# Some results on the bounds of signless Laplacian eigenvalues*

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**Abstract**: Let $G$ be a simple graph with $n$ vertices and $G^c$ be its complement. The matrix $Q(G) = D(G) + A(G)$ is called the signless Laplacian of $G$, where $D(G) = \mathrm{diag}(d(v_1), d(v_2), \ldots, d(v_n))$ and $A(G)$ denote the diagonal matrix of vertex degrees and the adjacency matrix of $G$, respectively. Let $q_1(G)$ be the largest eigenvalue of $Q(G)$. We first give some upper and lower bounds on $q_1(G) + q_1(G^c)$ for a graph $G$. Finally, lower and upper bounds are obtained for the clique number $\omega(G)$ and the independence number $\alpha(G)$, in terms of the eigenvalues of the signless Laplacian matrix of a graph $G$.

**Keywords**: Signless Laplacian eigenvales; Maximum and minimum degree; Bounds; Clique number; Independence number.

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## 1. Introduction

We consider only simple graphs (i.e. finite, undirected graphs without loops or multiple edges). Let $G = (V_G, E_G)$ be a simple graph on $n$ vertices and $m$ edges (so $n = |V_G|$ is its order, and $m = |E_G|$ is its size). For $v_i \in V_G$, the degree of $v_i$, written by $d(v_i)$ or $d_i$, is the number of edges incident with $v$. Let $\Delta = \max\{d_i : v_i \in V_G\}$ and $\delta = \min\{d_i : v_i \in V_G\}$. Spectral graph theory [3, 4, 10] studies properties of graphs using the spectrum of related matrices. The oldest and most studied matrix associated with $G$ appears to be adjacency matrix $A = (a_{ij})$ where $a_{ij} = 1$ if $v_i$ and $v_j$ of the graph $G$ are adjacent and 0 otherwise. Another much studied matrix is the Laplacian, defined by $L = D - A$ where $D(G) = \mathrm{diag}(d_1, d_2, \ldots, d_n)$ (see [1, 11, 17]). The matrix $Q = D + A$ is called the *signless Laplacian matrix* of $G$ (see [6]), which has recently attracted more and more researchers' attention. One reason for this is that the signless Laplacian spectrum seems to be more informative than the other commonly used graph matrices [6]. For more results on the signless Laplacian matrix one may refer to three survey papers [7, 8, 9].

For an $n\times n$ real symmetric matrix $M$, in view of Geršgorin's Theorem, its eigenvalues are nonnegative real numbers. In particular, if $M$ is equal to one of the matrices $A$, $L$ and $Q$ (associated to a graph $G$ on $n$ vertices), then the corresponding eigenvalues (or spectrum) are called the $A$-eigenvalues (or $A$-spectrum), $L$-eigenvalues (or $L$-spectrum) and $Q$- eigenvalues (or $Q$-spectrum), respectively. Throughout the paper, these eigenvalues will be denoted by $\lambda_1(G) \geqslant \lambda_2(G) \geqslant \cdots \geqslant \lambda_n(G)$, $\mu_1(G) \geqslant \mu_2(G) \geqslant \cdots \geqslant \mu_n(G)$ and $q_1(G) \geqslant q_2(G) \geqslant \cdots \geqslant q_n(G)$, respectively. They are the roots of the corresponding characteristic polynomials $P_G(x) = \det(xI-A)$, $L_G(x) = \det(xI-L)$ and $Q_G(x) = \det(xI-Q)$. The largest eigenvalues, i.e., $\lambda_1(G), \mu_1(G)$ and $q_1(G)$ are called the $A$-index, $L$-index and $Q$-index (of $G$), respectively.

Given a graph $G$, define $\omega(G)$ and $\alpha(G)$, the *clique number* and *independence number* of $G$ to be the numbers of vertices of the largest clique and the largest independent set in $G$, respectively. Obviously, $\omega(G) = \alpha(G^c)$, where $G^c$ is the complement of $G$.

The knowledge of the spectrum of a graph is important as spectral results which are relevant for the estimation of some parameters of graphs. Research on the bound involving eigenvalues of $A$ (resp. $L$, $Q$) attracts much attention [5, 20, 21]. For connected graph $G$, Chen and Wang [2] determined sharp upper and lower bounds on $q_1(G)$ involving maximum degree and minimum degree. Nosal [19] gave sharp lower and upper bounds of $\lambda_1(G) + \lambda_1(G^c)$. Li [13] gave another upper bound on $\lambda_1(G) + \lambda_1(G^c)$. Liu, Lu

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and Tian [14], presented an upper bound on $\mu_1(G)+\mu_1(G^c)$. Liu, Lu and Tian [16] presented lower and upper bounds for the independence number $\alpha(G)$ and the clique number $\omega(G)$ involving the Laplacian eigenvalues of the graph $G$; Liu and Liu [15] presented lower and upper bounds for the independence number $\alpha(G)$ and the clique number $\omega(G)$ involving the signless Laplacian eigenvalues of the graph $G$.

The remainder of the paper is organized as follows: in Section 2 we give some preliminary results needed later on; in Section 3 we gave new upper and lower bounds on $q_1(G)+q_1(G^c)$; in the last section we present improved lower and upper bounds for the independence number $\alpha(G)$ and the clique number $\omega(G)$ involving the signless Laplacian eigenvalues of the graph $G$.

## 2. Lemmas

In this section, we give some preliminary lemmas which will be used in the subsequent sections. Let $B$ be a matrix. Denote by $s_i(B)$ the $i$th row sum of $B$.

**Lemma 2.1** ([20])**.** *Let $B$ be a real symmetric $n\times n$ matrix, and let $\lambda$ be an eigenvalue of $B$ with an eigenvector $\mathbf{x}$ whose entries are all nonnegative. Let $p$ be any polynomial. Then*

$$\min_{1\leqslant i\leqslant n} s_i(p(B)) \leqslant p(\lambda) \leqslant \max_{1\leqslant i\leqslant n} s_i(p(B)).$$

*Moreover, if all entries of $\mathbf{x}$ are positive, then either of the equalities holds if and only if the row sums of $p(B)$ are all equal.*

**Lemma 2.2.** *Let $G$ be a simple graph with $n$ vertices and $m$ edges, and $\Delta$ and $\delta$ be the maximum degree and the minimum degree of the vertices of $G$, respectively. Then*

$$q_1(G) \leqslant \frac{\Delta+\delta-1+\sqrt{(\Delta+\delta-1)^2+8[2m-(n-1)\delta]}}{2},$$

*Moreover, if $G$ is connected, then the equality holds if and only if $G$ is a regular graph.*

*Proof.* If $G$ is connected, it is just Theorem 2.2 in [2]; If $G$ is not connected, in view of Lemma 2.1, by a similar discussion as in the proof of Theorem 2.2 in [2], the result also holds in this case. We omit the procedure here. ☐

**Lemma 2.3.** *Let $G$ be a simple graph with $n$ vertices and $m$ edges, and $\Delta$ and $\delta$ be the maximum degree and the minimum degree of the vertices of $G$, respectively. Then*

$$q_1(G) \leqslant \frac{\delta-1+\sqrt{(\delta-1)^2+8[2m+\Delta^2-(n-1)\delta]}}{2}.$$

*Moreover, if $G$ is connected, then the equality holds if and only if $G$ is a regular graph.*

*Proof.* If $G$ is connected, it is just Theorem 2.1 in [2]; If $G$ is not connected, in view of Lemma 2.1, by a similar discussion as in the proof of Theorem 2.1 in [2], the result also holds in this case. We omit the procedure here. ☐

**Lemma 2.4** ([18])**.** *Let $\mathcal{F}=\{(x_1,x_2,\ldots,x_n)^T : x_i\geq 0, \sum_{i=1}^n x_i=1\}$. Then*

$$1-\frac{1}{\omega(G)}=\max_{x\in\mathcal{F}}\langle x, Ax\rangle.$$

Consider two sequences of real numbers: $\xi_1\geqslant\xi_2\geqslant\cdots\geqslant\xi_n$ and $\eta_1\geq\eta_2\geqslant\cdots\geqslant\eta_m$ with $m<n$. The second sequence is said to *interlace* the first one whenever

$$\xi_i\geqslant\eta_i\geqslant\xi_{n-m+i}$$

for $i=1,2,\ldots,m$. The interlacing is called *tight* if there exists an integer $k\in[0,m]$ such that $\xi_i=\eta_i$ for $1\leqslant i\leqslant k$ and $\xi_{n-m+i}=\eta_i$ for $k+1\leqslant i\leqslant m$. Suppose rows and columns of the matrix $M$ are partitioned according to a partitioning of $\{1,2,\ldots,n\}$. The partition is called *regular* if each block of $M$ has constant row (and column) sum.

**Lemma 2.5** ([12])**.** *Let $B$ be the matrix whose entries are the average row sums of the blocks of a symmetric partitioned matrix of $M$. Then*

(i) *the eigenvalues of $B$ interlace the eigenvalues of $M$;*
(ii) *if the interlacing is tight, then the partition is regular.*

## 3. Upper and lower bounds on $q_1(G) + q_1(G^c)$

In this section, we give upper and lower bounds on $q_1(G) + q_1(G^c)$ involving maximum degree, minimum degree, order and size of $G$.

**Theorem 3.1.** *Let $G$ be a simple graph with $n$ vertices, and $\delta$ and $\Delta$ be the minimum degree and the maximum degree of $G$, respectively. Then*

$$q_1(G) + q_1(G^c) \leqslant n - 2 + \sqrt{(\Delta + \delta + 1 - n)^2 + n^2 + 4(\Delta - \delta)(n-1)}.$$

*Proof.* Suppose $|E_G| = m$. Note that $\Delta(G^c) = n - 1 - \delta$, $\delta(G^c) = n - 1 - \Delta$ and $|E_{G^c}| = \frac{n(n-1)}{2} - m$. By Lemma 2.2, we have

$$q_1(G) \leqslant \frac{\Delta + \delta - 1 + \sqrt{(\Delta + \delta - 1)^2 + 8[2m - (n-1)\delta]}}{2}$$

and

$$q_1(G^c) \leqslant \frac{(2n - 3 - \Delta - \delta) + \sqrt{(2n - 3 - \Delta - \delta)^2 + 8[(n-1)(\Delta + 1) - 2m]}}{2}.$$

This gives

$$q_1(G) + q_1(G^c) \leqslant n - 2 + \frac{1}{2} f(m), \tag{3.1}$$

where

$$f(m) = \sqrt{(\Delta + \delta - 1)^2 + 8[2m - (n-1)\delta]} + \sqrt{(2n - 3 - \Delta - \delta)^2 + 8[(n-1)(\Delta + 1) - 2m]}.$$

Since

$$\frac{df}{dm} = \frac{8}{\sqrt{(\Delta + \delta - 1)^2 + 8[2m - \delta(n-1)]}} - \frac{8}{\sqrt{(2n - 3 - \Delta - \delta)^2 + 8[(n-1)(\Delta + 1) - 2m]}},$$

it is easy to check that

$$\frac{df}{dm} \geqslant 0$$

if and only if

$$m \leqslant \frac{2(n-2)(n - 1 - \Delta - \delta) + 4(n-1)(\Delta + \delta + 1)}{16}.$$

Therefore

$$\begin{aligned} f(m) &\leqslant \sqrt{(\Delta + \delta - 1)^2 + 2(n-2)(n - 1 - \Delta - \delta) + 4(n-1)(\Delta - \delta + 1)} \\ &\quad + \sqrt{(2n - 3 - \Delta - \delta)^2 - 2(n-2)(n - 1 - \Delta - \delta) + 4(n-1)(\Delta - \delta + 1)} \\ &= 2\sqrt{(\Delta + \delta + 1 - n)^2 + n^2 + 4(\Delta - \delta)(n-1)}. \end{aligned}$$

In view of (3.1) we have

$$q_1(G) + q_1(G^c) \leqslant n - 2 + \sqrt{(\Delta + \delta + 1 - n)^2 + n^2 + 4(\Delta - \delta)(n-1)},$$

as desired. $\square$

**Theorem 3.2.** *Let $G$ be a simple graph with $n$ vertices, and $\delta$ and $\Delta$ be the minimum degree and the maximum degree of $G$, respectively. Then*

$$q_1(G) + q_1(G^c) \leqslant \frac{n+\delta-\Delta-3}{2} + \frac{1}{2}(f_1 + f_2),$$

*where*

$$f_1 = \sqrt{(\delta-1)^2 + 4\Delta^2 + 4(n-1)(\Delta+1-\delta) + 4(n-1-\delta)^2 + \frac{(n+\delta-\Delta-3)(n-\Delta-\delta-1)}{2}}, \quad (3.2)$$

$$f_2 = \sqrt{(n-2-\Delta)^2 + 4\Delta^2 + 4(n-1)(\Delta+1-\delta) + 4(n-1-\delta)^2 - \frac{(n+\delta-\Delta-3)(n-\Delta-\delta-1)}{2}}. \quad (3.3)$$

*Proof.* Suppose $|E_G| = m$. Note that $\Delta(G^c) = n-1-\delta$, $\delta(G^c) = n-1-\Delta$ and $|E_{G^c}| = \frac{n(n-1)}{2} - m$. By Lemma 2.3, we have

$$q_1(G) \leqslant \frac{\delta - 1 + \sqrt{(\delta-1)^2 + 8[2m + \Delta^2 - (n-1)\delta]}}{2},$$

and

$$q_1(G^c) \leqslant \frac{(n-\Delta-2) + \sqrt{(n-2-\Delta)^2 + 8[(n-1)(\Delta+1) - 2m + (n-1-\delta)^2]}}{2}.$$

This gives

$$q_1(G) + q_1(G^c) \leqslant \frac{n+\delta-\Delta-3}{2} + \frac{1}{2}g(m), \quad (3.4)$$

where

$$g(m) = \sqrt{(\delta-1)^2 + 8[2m + \Delta^2 - (n-1)\delta]} + \sqrt{(n-2-\Delta)^2 + 8[(n-1)(\Delta+1) - 2m + (n-1-\delta)^2]}.$$

Since

$$\frac{dg}{dm} = \frac{8}{\sqrt{(\delta-1)^2 + 8[2m + \Delta^2 - (n-1)\delta]}} - \frac{8}{\sqrt{(n-2-\Delta)^2 + 8[(n-1)(\Delta+1) - 2m + (n-1-\delta)^2]}},$$

it is easy to check that

$$\frac{dg}{dm} \geqslant 0$$

if and only if

$$m \leqslant \frac{(n+\delta-\Delta-3)(n-\Delta-\delta-1) + 8(n-1)(\Delta+\delta+1) + 8(n-\Delta-\delta-1)(n+\Delta-\delta-1)}{32}.$$

Therefore

$$g(m) \leqslant f_1 + f_2,$$

where $f_1$ and $f_2$ are defined in (3.2) and (3.3), respectively.

In view of (3.4) we have

$$q_1(G) + q_1(G^c) \leqslant \frac{n+\delta-\Delta-3}{2} + \frac{1}{2}(f_1 + f_2),$$

where $f_1$ and $f_2$ are defined in (3.2) and (3.3), respectively. This completes the proof. □

## 4. Upper and lower bounds for $\omega(G)$ and $\alpha(G)$

In this section, we give upper and lower bounds for clique number and independence number of (regular) graph $G$ involving signless Laplacian eigenvalues.

**Theorem 4.1.** *Let $G$ be a simple graph with $n$ vertices, $m$ edges and maximum degree $\Delta$. Then*

$$\omega(G) \geqslant \frac{2m}{2m-(q_1-\Delta)^2}, \tag{4.1}$$

*where $q_1$ is the largest eigenvalue of $Q(G)$.*

*Proof.* Let $(x_1, x_2, \ldots, x_n)^T$ be the normalized eigenvector corresponding to $q_1(G)$. Then

$$\begin{aligned} q_1(G) &= \sum_{1\leqslant i<j\leqslant n, a_{ij}=1} (x_i+x_j)^2 \\ &= \sum_{i=1}^{n} d_i x_i^2 + \sum_{1\leqslant i<j\leqslant n, a_{ij}=1} 2x_i x_j \\ &\leqslant \Delta \sum_{i=1}^{n} x_i^2 + \sum_{1\leqslant i<j\leqslant n, a_{ij}=1} 2x_i x_j \\ &= \Delta + \sum_{1\leqslant i<j\leqslant n, a_{ij}=1} 2x_i x_j. \end{aligned}$$

Since $q_1(G) \geqslant \Delta + 1$ (see (3) in [2]), by the Cauchy inequality we have

$$(q_1(G)-\Delta)^2 \leqslant \left( \sum_{1\leqslant i<j\leqslant n, a_{ij}=1} 2x_i x_j \right)^2 \leqslant 2m \left( 2 \sum_{1\leqslant i<j\leqslant n, a_{ij}=1} x_i^2 x_j^2 \right).$$

Note that $(x_1^2, x_2^2, \ldots, x_n^2)^T \geqslant 0$ and $x_1^2 + x_2^2 + \cdots + x_n^2 = 1$; hence in view of Lemma 2.4 we have

$$\sum_{1\leqslant i<j\leqslant n, a_{ij}=1} 2x_i^2 x_j^2 \leqslant 1 - \frac{1}{\omega(G)}.$$

Therefore

$$\frac{(q_1(G)-\Delta)^2}{2m} \leqslant 1 - \frac{1}{\omega(G)},$$

that is

$$\omega(G) \geqslant \frac{2m}{2m-(q_1-\Delta)^2}.$$

This completes the proof. □

**Note.** In [14] it was proved that

$$\omega(G) \geqslant \frac{2m}{2m-(\mu_1-\Delta)^2}, \tag{4.2}$$

where $\mu_1$ is the largest eigenvalue of $L(G)$. Note that $\Delta + 1 \leqslant \mu_1(G) \leqslant q_1(G)$, hence the lower bound (4.1) is better than (4.2).

**Theorem 4.2.** *Let $G$ be a simple graph of order $n$ with at least one edge, and minimum degree $\delta$ and maximum degree $\Delta$. Let $q_1$ and $q_2$ be the first and the second largest eigenvalues of $Q(G)$. If $q_1 + q_2 - 3\delta \leq 0$, then*

$$\alpha(G) \geqslant \frac{q_1+q_2-3\delta}{\delta} \cdot \frac{n\Delta}{q_1+q_2-4\Delta}. \tag{4.3}$$

*Proof.* Let $G$ be a simple graph with order $n$ and a partition $V_G = V_1 \cup V_2$. Let $G_i$ $(i = 1, 2)$ be the subgraph of $G$ induced by $V_i$ with $n_i < n$ vertices and average degree $r_i$ $(n_1 + n_2 = n)$. Let $\bar{d}_i = \sum_{v \in V_i} d_G(v)/n_i$ for $i = 1, 2$. Note that $Q(G) = \begin{pmatrix} Q_{11} & Q_{12} \\ Q_{21} & Q_{22} \end{pmatrix} = \begin{pmatrix} D_{11} + A(G_1) & A_{12} \\ A_{21} & D_{22} + A(G_2) \end{pmatrix}$, where $D_{11} = \operatorname{diag}(d_G(v_1), \ldots, d_G(v_{n_1}))$, $D_{12} = \operatorname{diag}(d_G(v_{n_1+1}), \ldots, d_G(v_n))$ and $A_{21} = A_{12}^T$. Put $B = (\frac{b_{ij}}{n_i})$, where $b_{ij}$ is the sum of the entries in $Q_{ij}$. Then

$$B = \begin{pmatrix} \bar{d}_1 + r_1 & \bar{d}_1 - r_1 \\ \bar{d}_2 - r_2 & \bar{d}_2 + r_2 \end{pmatrix}$$

and $|\varphi I - B| = \varphi^2 - (r_1 + r_2 + \bar{d}_1 + \bar{d}_2)\varphi + 2(r_2\bar{d}_1 + r_1\bar{d}_2)$. Then by Lemma 2.5, we have $q_{n-1}(G) \leqslant \varphi_1(B) \leqslant q_1(G)$, $q_n(G) \leqslant \varphi_2(B) \leqslant q_2(G)$. Then

$$\varphi_1(B) + \varphi_2(B) = r_1 + r_2 + \bar{d}_1 + \bar{d}_2 \leqslant q_1(G) + q_2(G).$$

Note that $2(n_2\bar{d}_2 - n_1\bar{d}_1) = n_2(\bar{d}_2 + r_2) - n_1(\bar{d}_1 + r_1)$, and hence $n_2\bar{d}_2 - n_1\bar{d}_1 = n_2r_2 - n_1r_1$.

Let $V_{G_1}$ be the largest independent set of $G$, then $r_1 = 0$ and $\alpha(G) = n_1$. We have $r_2 = \bar{d}_2 - \frac{n_1}{n_2}\bar{d}_1$, and

$$\bar{d}_1 + 2\bar{d}_2 - \frac{n_1}{n_2}\bar{d}_1 \leqslant q_1 + q_2.$$

By $n = n_1 + n_2$, we have

$$\frac{q_1 + q_2 - 2\bar{d}_2 - \bar{d}_1}{\bar{d}_1} n = \frac{q_1 + q_2 - 2\bar{d}_2 - 2\bar{d}_1}{\bar{d}_1} n_1.$$

Since $G$ has at least one edge, $n_1 < n$. Note that $\delta \leqslant \bar{d}_1 \leqslant \Delta$, $\delta \leqslant \bar{d}_2 \leqslant \Delta$, and hence

$$\frac{q_1 + q_2 - 4\Delta}{\Delta} n_1 \leqslant \frac{q_1 + q_2 - 3\delta}{\delta} n.$$

Thus we get

$$\alpha(G) = n_1 \geqslant \frac{q_1 + q_2 - 3\delta}{\delta} \cdot \frac{n\Delta}{q_1 + q_2 - 4\Delta},$$

as required. $\square$

**Remark 1.** *Note that if $q_1 + q_2 - 3\delta > 0$, then $\frac{q_1+q_2-3\delta}{\delta} \cdot \frac{n\Delta}{q_1+q_2-4\Delta} < 0$, and the inequality in (4.3) is trivial. Hence, we add the restriction $q_1 + q_2 - 3\delta \leq 0$ in Theorem 4.2. In fact, there exists graph, say $G$, such that $q_1(G) + q_2(G) - 3\delta(G) \leq 0$. For example, $q_1(K_2) + q_2(K_2) - 3\delta(K_2) = 2 + 0 - 3 < 0$.*

If $G$ is a $d$-regular graph, then $q_1 = 2d, \Delta = \delta = d$. Hence, by Theorem 4.2 we have

**Corollary 4.3** ([15]). *Let $G$ be a simple $d$-regular graph of order $n$ with at least one edge. Then*

$$\alpha(G) \geqslant \frac{n(d - q_2)}{2d - q_2},$$

*where $q_2$ is the second largest eigenvalue of $Q(G)$.*

**Theorem 4.4.** *Let $G$ be a $d$-regular graph with order $n$ $(n \geq 3)$. Then*

$$\omega(G) \geqslant \frac{n^2}{n^2 - nd + (d - q_{n-1})M^2}, \tag{4.4}$$

*where $M = \min_{y_i \neq 0} \frac{1}{|y_i|}$ and $\mathbf{u}_{n-1} = (y_1, y_2, \ldots, y_n)^T$ is the normalized eigenvector corresponding to $q_{n-1}$ (the second least eigenvalue of $Q(G)$).*

*Proof.* Since $G$ is a $d$-regular graph, we have $q_1 = 2d$. And the normalized eigenvector corresponding to $q_1$ is $\mathbf{u}_1 = \frac{\mathbf{e}}{\sqrt{n}}$, where $\mathbf{e} = (1, 1, \ldots, 1)^T$. Let $\theta = \frac{M}{n}$ and $\mathbf{x} = \frac{1}{n}\mathbf{e} + \theta \mathbf{u}_{n-1}$. Then $\theta y_i \geqslant -\frac{1}{n}$ $(i = 1, 2, \ldots, n)$. Since $\sum_{i=1}^{n} q_i = 2m = nd$ and $n \geq 3$, we have $q_1 \neq q_{n-1}$ and $\langle \mathbf{e}, \mathbf{u}_{n-1} \rangle = 0$. So $\mathbf{x} \in \mathcal{F} = \{(x_1, x_2, \ldots, x_n)^T : x_i \geq 0, \sum_{i=1}^{n} x_i = 1\}$. By Lemma 2.4, we have

$$
\begin{aligned}
\langle \mathbf{x}, Q\mathbf{x} \rangle &= \langle \mathbf{x}, D\mathbf{x} \rangle + \langle \mathbf{x}, A\mathbf{x} \rangle \\
&\leqslant d\langle \mathbf{x}, \mathbf{x} \rangle + \left(1 - \frac{1}{\omega(G)}\right) \\
&= d(\frac{1}{n} + \theta^2) + \left(1 - \frac{1}{\omega(G)}\right).
\end{aligned}
$$

On the other hand,

$$
\begin{aligned}
\langle \mathbf{x}, Q\mathbf{x} \rangle &= \left\langle \frac{\mathbf{e}}{n} + \theta \mathbf{u}_{n-1}, Q\left(\frac{\mathbf{e}}{n} + \theta \mathbf{u}_{n-1}\right) \right\rangle \\
&= \left\langle \frac{\mathbf{e}}{n}, Q\frac{\mathbf{e}}{n} \right\rangle + \left\langle \frac{\mathbf{e}}{n}, Q\theta \mathbf{u}_{n-1} \right\rangle + \left\langle \theta \mathbf{u}_{n-1}, Q\frac{\mathbf{e}}{n} \right\rangle + \langle \theta \mathbf{u}_{n-1}, Q\theta \mathbf{u}_{n-1} \rangle \\
&= \frac{1}{n^2}\langle \mathbf{e}, Q\mathbf{e} \rangle + 0 + 0 + \theta^2 \langle \mathbf{u}_{n-1}, Q\mathbf{u}_{n-1} \rangle \\
&= \frac{2nd}{n^2} + \theta^2 q_{n-1}.
\end{aligned}
$$

Therefore $\frac{2d}{n} + \theta^2 q_{n-1} \leqslant d(\frac{1}{n} + \theta^2) + 1 - \frac{1}{\omega}$, that is

$$
\omega(G) \geqslant \frac{1}{1 - \frac{d}{n} + \theta^2(d - q_{n-1})}.
$$

Since $\theta = \frac{M}{n}$, $M = \min_{y_i \neq 0} \frac{1}{|y_i|}$, we find

$$
\omega(G) \geqslant \frac{n^2}{n^2 - nd + (d - q_{n-1})M^2}.
$$

This completes the proof. $\square$

**Remark.** For a $d$-regular graph $G$, in [15] it was proved that

$$
\omega(G) \geqslant \frac{n^2}{n^2 - nd + (d - q_n)M^2}, \tag{4.5}
$$

where $q_n$ is the least eigenvalue of $Q(G)$. Note that $q_{n-1} \geqslant q_n$, hence the lower bound (4.4) is better than (4.5).

For a $d$-regular graph $G$, when $i = \begin{cases} \lfloor \frac{n-1}{2} \rfloor, & n \text{ is even;} \\ \lfloor \frac{n}{2} \rfloor, & n \text{ is odd,} \end{cases}$ we have $q_1 > q_{n-i} \geqslant q_{n-i+1} \geqslant \cdots \geqslant q_{n-1} \geqslant q_n$. By a similar discussion as in the proof of Theorem 4.4, we can obtain another improved lower bound on $\omega(G)$ as follows:

$$
\omega(G) \geqslant \frac{n^2}{n^2 - nd + (d - q_{n-i})M^2},
$$

where $M = \min_{y_i \neq 0} \frac{1}{|y_i|}$ and $\mathbf{u}_{n-i} = (y_1, y_2, \ldots, y_n)^T$ is the normalized eigenvector corresponding to $q_{n-i}$.


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