infty} a_m z^m$, $a_0 = 0$, $a_1 = 1$. By the definition of the class $\mathcal{SL}^k$, there exists a function ω, $|\omega(z)| < 1$ for $z \in \mathbb{D}$ such that

$$\frac{zf'(z)}{f(z)} = \frac{1+\tau_k^2\omega^2(z)}{1-k\tau_k\omega(z)-\tau_k^2\omega^2(z)}.$$

We get

$$zf'(z) - f(z) = k\tau_k\omega(z)zf'(z) + \tau_k^2\omega^2(z)\left[zf'(z)+f(z)\right],$$

$$\sum_{m=1}^{\infty}(m-1)a_m z^m = k\tau_k\omega(z)\sum_{m=1}^{\infty} m a_m z^m + \tau_k^2\omega^2(z)\sum_{m=1}^{\infty}(m+1)a_m z^m$$

and so

$$\sum_{m=1}^{n}(m-1)a_m z^m + \sum_{m=n+1}^{\infty} c_m z^m = k\tau_k\omega(z)\sum_{m=1}^{n-1} m a_m z^m + \tau_k^2\omega^2(z)\sum_{m=1}^{n-2}(m+1)a_m z^m.$$

For $n \geq 2$, we find

$$
\begin{aligned}
& \left|\sum_{m=1}^{n}(m-1)a_m z^m + \sum_{m=n+1}^{\infty} c_m z^m\right|^2 \\
= & \left|k\tau_k\omega(z)\sum_{m=1}^{n-1} m a_m z^m + \tau_k^2\omega^2(z)\sum_{m=1}^{n-2}(m+1)a_m z^m\right|^2 \\
\leq & \left|k\tau_k\sum_{m=1}^{n-1} m a_m z^m + \tau_k^2\omega(z)\sum_{m=1}^{n-1} m a_{m-1} z^{m-1}\right|^2 \\
\leq & \sum_{m=1}^{n-1}\left|k\tau_k m a_m z^m + \tau_k^2\omega(z) m a_{m-1} z^{m-1}\right|^2 \\
\leq & \sum_{m=1}^{n-1}\left(\left|k\tau_k m a_m z^m\right|^2 + \left|\tau_k^2 m a_{m-1} z^{m-1}\right|^2 + 2\left|k\tau_k^3 m^2 a_m a_{m-1} z^{2m-1}\right|\right).
\end{aligned}
$$

Integrating the both sides of this inequality around $z = re^{im\varphi}$ and taking limit $r \to 1^-$ we obtain

$$
\begin{aligned}
& \sum_{m=1}^{n}(m-1)^2\left|a_m\right|^2 + \sum_{m=n+1}^{\infty}\left|c_m\right|^2 \\
\leq\ & k^2\tau_k^2\sum_{m=1}^{n-1} m^2\left|a_m\right|^2 + \tau_k^4\sum_{m=1}^{n-1} m^2\left|a_{m-1}\right|^2 + 2k\left|\tau_k\right|^3\sum_{m=1}^{n-1} m^2\left|a_m\right|\left|a_{m-1}\right|
\end{aligned}
$$

and hence we find

$$
\begin{aligned}
& (n-1)^2\left|a_n\right|^2 \\
\leq\ & \sum_{m=1}^{n-1}\left\{k^2\tau_k^2 m^2-(m-1)^2\right\}\left|a_m\right|^2 + \sum_{m=1}^{n-1}\tau_k^4 m^2\left|a_{m-1}\right|^2 + \sum_{m=1}^{n-1} 2k|\tau_k|^3 m^2\left|a_m\right|\left|a_{m-1}\right| \quad (2.12)
\end{aligned}
$$

The inequality (2.11) holds for $n = 1$. Assume that the estimation (2.11) holds for all natural numbers less or equal to n. Then from (2.12) and from (2.11) we have

$$\begin{aligned}
& n^2 \left|a_{n+1}\right|^2 \\
\leq\ & \sum_{m=1}^{n} \left\{k^2\tau_k^2 m^2 - (m-1)^2\right\} \left|a_m\right|^2 + \tau_k^4 \sum_{m=1}^{n} m^2 \left|a_{m-1}\right|^2 + 2k \left|\tau_k\right|^3 \sum_{m=1}^{n} m^2 \left|a_m\right| \left|a_{m-1}\right| \\
\leq\ & \sum_{m=1}^{n} \left\{k^2\tau_k^2 m^2 - (m-1)^2\right\} \left\{\left|\tau_k\right|^{m-1} F_{k,m}\right\}^2 + \tau_k^4 \sum_{m=1}^{n} m^2 \left\{\left|\tau_k\right|^{m-2} F_{k,m-1}\right\}^2 \\
& + 2k \left|\tau_k\right|^3 \sum_{m=1}^{n} m^2 \left\{\left|\tau_k\right|^{m-1} F_{k,m}\right\} \left\{\left|\tau_k\right|^{m-2} F_{k,m-1}\right\} \\
=\ & \sum_{m=1}^{n} \left[\left\{m\tau_k^m \left(kF_{k,m} + F_{k,m-1}\right)\right\}^2 - (m-1)^2 \left\{\left|\tau_k\right|^{m-1} F_{k,m}\right\}^2\right] \\
=\ & \sum_{m=1}^{n} \left[\left\{m\tau_k^m F_{k,m+1}\right\}^2 - (m-1)^2 \left\{\left|\tau_k\right|^{m-1} F_{k,m}\right\}^2\right] \\
=\ & n^2 |\tau_k|^{2n} \left\{F_{k,n+1}\right\}^2 .
\end{aligned} \tag{2.13}$$

In this way we have proved by induction the inequality (2.11) for all $n \in \mathbb{N}$. □

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