Multiplication and Composition Operators on Weak L_p spaces

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Abstract

In a self-contained presentation, we discuss the Weak L_p spaces. Invertible and compact multiplication operators on Weak L_p are characterized. Boundedness of the composition operator on Weak L_p is also characterized.
Keywords: Compact operator, multiplication and composition operator, distribution function, Weak L_p spaces.

Contents

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1 Introduction

One of the real attraction of Weak L_p space is that the subject is sufficiently concrete and yet the spaces have fine structure of importance for applications. WeakL_p spaces are function spaces which are closely related to L_p spaces. We do not know the exact origin of Weak L_p spaces, which is a apparently part of the folklore. The Book by Colin Benett and Robert Sharpley[4] contains a good presentation of Weak L_p but from the point of view of rearrangement function. In the present paper we study the Weak L_p space from the point of view of distribution function. This circumstance motivated us to undertake a preparation of the present paper containing a detailed exposition of these function spaces. In section 6 of the present paper we first prove a characterization of the boundedness of M_u in terms of u, and show that the set of multiplication operators on Weak L_p is a maximal abelian subalgebra of $B\big(\text{Weak}\,L_p\big)$, the Banach algebra of all bounded linear operators on Weak L_p. For the systemic study of the multiplication operator on different spaces we refereed to ([1], [2], [5] [3], [10], [18], [21]).
We use it to characterize the invertibility of M_u on Weak L_p. The compact multiplication operators are also characterized in this section.
In section 7 a necessary and sufficient condition for the boundedness of composition operator C_T is given. For the study of composition operator on different function spaces we refereed to ([6], [11], [12], [16], [18], [19]).

2 Weak L_p spaces

Definition 2.1. *For f a measurable function on X, the distribution function of f is the function D_f defined on $[0, \infty)$ as follows:*

$$D_f(\lambda) := \mu\Big(\{x \in X : |f(x)| > \lambda\}\Big). \tag{1}$$

The distribution function D_f provides information about the size of f but not about the behavior of f itself near any given point. For instance, a function on $\mathbb{R}^n$ and each of its translates have the same distribution function. It follows from definition 2.1 that D_f is a decreasing function of λ (not necessarily strictly).

Let (X, μ) be a measurable space and f and g be a measurable functions on (X, μ) then D_f enjoy the following properties: For all λ_1, $\lambda_2 > 0$:

1. $|g| \le |f|$ μ-a.e. implies that $D_g \le D_f$;

2. $D_{cf}(\lambda_1) = D_f\left(\frac{\lambda_1}{|c|}\right)$ for all $c \in \mathbb{C}/\{0\}$;

3. $D_{f+g}(\lambda_1 + \lambda_2) \le D_f(\lambda_1) + D_g(\lambda_2)$;

4. $D_{fg}(\lambda_1\lambda_2) \le D_f(\lambda_1) + D_g(\lambda_2)$.

For more details on distribution function see ([7] and [15]).
Next, Let (X, μ) be a measurable space, for $0 < p < \infty$, we consider

$$\text{Weak } L_p := \left\{ f : \mu(\{x \in X : |f(x)| > \lambda\}) \le \left(\frac{C}{\lambda}\right)^p \right\},$$

for some $C > 0$. Observe that Weak $L_\infty = L_\infty$.

Weak L_p as a space of functions is denoted by $L_{(p,\infty)}$.

Proposition 2.1. *Let $f \in$ Weak L_p with $0 < p < \infty$. Then*

$$\begin{aligned} \|f\|_{L_{(p,\infty)}} &= \inf\left\{ C > 0 : D_f(\lambda) \le \left(\frac{C}{\lambda}\right)^p \right\} \\ &= \left(\sup_{\lambda > 0} \lambda^p D_f(\lambda) \right)^{1/p} \\ &= \sup_{\lambda > 0} \lambda \left\{ D_f(\lambda) \right\}^{1/p}. \end{aligned}$$

Proof. Let us define

$$\lambda = \inf\left\{ C > 0 : D_f(\alpha) \le \left(\frac{C}{\alpha}\right)^p \right\},$$

and

$$B = \left(\sup_{\alpha > 0} \alpha^p D_f(\alpha) \right)^{1/p}.$$

Since $f \in$ Weak L_p, then

$$D_f(\alpha) \le \left(\frac{C}{\alpha}\right)^p,$$

for some $C > 0$, then

$$\left\{C > 0 : D_f(\alpha) \leq \left(\frac{C}{\alpha}\right)^p \quad \forall\, \alpha > 0\right\} \neq \emptyset.$$

On the other hand

$$\alpha^p D_f(\alpha) \leq B^p,$$

thus $\{\alpha^p D_f(\alpha) : \alpha > 0\}$ is bounded above by B^p and so $B \in \mathbb{R}$.
Therefore

$$\lambda = \inf\left\{C > 0 : D_f(\alpha) \leq \left(\frac{C}{\alpha}\right)^p \quad \alpha > 0\right\} \leq B. \tag{2}$$

Now, let $\epsilon > 0$, then there exists C such that

$$\lambda \leq C < \lambda + \epsilon,$$

and thus

$$D_f(\lambda) \leq \frac{C^p}{\lambda^p} < \frac{(\lambda+\epsilon)^p}{\lambda^p},$$

then

$$\begin{aligned} \sup_{\lambda>0} \lambda^p D_f(\lambda) &< (\lambda+\epsilon)^p \\ \left(\sup_{\lambda>0} \lambda^p D_f(\lambda)\right)^{1/p} &\leq \lambda \\ B &< \lambda, \end{aligned} \tag{3}$$

by (2) and (3) $B = \lambda$. □

Definition 2.2. *For $0 < p < \infty$ the space $L_{(p,\infty)}$ is defined as the set of all $\mu-$measurable functions f such that*

$$\begin{aligned} \|f\|_{L_{(p,\infty)}} &= \inf\left\{C > 0 : D_f(\lambda) \leq \left(\frac{C}{\lambda}\right)^p \quad \forall\, \lambda > 0\right\} \\ &= \left(\sup_{\lambda>0} \lambda^p D_f(\lambda)\right)^{1/p} \\ &= \sup_{\lambda>0} \lambda \left\{D_f(\lambda)\right\}^{1/p}, \end{aligned}$$

is finite. Two functions in $L_{(p,\infty)}$ will be considered equal if they are equal $\mu-a.e.$

The Weak $L_p = L_{(p,\infty)}$ are larger than the L_p spaces, we have the following.

Proposition 2.2. *For any $0 < p < \infty$ and any $f \in L_p$ we have*

$$L_p \subset L_{(p,\infty)},$$

and hence

$$\left\|f\right\|_{L_{(p,\infty)}} \leq \left\|f\right\|_{L_p}.$$

(This is just a restatement of the Chebyshev inequality).

Proof. If $f \in L_p$, then

$$\lambda^p \mu\Big(\{x \in X : |f(x)| > \lambda\}\Big) \leq \int\limits_{\{|f|>\lambda\}} |f|^p\, du \leq \int\limits_X |f|^p\, du = \left\|f\right\|_{L_p}^p,$$

therefore

$$\mu\Big(\{x \in X : |f(x)| > \lambda\}\Big) \leq \left(\frac{\|f\|_{L_p}}{\lambda}\right)^p. \tag{4}$$

Hence $f \in$ Weak $L_p = L_{(p,\infty)}$, which means that

$$L_p \subset L_{(p,\infty)}. \tag{5}$$

Next, from (4) we have

$$\left(\sup_{\lambda>0}\{\lambda^p D_f(\lambda)\}\right)^{1/p} \leq \left\|f\right\|_{L_p}$$
$$\left\|f\right\|_{L_{(p,\infty)}} \leq \left\|f\right\|_{L_p}.$$

□

Remark 2.1. The inclusion (5) is strict, indeed, let $f(x) = x^{-1/p}$ on $(0, \infty)$ (with the Lebesgue measure). Note

$$m\left(\left\{x \in (0,\infty) : \frac{1}{|x|^{1/p}} > \lambda\right\}\right) = m\left(\left\{x \in (0,\infty) : |x| < \frac{1}{\lambda^p}\right\}\right) = 2\lambda^{-p}.$$

Thus $f \in$ Weak $L_p(0,\infty)$, but

$$\int_0^\infty \left(\frac{1}{x^{1/p}}\right)^p dx = \int_0^\infty \frac{dx}{x} \to \infty,$$

then $f \notin L_p(0,\infty)$.

Proposition 2.3. *Let $f, g \in L_{(p,\infty)}$. Then*

1. $\big\|cf\big\|_{L_{(p,\infty)}} = |c|\big\|f\big\|_{L_{(p,\infty)}}$ *for any constant c,*

2. $\big\|f+g\big\|_{L_{(p,\infty)}} \leq 2\left(\big\|f\big\|^p_{L_{(p,\infty)}} + \big\|g\big\|^p_{L_{(p,\infty)}}\right)^{1/p}$.

Proof. (1) For $c > 0$ we have

$$\mu\Big(\{x \in X : |cf(x)| > \lambda\}\Big) = \mu\Big(\left\{x \in X : |f(x)| > \frac{\lambda}{c}\right\}\Big),$$

thus

$$D_{cf}(\lambda) = D_f\left(\frac{\lambda}{c}\right).$$

And thus

$$\begin{aligned}
\|cf\|_{L_{(p,\infty)}} &= \left(\sup_{\lambda>0} \lambda^p D_{cf}(\lambda)\right)^{1/p} \\
&= \left(\sup_{\lambda>0} \lambda^p D_f\left(\frac{\lambda}{c}\right)\right)^{1/p} \\
&= \left(\sup_{cw>0} c^p w^p D_f(w)\right)^{1/p} = c\left(\sup_{cw>0} w^p D_f(w)\right)^{1/p},
\end{aligned}$$

then

$$\|cf\|_{L_{(p,\infty)}} = c\|f\|_{L_{(p,\infty)}}.$$

(2) Note that

$$\Big\{x \in X : |f(x)+g(x)| > \lambda\Big\} \subseteq \left\{x \in X : |f(x)| > \frac{\lambda}{2}\right\} \bigcup \left\{x \in X : |g(x)| > \frac{\lambda}{2}\right\}.$$

Hence

$$\begin{aligned}
&\mu\Big(\{x \in X : |f(x)+g(x)| > \lambda\}\Big) \\
&\quad \leq \mu\left(\left\{x \in X : |f(x)| > \frac{\lambda}{2}\right\}\right) + \mu\left(\left\{x \in X : |g(x)| > \frac{\lambda}{2}\right\}\right),
\end{aligned}$$

then

$$\begin{aligned}
\lambda^p D_{f+g}(\lambda) \leq& \lambda^p D_f\left(\frac{\lambda}{2}\right) + \lambda^p D_g\left(\frac{\lambda}{2}\right) \\
\lambda^p D_{f+g}(\lambda) \leq& 2^p\left[\sup_{\lambda>0} \lambda^p D_f(\lambda) + \sup_{\lambda>0} \lambda^p D_g(\lambda)\right],
\end{aligned}$$

therefore

$$\left(\sup_{\lambda>0}\lambda^p D_{f+g}(\lambda)\right)^{1/p} \leq 2\left(\|f\|^p_{L_{(p,\infty)}} + \|g\|^p_{L_{(p,\infty)}}\right)^{1/p}$$
$$\|f+g\|_{L_{(p,\infty)}} \leq 2\left(\|f\|^p_{L_{(p,\infty)}} + \|g\|^p_{L_{(p,\infty)}}\right)^{1/p}.$$

□

Remark 2.2. Proposition 2.3 (2) tell us that $\|.\|_{L_{(p,\infty)}}$ define a quasi-norm on $L_{(p,\infty)}$.

Definition 2.3. *A quasi-norm is a functional that is like a norm except that it does only satisfy the triangle inequality with a constant $C \geq 1$, that is*

$$\|f+g\| \leq C\Big(\|f\| + \|g\|\Big).$$

3 Convergence in measure

Next, we discus some convergence notions. The following notion is of importance in probability theory.

Definition 3.1. *Let f, f_n $(n = 1, 2, 3, ...)$ be a measurable functions on the measurable space (X, μ). The sequence $\{f_n\}_{n\in\mathbb{N}}$ is said to converge in measure to f ($f_n \overset{\mu}{\to} f$) if for all $\epsilon > 0$ there exists an $n_0 \in \mathbb{N}$ such that*

$$\mu\Big(\{x \in X : |f_n(x) - f(x)| > \epsilon\}\Big) < \epsilon \quad \textit{for all} \quad n \geq n_0. \tag{6}$$

Remark 3.1. The preceding definition is equivalent to the following statement.
For all $\epsilon > 0$,

$$\lim_{n\to\infty}\mu\Big(\{x \in X : |f_n(x) - f(x)| > \epsilon\}\Big) = 0. \tag{7}$$

Cleary (7) implies (6). To see the convergence given $\epsilon > 0$, pick $0 < \delta < \epsilon$ and apply (6) for this δ.
There exists an $n_0 \in \mathbb{N}$ such that

$$\mu\Big(\{x \in X : |f_n(x) - f(x)| > \delta\}\Big) < \delta,$$

holds for $n \geq n_0$. Since

$$\mu\Big(\{x \in X : |f_n(x) - f(x)| > \epsilon\}\Big) \leq \mu\Big(\{x \in X : |f_n(x) - f(x)| > \delta\}\Big).$$

We concluded that

$$\mu\Big(\{x \in X : |f_n(x) - f(x)| > \epsilon\}\Big) < \delta,$$

for all $n \geq n_0$. Let $n \to \infty$ to deduce that

$$\limsup_{n\to\infty} \mu\Big(\{x \in X : |f_n(x) - f(x)| > \epsilon\}\Big) \leq \delta. \tag{8}$$

Since (8) holds for all $0 < \delta < \epsilon$ (7) follows by letting $\delta \to 0$.

Remark 3.2. Convergence in measure is a more general property than convergence in either L_p or $L_{(p,\infty)}$, $0 < p < \infty$, as the following proposition indicates:

Proposition 3.1. *Let $0 < p \leq \infty$ and f_n, f be in $L_{(p,\infty)}$.*

1. *If f_n, f are in L_p and $f_n \to f$ in L_p, then $f_n \to f$ in $L_{(p,\infty)}$.*
2. *If $f_n \to f$ in $L_{(p,\infty)}$ then $f_n \overset{\mu}{\to} f$.*

Proof. (1) Fix $0 < p < \infty$. Proposition 2.2 gives that for all $\epsilon > 0$ we have:

$$\begin{aligned} \mu\Big(\{x \in X : |f_n(x) - f(x)| > \epsilon\}\Big) &\leq \frac{1}{\epsilon^p} \int_X |f_n - f|^p \, d\mu \\ \epsilon^p \mu\Big(\{x \in X : |f_n(x) - f(x)| > \epsilon\}\Big) &\leq \|f_n - f\|_{L_p}^p \\ \sup_{\lambda>0} \lambda^p D_{f_n - f}(\lambda) &\leq \|f_n - f\|_{L_p}^p, \end{aligned}$$

and thus

$$\|f_n - f\|_{L_{(p,\infty)}} \leq \|f_n - f\|_{L_p}.$$

This shows that convergence in L_p implies convergence in Weak L_p. The case $p = \infty$ is tautological.

(2) Give $\epsilon > 0$ find an $n_0 \in \mathbb{N}$ such that for $n > n_0$, we have

$$\|f_n - f\|_{L_{(p,\infty)}} = \left(\sup_{\lambda>0} \lambda^p D_{f_n-f}(\lambda)\right)^{1/p} < \epsilon^{\frac{1}{p}+1},$$

then taking $\lambda = \epsilon$, we conclude that

$$\epsilon^p \mu\Big(\{x \in X : |f_n(x) - f(x)| > \epsilon\}\Big) < \epsilon^{p+1},$$

for $n > n_0$.
Hence

$$\mu\Big(\{x \in X : |f_n(x) - f(x)| > \epsilon\}\Big) < \epsilon \quad \text{for} \quad n > n_0.$$

$\square$

Example 3.1. Fix $0 < p < \infty$. On $[0, 1]$ define the functions

$$f_{k,j} = k^{1/p} \chi_{\left(\frac{j-1}{k}, \frac{j}{k}\right)} \quad k \geq 1, \quad 1 \leq j \leq k.$$

Consider the sequence $\{f_{1,1}, f_{2,1}, f_{2,2}, f_{3,1}, f_{3,2}, f_{3,3}, ...\}$.
Observe that

$$m\Big(\{x \in [0, 1] : f_{k,j}(x) > 0\}\Big) = \frac{1}{k},$$

thus

$$\lim_{k\to\infty} m\Big(\{x \in [0, 1] : f_{k,j}(x) > 0\}\Big) = 0,$$

that is $f_{k,j} \xrightarrow{m} 0$.

Likewise, Observe that

$$\|f_{k,j}\|_{L_{(p,\infty)}} = \left(\sup_{\lambda>0} \lambda^p m\Big(\{x \in [0, 1] : f_{k,j}(x) > \lambda\}\Big)\right)^{1/p}$$
$$\geq \left(\sup_{k\geq1} \frac{k-1}{k}\right)^{1/p} = 1.$$

Which implies that $f_{k,j}$ does not converge to 0 in $L_{(p,\infty)}$.

It turns out that every sequence convergent in $L_{(p,\infty)}$ has a subsequence that converges μ-a.e. to the same limit.

Theorem 3.1. *Let f_n and f be a complex-valued measurable functions on a measure space $(X, \mathcal{A}, \mu)$ and suppose $f_n \xrightarrow{\mu} f$. Then some subsequence of f_n converges to f μ−a.e.*

Proof. For all $k = 1, 2, ...$ choose inductively n_k such that

$$\mu\Big(\{x \in X : |f_n(x) - f(x)| > 2^{-k}\}\Big) < 2^{-k}, \tag{9}$$

and such that $n_1 < n_2 < ... < n_k < ...$ Define the sets

$$A_k = \Big\{x \in X : |f_{n_k}(x) - f(x)| > 2^{-k}\Big\},$$

(9) implies that

$$\mu\left(\bigcup_{k=m}^{\infty} A_k\right) \leq \sum_{k=m}^{\infty} \mu(A_k) \leq \sum_{k=m}^{\infty} 2^{-k} = 2^{1-m}, \tag{10}$$

for all $m = 1, 2, 3, ...$ It follows from (10) that

$$\mu\left(\bigcup_{k=1}^{\infty} A_k\right) \leq 1 < \infty. \tag{11}$$

Using (10) and (11), we conclude that the sequence of the measure of the sets $\left\{\bigcup\limits_{k=m}^{\infty} A_k\right\}_{m \in \mathbb{N}}$ converges as $m \to \infty$ to

$$\mu\left(\bigcap_{m=1}^{\infty} \bigcup_{k=m}^{\infty} A_k\right) = 0. \tag{12}$$

To finish the proof, observe that the null set in (12) contains the set of all $x \in X$ for which $f_{n_k}(x)$ does not converge to $f(x)$. □

Remark 3.3. In many situations we are given a sequence of functions and we would like to extract a convergent subsequence. One way to achieve this is via the next theorem which is a useful variant of theorem 3.1. We first give a relevant definition.

Definition 3.2. *We say that a sequence of measurable functions $\{f_n\}_{n \in \mathbb{N}}$ on the measure space $(X, \mathcal{A}, \mu)$ is Cauchy in measure if for every $\epsilon > 0$ there exists an $n_0 \in \mathbb{N}$ such that for n, $m > n_0$ we have*

$$\mu\Big(\{x \in X : |f_n(x) - f_m(x)| > \epsilon\}\Big) < \epsilon.$$

Theorem 3.2. *Let $(X, \mathcal{A}, \mu)$ be a measure space and let $\{f_n\}_{n\in\mathbb{N}}$ be a complex valued sequence on X, that is Cauchy in measure. Then some subsequence of f_n converges $\mu-$a.e.*

Proof. The proof is very similar to that of theorem 3.1 for all $k = 1, 2, 3, ...$ choose n_k inductively such that

$$\mu\Big(\{x \in X : |f_{n_k}(x) - f_{n_{k+1}}(x)| > 2^{-k}\}\Big) < 2^{-k}, \tag{13}$$

and such that $n_1 < n_2 < n_3 < ... < n_k < n_{k+1} < ...$ Define

$$A_k = \Big\{x \in X : |f_{n_k}(x) - f_{n_{k+1}}(x)| > 2^{-k}\Big\}.$$

As shown in the proof of theorem 3.1 (13) implies that

$$\mu\left(\bigcap_{m=1}^{\infty}\bigcup_{k=m}^{\infty} A_k\right) = 0, \tag{14}$$

for $x \notin \bigcup\limits_{k=m}^{\infty} A_k$ and $i \geq j \geq j_0 \geq m$ (and j_0 large enough) we have

$$|f_{n_i}(x) - f_{n_j}(x)| \leq \sum_{l=j}^{i-1} |f_{n_l}(x) - f_{n_{l+1}}(x)| \leq \sum_{l=j}^{i} 2^{-l} \leq 2^{1-j} \leq 2^{1-j_0}.$$

This implies that the sequence $\{f_{n_i}(x)\}_{i\in\mathbb{N}}$ is Cauchy for every x in the set $\left(\bigcup\limits_{k=m}^{\infty} A_k\right)^c$ and therefore converges for all such x. We define a function

$$f(x) = \begin{cases} \lim\limits_{j\to\infty} f_{n_j}(x) & \text{when } x \notin \bigcap\limits_{m=1}^{\infty}\bigcup\limits_{k=m}^{\infty} A_k \\ \\ 0 & \text{when } x \in \bigcap\limits_{m=1}^{\infty}\bigcup\limits_{k=m}^{\infty} A_k \end{cases}$$

Then $f_{n_j} \to f$ almost everywhere. □

Proposition 3.2. *If $f \in Weak\, L_p$ and $\mu\Big(\{x \in X : f(x) \neq 0\}\Big) < \infty$, then $f \in L_q$ for all $q < p$. On the other hand, if $f \in Weak\, L_p \cap L_\infty$ then $f \in L_q$ for all $q > p$.*

Proof. If $p < \infty$, we write

$$\int_X |f(x)|^q \, d\mu = q \int_0^\infty \lambda^{q-1} D_f(\lambda) \, d\lambda$$
$$= q \int_0^1 \lambda^{q-1} D_f(\lambda) \, d\lambda + q \int_1^\infty \lambda^{q-1} D_f(\lambda) \, d\lambda.$$

Note that

$$\mu\Big(\{x \in X : |f(x)| > \lambda\}\Big) \le \mu\Big(\{x \in X : f(x) \neq 0\}\Big).$$

Therefore $\mu\Big(\{x \in X : |f(x)| > \lambda\}\Big) \le C$, then

$$\int_X |f(x)|^q \, d\mu \le qC \int_0^1 \lambda^{q-1} \, d\lambda + qC \int_1^\infty \lambda^{q-p-1} \, d\lambda = C + \frac{qC\lambda^{q-p}}{q-p}\Big]_1^\infty < \infty$$

Therefore $f \in L_q$.

If $f \in$ Weak $L_p \cap L_\infty$. Then

$$\int_X |f(x)|^q \, d\mu = q \int_0^\infty \lambda^{q-1} D_f(\lambda) \, d\lambda$$
$$= q \int_0^M \lambda^{q-1} D_f(\lambda) \, d\lambda + q \int_M^\infty \lambda^{q-1} D_f(\lambda) \, d\lambda,$$

where $M = esssup|f(x)|$. Note that

$$\mu\Big(\{x \in X : |f(x)| > \lambda\}\Big) = 0 \quad \text{for} \quad \lambda > M,$$

since $f \in$ Weak $L_p \cap L_\infty$, therefore

$$q \int_M^\infty \lambda^{q-1} D_f(\lambda) \, d\lambda = 0 \quad \text{and} \quad D_f(\lambda) \le \frac{\big\|f\big\|^p_{L_{(p,\infty)}}}{\lambda^p}.$$

Then

$$\int_X |f(x)|^q \, d\mu = q \int_0^M \lambda^{q-1} D_f(\lambda) \, d\lambda \leq q \|f\|_{L_{(p,\infty)}}^p \int_0^M \lambda^{q-p-1} \, d\lambda$$
$$= \frac{q \|f\|_{L_{(p,\infty)}}^p M^{q-p}}{q-p} < \infty,$$

then

$$\int_X |f(x)|^q \, d\mu \leq \infty.$$

Thus $f \in L_q$. □

Proposition 3.3. *Let $f \in$ Weak $L_{p_0} \cap$ Weak L_{p_1} with $p_0 < p < p_1$. Then $f \in L_p$.*

Proof. Let us write

$$f = f\chi_{\{|f| \leq 1\}} + f\chi_{\{|f|>1\}} = f_1 + f_2.$$

Observe that $f_1 \leq f$ and $f_2 \leq f$. In particular $f_1 \in$ Weak L_{p_0} and $f_2 \in$ Weak L_{p_1}. Also, write that f_1 is bounded and

$$\mu\Big(\{x \in X : f_2(x) \neq 0\}\Big) = \mu\Big(\{x \in X : |f(x)| > 1\}\Big) < C < \infty.$$

Therefore by proposition 3.2, we have $f_1 \in L_p$ and $f_2 \in L_p$. Since L_p is a linear vector space, we conclude that $f \in L_p$. □

4 An interpolation result

It is a useful fact that if a function is in $L_p(X,\mu) \cap L_q(X,\mu)$, then it also lies in $L_r(X,\mu)$ for all $p < r < q$. The usefulness of the spaces $L_{(p,\infty)}$ can be seen from the following sharpening of this statement:

Proposition 4.1. *Let $0 < p < q \leq \infty$ and let f in $L_{(p,\infty)} \cap L_{(q,\infty)}$. Then f is in L_r for all $p < r < q$ and*

$$\|f\|_{L_r} \leq \left(\frac{r}{r-p} + \frac{r}{q-r}\right)^{1/r} \|f\|_{L_{(p,\infty)}}^{\frac{\frac{1}{r}-\frac{1}{q}}{\frac{1}{p}-\frac{1}{q}}} \|f\|_{L_{(q,\infty)}}^{\frac{\frac{1}{p}-\frac{1}{r}}{\frac{1}{p}-\frac{1}{q}}} \tag{15}$$

whit the suitable interpolation when $q = \infty$.

Proof. Let us take first $q < \infty$. We know that

$$D_f(\lambda) \leq \min\left(\frac{\|f\|^p_{L_{p,\infty}}}{\lambda^p}, \frac{\|f\|^q_{L_{q,\infty}}}{\lambda^q}\right), \tag{16}$$

set

$$B = \left(\frac{\|f\|^q_{L_{q,\infty}}}{\|f\|^p_{L_{p,\infty}}}\right)^{\frac{1}{q-p}}. \tag{17}$$

We now estimate the L_r norm of f. By (16), (17), we have

$$\begin{aligned}
\|f\|^r_{L_r} &= r\int_0^\infty \lambda^{r-1} D_f(\lambda)\, d\lambda \\
&\leq r\int_0^\infty \lambda^{r-1} \min\left(\frac{\|f\|^p_{L_{p,\infty}}}{\lambda^p}, \frac{\|f\|^q_{L_{q,\infty}}}{\lambda^q}\right) d\lambda \\
&= r\int_0^B \lambda^{r-1-p}\|f\|^p_{L_{p,\infty}}\, d\lambda + r\int_B^\infty \lambda^{r-1-q}\|f\|^q_{L_{q,\infty}}\, d\lambda \qquad (18) \\
&= \frac{r}{r-q}\|f\|^p_{L_{p,\infty}} B^{r-p} + \frac{r}{q-r}\|f\|^q_{L_{q,\infty}} B^{r-q} \\
&= \left(\frac{r}{r-p} + \frac{r}{q-r}\right)\left(\|f\|^p_{L_{(p,\infty)}}\right)^{\frac{q-r}{q-p}}\left(\|f\|^q_{L_{(q,\infty)}}\right)^{\frac{r-p}{q-p}}.
\end{aligned}$$

Observe that the integrals converge, since $r - p > 0$ and $r - q < 0$.
The case $q = \infty$ is easier. Since $D_f(\lambda) = 0$ for $\lambda > \|f\|_{L_\infty}$ we need to use only the inequality

$$D_f(\lambda) \leq \lambda^{-p}\|f\|^p_{L_{(p,\infty)}},$$

for $\lambda \leq \|f\|_{L_\infty}$ in estimating the first integral in (18). We obtain

$$\|f\|^r_{L_r} \leq \frac{r}{r-p}\|f\|^p_{L_{(p,\infty)}}\|f\|^{r-p}_{L_\infty},$$

Which is nothing other than (15) when $q = \infty$. This complete the proof. □

Note that (15) holds with constant 1 if $L_{(p,\infty)}$ and $L_{(q,\infty)}$ are replaced by L_p and L_q, respectively. It is often convenient to work with functions that are only locally in some L_p space. This leads to the following definition.

Definition 4.1. *For $0 < p < \infty$, the space $L^p_{loc}(\mathbb{R}^n, |.|)$ or simply $L^p_{loc}(\mathbb{R}^n)$ (where $|.|$ denote the Lebesgue measure) is the set of all Lebesgue-measurable functions f on $\mathbb{R}^n$ that satisfy*

$$\int_K |f(x)|^p \, dx < \infty, \tag{19}$$

for any compact subset K of $\mathbb{R}^n$. Functions that satisfy (19) with $p = 1$ are called locally integrable functions on $\mathbb{R}^n$.

The union of all $L_p(\mathbb{R}^n)$ spaces for $1 \leq p \leq \infty$ is contained in $L^1_{loc}(\mathbb{R}^n)$.

More generally, for $0 < p < q < \infty$ we have the following:

$$L_q(\mathbb{R}^n) \subseteq L^q_{loc}(\mathbb{R}^n) \subseteq L^p_{loc}(\mathbb{R}^n).$$

Functions in $L_p(\mathbb{R}^n)$ for $0 < p < 1$ may not be locally integrable. For example, take $f(x) = |x|^{-\alpha-n}\chi_{\{x:|x|\leq 1\}}$ which is in $L_p(\mathbb{R}^n)$ when $p < n/(n+\alpha)$, and observe that f is not integrable over any open set in $\mathbb{R}^n$ containing the origen.

In what follows we will need the following useful result.

Proposition 4.2. *Let $\{a_j\}_{j\in\mathbb{N}}$ be a sequence of positives reals.*

a) $\left(\sum_{j=1}^{\infty} a_j\right)^{\theta} \leq \sum_{j=1}^{\infty} a_j^{\theta}$ *for any $0 \leq \theta \leq 1$. If $\sum_{j=1}^{\infty} a_j^{\theta} < \infty$.*

b) $\sum_{j=1}^{\infty} a_j^{\theta} \leq \left(\sum_{j=1}^{\infty} a_j\right)^{\theta}$ *for any $1 \leq \theta < \infty$. If $\sum_{j=1}^{\infty} a_j < \infty$.*

c) $\left(\sum_{j=1}^{N} a_j\right)^{\theta} \leq N^{\theta-1} \sum_{j=1}^{N} a_j^{\theta}$ *when $1 \leq \theta < \infty$.*

d) $\left(\sum_{j=1}^{N} a_j^{\theta}\right) \leq N^{1-\theta} \left(\sum_{j=1}^{N} a_j\right)^{\theta}$ *when $0 \leq \theta \leq 1$.*

Proof. (a) We proceed by induction. Note that if $0 \leq \theta \leq 1$, then $\theta - 1 \leq 0$, also $a_1 + a_2 \geq a_1$ and $a_1 + a_2 \geq a_2$ from this we have $(a_1 + a_2)^{\theta-1} \leq a_1^{\theta-1}$ and $(a_1 + a_2)^{\theta-1} \leq a_2^{\theta-1}$ and thus

$$a_1(a_1 + a_2)^{\theta-1} \leq a_1^{\theta} \quad \text{and} \quad a_2(a_1 + a_2)^{\theta-1} \leq a_2^{\theta}.$$

Hence

$$a_1(a_1 + a_2)^{\theta-1} + a_2(a_1 + a_2)^{\theta-1} \leq a_1^{\theta} + a_2^{\theta},$$

next, pulling out the common factor on the left hand side of the above inequality, we have

$$(a_1 + a_2)^{\theta-1}(a_1 + a_2) \leq a_1^{\theta} + a_2^{\theta},$$
$$(a_1 + a_2)^{\theta} \leq a_1^{\theta} + a_2^{\theta}.$$

Now, suppose that

$$\left(\sum_{j=1}^{n} a_j\right)^{\theta} \leq \sum_{j=1}^{n} a_j^{\theta},$$

holds. Since

$$\sum_{j=1}^{n} a_j + a_{n+1} \geq a_{n+1},$$

and

$$\sum_{j=1}^{n} a_j + a_{n+1} \geq \sum_{j=1}^{n} a_j,$$

we have

$$\left(\sum_{j=1}^{n} a_j + a_{n+1}\right)^{\theta-1} \leq a_{n+1}^{\theta-1},$$

and

$$\left(\sum_{j=1}^{n} a_j + a_{n+1}\right)^{\theta-1} \leq \left(\sum_{j=1}^{n} a_j\right)^{\theta-1}.$$

Hence

$$\begin{aligned}\left(\sum_{j=1}^{n} a_j + a_{n+1}\right)^{\theta-1}\left(\sum_{j=1}^{n} a_j + a_{n+1}\right) &\leq a_{n+1}^{\theta} + \left(\sum_{j=1}^{n} a_j\right)^{\theta}\\ \left(\sum_{j=1}^{n} a_j + a_{n+1}\right)^{\theta} &\leq a_{n+1}^{\theta} + \left(\sum_{j=1}^{n} a_j\right)^{\theta}\\ &\leq a_{n+1}^{\theta} + \sum_{j=1}^{n} a_j^{\theta} = \sum_{j=1}^{n+1} a_j^{\theta}.\end{aligned}$$

Since $\sum_{j=1}^{\infty} a_j^{\theta} < \infty$, we have

$$\left(\sum_{j=1}^{\infty} a_j\right)^{\theta} \leq \sum_{j=1}^{\infty} a_j^{\theta}.$$

(b) Since $\sum_{j=1}^{\infty} a_j < \infty$, then $\lim_{j\to\infty} a_j = 0$,
which implies that there exists $n_0 \in \mathbb{N}$ such that

$$0 < a_j < 1 \quad \text{if} \quad j \geq n_0, \quad \text{since} \quad 1 \leq \theta < \infty,$$

we obtain

$$a_j^{\theta} < a_j \quad \text{for all} \quad j \geq n_0.$$

From this we have

$$\sum_{j=1}^{\infty} a_j^{\theta} < \infty.$$

Consider the sequence $\{a_j^{\theta}\}_{j\in\mathbb{N}}$, since $1 \leq \theta$, then $0 < \dfrac{1}{\theta} \leq 1$ by part (a)

$$\left(\sum_{j=1}^{\infty} a_j^{\theta}\right)^{\frac{1}{\theta}} \leq \sum_{j=1}^{\infty} \left(a_j^{\theta}\right)^{\frac{1}{\theta}} = \sum_{j=1}^{\infty} a_j,$$

and thus

$$\sum_{j=1}^{\infty} a_j^{\theta} \leq \left(\sum_{j=1}^{\infty} a_j\right)^{\theta}.$$

(c) By Hölder's inequality we have

$$\sum_{j=1}^{N} a_j \leq \left(\sum_{j=1}^{N} 1\right)^{1-\frac{1}{\theta}} \left(\sum_{j=1}^{N} a_j^{\theta}\right)^{\frac{1}{\theta}} = N^{\frac{\theta-1}{\theta}} \left(\sum_{j=1}^{N} a_j^{\theta}\right)^{\frac{1}{\theta}},$$

then

$$\left(\sum_{j=1}^{N} a_j\right)^{\theta} \leq N^{\theta-1} \sum_{j=1}^{N} a_j^{\theta}.$$

(d) On more time, by Hölder's inequality

$$\sum_{j=1}^{N} a_j^{\theta} \leq \left(\sum_{j=1}^{N} 1\right)^{1-\theta} \left(\sum_{j=1}^{N} \left(a_j^{\theta}\right)^{\frac{1}{\theta}}\right)^{\theta} = N^{1-\theta} \left(\sum_{j=1}^{N} a_j\right)^{\theta}.$$

□

Proposition 4.3. *Let $f_1, \ldots, f_N$ be in $L_{(p,\infty)}$ then*

a) $\left\| \sum_{j=1}^{N} f_j \right\|_{L_{(p,\infty)}} \leq N \sum_{j=1}^{N} \|f_j\|_{L_{(p,\infty)}}$ *for* $1 \leq p < \infty$.

b) $\left\| \sum_{j=1}^{N} f_j \right\|_{L_{(p,\infty)}} \leq N^{\frac{1}{p}} \sum_{j=1}^{N} \|f_j\|_{L_{(p,\infty)}}$ *for* $0 < p < 1$.

Proof. First of all, note that for $\alpha > 0$ and $N \geq 1$

$$|f_1| + ... + |f_N| \geq |f_1 + f_2 + ... + f_N| > \alpha \geq \frac{\alpha}{N}.$$

Thus

$$\Big\{x \in X : |f_1 + f_2 + \ldots + f_N| > \alpha\Big\}$$
$$\subset \left\{x \in X : |f_1| > \frac{\alpha}{N}\right\} \cup \left\{x \in X : |f_1| > \frac{\alpha}{N}\right\} \cup \ldots \cup \left\{x \in X : |f_N| > \frac{\alpha}{N}\right\}.$$

Then

$$\mu\Big(\{x \in X : |f_1 + f_2 + ... + f_N| > \alpha\}\Big) \leq \sum_{j=1}^{N} \mu\left(\left\{x \in X : |f_j| > \frac{\alpha}{N}\right\}\right),$$

that is

$$D_{\sum f_j}(\alpha) \leq \sum_{j=1}^{N} D_{f_j}\left(\frac{\alpha}{N}\right).$$

Hence

$$\begin{aligned}
\Big\|\sum_{j=1}^{N} f_j\Big\|_{L_{(p,\infty)}}^{p} = \sup_{\alpha>0} \alpha^p D_{\sum f_j}(\alpha) &\leq \sum_{j=1}^{N} \sup_{\alpha>0} \alpha^p D_{f_j}\left(\frac{\alpha}{N}\right) \\
&= \sum_{j=1}^{N} \sup_{\alpha>0} \alpha^p D_{N f_j}(\alpha) \\
&= \sum_{j=1}^{N} \|N f_j\|_{L_{(p,\infty)}}^{p} = N^p \sum_{j=1}^{N} \|f_j\|_{L_{(p,\infty)}}^{p},
\end{aligned}$$

thus

$$\Big\|\sum_{j=1}^{N} f_j\Big\|_{L_{(p,\infty)}} \leq N \left(\sum_{j=1}^{N} \|f_j\|_{L_{(p,\infty)}}^{p}\right)^{\frac{1}{p}}.$$

By proposition (4.2) (a) since $0 < \frac{1}{p} < 1$ we have

$$\Big\|\sum_{j=1}^{N} f_j\Big\|_{L_{(p,\infty)}} \leq N \left(\sum_{j=1}^{N} \|f_j\|_{L_{(p,\infty)}}\right).$$

(b) As in part (a) we have

$$\Big\|\sum_{j=1}^{N} f_j\Big\|_{L_{(p,\infty)}}^{p} \leq N^p \left(\sum_{j=1}^{N} \big\|f_j\big\|_{L_{(p,\infty)}}^{p}\right).$$

Since $0 < p < 1$, then $1 < \frac{1}{p}$, next by proposition 4.2 (c) we have

$$\begin{aligned}
\Big\|\sum_{j=1}^{N} f_j\Big\|_{L_{(p,\infty)}} \leq N \left(\sum_{j=1}^{N} \big\|f_j\big\|_{L_{(p,\infty)}}^{p}\right)^{\frac{1}{p}} &\leq N(N^{\frac{1}{p}-1}) \sum_{j=1}^{N} \left(\big\|f_j\big\|_{L_{(p,\infty)}}^{p}\right)^{\frac{1}{p}} \\
&= N^{\frac{1}{p}} \sum_{j=1}^{N} \big\|f_j\big\|_{L_{(p,\infty)}}.
\end{aligned}$$

□

Proposition 4.4. *Give a measurable function f on (X,μ) and $\lambda>0$, define $f_\lambda = f\chi_{\{|f|>\lambda\}}$ and $f^\lambda = f - f_\lambda = f_\lambda = f\chi_{\{|f|\le\lambda\}}$.*

a) Then

$$D_{f_\lambda}(\alpha) = \begin{cases} D_f(\alpha) & \text{when } \alpha>\lambda \\ D_f(\lambda) & \text{when } \alpha\le\lambda. \end{cases}$$

$$D_{f^\lambda}(\alpha) = \begin{cases} 0 & \text{when } \alpha\ge\lambda \\ D_f(\alpha)-D_f(\lambda) & \text{when } \alpha<\lambda \end{cases}$$

b) If $f\in L_p(X,\mu)$. Then

$$\|f_\lambda\|_{L_p}^p = p\int_\lambda^\infty \alpha^{p-1}D_f(\alpha)\,d\alpha + \lambda^p D_f(\lambda),$$

$$\|f^\lambda\|_{L_p}^p = p\int_0^\lambda \alpha^{p-1}D_f(\alpha)\,d\alpha - \lambda^p D_f(\lambda),$$

$$\int_{\lambda<|f|\le\delta} |f|^p\,d\mu = p\int_\lambda^\delta \alpha^{p-1}D_f(\alpha)\,d\alpha - \delta^p D_f(\alpha) + \lambda^p D_f(\lambda).$$

c) If f is in $L_{(p,\infty)}$ then f^λ is in $L_q(X,\mu)$ for any $q>p$ and f_λ is in $L_q(X,\mu)$ for any $q<p$. Thus $L_{(p,\infty)}\subseteq L_{p_0}+L_{p_1}$ when $0<p_0<p<p_1\le\infty$.

Proof. (a) Note

$$D_{f_\lambda}(\alpha) = \mu\Big(\{x:|f(x)|\chi_{\{|f|>\lambda\}}(x)>\alpha\}\Big) = \mu\Big(\{x:|f(x)|>\alpha\}\cap\{x:|f|>\lambda\}\Big),$$

if $\alpha>\lambda$, then $\{x:|f(x)|>\alpha\}\subseteq\{x:|f|>\lambda\}$, thus

$$D_{f_\lambda}(\alpha) = \mu\Big(\{x:|f(x)|>\alpha\}\cap\{x:|f|>\lambda\}\Big) = \mu\Big(\{x:|f(x)|>\alpha\}\Big) = D_f(\alpha).$$

If $\alpha\le\lambda$, then $\{x:|f(x)|>\lambda\}\subseteq\{x:|f|>\alpha\}$, thus

$$D_{f_\lambda}(\alpha) = \mu\Big(\{x:|f(x)|>\alpha\}\cap\{x:|f|>\lambda\}\Big) = \mu\Big(\{x:|f(x)|>\lambda\}\Big) = D_f(\lambda).$$

And thus

$$D_{f_\lambda}(\alpha) = \begin{cases} D_f(\alpha) & \text{when } \alpha > \lambda \\ D_f(\lambda) & \text{when } \alpha \le \lambda. \end{cases} \tag{20}$$

Next, consider

$$\begin{aligned} D_{f^\lambda}(\alpha) &= \mu\Big(\{x : |f(x)|\chi_{\{|f|\le\lambda\}}(x) > \alpha\}\Big) \\ &= \mu\Big(\{x : |f(x)| > \alpha\} \cap \{x : |f| \le \lambda\}\Big), \end{aligned}$$

if $\alpha \ge \lambda$ then $\{x : |f| > \alpha\} \cap \{x : |f(x)| \le \lambda\} = \emptyset$, thus $D_{f^\lambda}(\alpha) = 0$.
If $\alpha < \lambda$, then

$$\begin{aligned} D_{f^\lambda}(\alpha) &= \mu\Big(\{x : |f(x)| > \alpha\} \cap \{x : |f(x)| \le \lambda\}\Big) \\ &= \mu\Big(\{x : |f(x)| > \alpha\} \cap \{x : |f(x)| > \lambda\}^c\Big) \\ &= \mu\Big(\{x : |f(x)| > \alpha\} \setminus \{x : |f(x)| > \lambda\}\Big) \\ &= \mu\Big(\{x : |f(x)| > \alpha\}\Big) - \mu\Big(\{x : |f(x)| > \lambda\}\Big) = D_f(\alpha) - D_f(\lambda). \end{aligned}$$

And hence

$$D_{f^\lambda}(\alpha) = \begin{cases} 0 & \text{when } \alpha \ge \lambda \\ D_f(\alpha) - D_f(\lambda) & \text{when } \alpha < \lambda. \end{cases} \tag{21}$$

(b) If $f \in L_p(X, \mu)$, then

$$\|f_\lambda\|_{L_p}^p = p\int_0^\infty \alpha^{p-1} D_{f_\lambda}(\alpha)\,d\alpha = p\int_0^\lambda \alpha^{p-1} D_{f_\lambda}(\alpha)\,d\alpha + p\int_\lambda^\infty \alpha^{p-1} D_{f_\lambda}(\alpha)\,d\alpha,$$

By part (a)(20) we have

$$\begin{aligned} \|f_\lambda\|_{L_p}^p &= p\int_0^\lambda \alpha^{p-1} D_f(\lambda)\,d\alpha + p\int_\lambda^\infty \alpha^{p-1} D_f(\alpha)\,d\alpha \\ &= \lambda^p D_f(\lambda) + p\int_\lambda^\infty \alpha^{p-1} D_f(\alpha)\,d\alpha. \end{aligned}$$

Also

$$\left\|f^{\lambda}\right\|_{L_p}^p = p\int_0^{\infty} \alpha^{p-1} D_{f^{\lambda}}(\alpha)\,d\alpha = p\int_0^{\lambda} \alpha^{p-1} D_{f^{\lambda}}(\alpha)\,d\alpha + p\int_{\lambda}^{\infty} \alpha^{p-1} D_{f^{\lambda}}(\alpha)\,d\alpha,$$

by part (a) (21) we obtain

$$\left\|f^{\lambda}\right\|_{L_p}^p = p\int_0^{\lambda} \alpha^{p-1}\Big(D_f(\alpha) - D_f(\lambda)\Big)\,d\alpha = p\int_0^{\lambda} \alpha^{p-1} D_f(\alpha)\,d\alpha - \lambda^p D_f(\lambda).$$

Next,

$$\begin{aligned}
&\int_{\lambda<|f|\le\delta} |f|^p\,d\mu \\
&\qquad = \int_{|f|>\lambda} |f|^p\,d\mu - \int_{|f|>\delta} |f|^p\,d\mu \\
&\qquad = \int_X |f|^p \chi_{\{|f|>\lambda\}}\,d\mu - \int_X |f|^p \chi_{\{|f|>\delta\}}\,d\mu \\
&\qquad = \int_X |f_\lambda|^p\,d\mu - \int_X |f_\delta|^p\,d\mu \\
&\qquad = p\int_{\lambda}^{\infty} \alpha^{p-1} D_f(\alpha)\,d\alpha + \lambda^p D_f(\lambda) - p\int_{\delta}^{\infty} \alpha^{p-1} D_f(\alpha)\,d\alpha - \delta^p D_f(\delta) \\
&\qquad = p\left(\int_{\lambda}^{\infty} \alpha^{p-1} D_f(\alpha)\,d\alpha - \int_{\delta}^{\infty} \alpha^{p-1} D_f(\alpha)\,d\alpha\right) + \lambda^p D_f(\lambda) - \delta^p D_f(\delta) \\
&\qquad = p\int_{\lambda}^{\delta} \alpha^{p-1} D_f(\alpha)\,d\alpha - \delta^p D_f(\alpha) + \lambda^p D_f(\lambda).
\end{aligned}$$

(c) We known that

$$D_f(\alpha) \le \frac{\left\|f\right\|_{L_{(p,\infty)}}^p}{\alpha^p},$$

then if $q > p$

$$\begin{aligned}\|f^\lambda\|_{L_q}^q &= q\int_0^\lambda \alpha^{q-1}D_f(\alpha)\,d\alpha - \lambda^q D_f(\lambda)\\ &\le q\int_0^\lambda \alpha^{q-1}\frac{\|f\|_{L_{(p,\infty)}}^p}{\alpha^p}\,d\alpha - \lambda^q D_f(\lambda)\\ &= q\|f\|_{L_{(p,\infty)}}^p\frac{\lambda^{q-p}}{q-p} - \lambda^q D_f(\lambda) \le q\|f\|_{L_{(p,\infty)}}^p\frac{\lambda^{q-p}}{q-p} < \infty.\end{aligned}$$

And thus $f^\lambda \in L_q$ if $q > p$.

Now, if $q < p$, then

$$\begin{aligned}\|f_\lambda\|_{L_q}^q &= q\int_\lambda^\infty \alpha^{q-1}D_f(\alpha)\,d\alpha + \lambda^q D_f(\lambda)\\ &\le q\|f\|_{L_{(p,\infty)}}^p\int_\lambda^\infty \alpha^{q-p-1}\,d\alpha + \lambda^q D_f(\lambda)\\ &= q\frac{\lambda^{q-p}}{p-q}\|f\|_{L_{(p,\infty)}}^p + \lambda^q D_f(\lambda) < \infty.\end{aligned}$$

Thus $f_\lambda \in L_q$ if $q < p$.
Finally, since $f \in L_{(p,\infty)}$ and

$$f = f^\lambda + f_\lambda,$$

where $f^\lambda \in L_{p_1}$ if $p < p_1$ and $f_\lambda \in L_{p_0}$ if $p_0 < p$. Then

$$L_{(p,\infty)} \subseteq L_{p_0} + L_{p_1} \quad \text{when} \quad 0 < p_0 < p < p_1 \le \infty.$$

□

Proposition 4.5. *Let (X,μ) be a measure space and let E be a subset of X with $\mu(E) < \infty$. Then*

a) for $0 < q < p$ we have

$$\int_E |f(x)|^q\,d\mu \le \frac{p}{p-q}\big[\mu(E)\big]^{1-\frac{q}{p}}\|f\|_{L_{(p,\infty)}}^q \quad \textit{for} \quad f \in L_{(p,\infty)}.$$

b) Conclude that if $\mu(X) < \infty$ and $0 < q < p$, then

$$L_p(X,\mu) \subseteq L_{(p,\infty)} \subseteq L_q(X,\mu).$$

Proof. Let $f \in L_{(p,\infty)}$, then

$$\int_E |f|^q \, d\mu$$

$$= q \int_0^\infty \lambda^{q-1} \mu\Big(\{x \in E : |f(x)| > \lambda\}\Big) \, d\lambda$$

$$\leq q \int_0^{\left[\mu(E)\right]^{-\frac{1}{p}} \|f\|_{L_{(p,\infty)}}} \lambda^{q-1} \mu(E) \, d\lambda + q \int_{\left[\mu(E)\right]^{-\frac{1}{p}} \|f\|_{L_{(p,\infty)}}}^\infty \lambda^{q-1} D_f(\lambda) \, d\lambda$$

$$\leq q \int_0^{\left[\mu(E)\right]^{-\frac{1}{p}} \|f\|_{L_{(p,\infty)}}} \lambda^{q-1} \mu(E) \, d\lambda + q \int_{\left[\mu(E)\right]^{-\frac{1}{p}} \|f\|_{L_{(p,\infty)}}}^\infty \lambda^{q-1} \frac{\|f\|^p_{L_{(p,\infty)}}}{\lambda^p} \, d\lambda$$

$$= \left(\left[\mu(E)\right]^{-\frac{1}{p}} \|f\|_{L_{(p,\infty)}}\right)^q \mu(E) + \frac{q}{p-q} \left(\left[\mu(E)\right]^{-\frac{1}{p}} \|f\|_{L_{(p,\infty)}}\right)^{q-p} \|f\|^p_{L_{(p,\infty)}}$$

$$= \left[\mu(E)\right]^{1-\frac{q}{p}} \|f\|^q_{L_{(p,\infty)}} + \frac{q}{p-q} \left[\mu(E)\right]^{1-\frac{q}{p}} \|f\|^q_{L_{(p,\infty)}}$$

$$= \frac{p}{p-q} \left[\mu(E)\right]^{1-\frac{q}{p}} \|f\|^q_{L_{(p,\infty)}}.$$

And thus

$$\int_E |f|^q \, d\mu \leq \frac{p}{p-q} \left[\mu(E)\right]^{1-\frac{q}{p}} \big\|f\big\|^q_{L_{(p,\infty)}}.$$

(b) If $\mu(X) < \infty$, then

$$\int_X |f|^q \, d\mu \leq \frac{p}{p-q} \left[\mu(X)\right]^{1-\frac{q}{p}} \big\|f\big\|^q_{L_{(p,\infty)}}.$$

Hence

$$L_p \subseteq L_{(p,\infty)} \subseteq L_q.$$

□

Corolario 4.1. *Let* (X, μ) *be a measurable space and let* E *be a subset of* X *with* $\mu(E) < \infty$*. Then*

$$\|f\|_{p/2} \leq \left[4\mu(E)\right]^{1/p} \left\|f\right\|_{L_{(p,\infty)}}.$$

And thus $L_{(p,\infty)} \subseteq L_{p/2}$*.*

Proof. Since $0 < \frac{p}{2} < p$ we can apply proposition 4.5 to obtain

$$\begin{aligned} \int_E |f|^{p/2}\, d\mu &\leq \frac{p}{p - \frac{p}{2}} [\mu(E)]^{1-\frac{p/2}{p}} \left\|f\right\|_{L_{(p,\infty)}}^{p/2} \\ &= 2[\mu(E)]^{1/2} \left\|f\right\|_{L_{(p,\infty)}}^{p/2} \\ \|f\|_{p/2} &\leq 2^{2/p} [\mu(E)]^{1/p} \left\|f\right\|_{L_{(p,\infty)}} \\ &= \left[4\mu(E)\right]^{1/p} \left\|f\right\|_{L_{(p,\infty)}}. \end{aligned}$$

From this last result one can see that

$$L_{(p,\infty)} \subseteq L_{p/2}.$$

□

5 Normability of Weak L_p for $p > 1$

Let $(X, \mathcal{A}, \mu)$ be a measure space and let $0 < p < \infty$. Pick $0 < r < p$ and define

$$\left|\left|\left|f\right|\right|\right|_{L_{(p,\infty)}} = \sup_{0<\mu(E)<\infty} \left[\mu(E)\right]^{-\frac{1}{r}+\frac{1}{p}} \left(\int_E |f|^r \, d\mu \right)^{\frac{1}{r}},$$

where the supremum is taken over all measurable subsets E of X of finite measure.

Proposition 5.1. *Let f be in $L_{(p,\infty)}$. Then*

$$\|f\|_{L_{(p,\infty)}} \leq |||f|||_{L_{(p,\infty)}} \leq \left(\frac{p}{p-r}\right)^{\frac{1}{r}} \|f\|_{L_{(p,\infty)}}.$$

Proof. By proposition 4.5 with $q = r$ we have

$$\begin{aligned}
|||f|||_{L_{(p,\infty)}} &= \sup_{0<\mu(E)<\infty} \big[\mu(E)\big]^{-\frac{1}{r}+\frac{1}{p}} \left(\int_E |f|^r \, d\mu\right)^{\frac{1}{r}} \\
&\leq \sup_{0<\mu(E)<\infty} \big[\mu(E)\big]^{-\frac{1}{r}+\frac{1}{p}} \left(\frac{p}{p-r}\big[\mu(E)\big]^{1-\frac{r}{p}} \|f\|^r_{L_{(p,\infty)}}\right)^{\frac{1}{r}} \\
&= \sup_{0<\mu(E)<\infty} \big[\mu(E)\big]^{-\frac{1}{r}+\frac{1}{p}} \left(\frac{p}{p-r}\right)^{\frac{1}{r}} \big[\mu(E)\big]^{\frac{1}{r}-\frac{1}{p}} \|f\|_{L_{(p,\infty)}} \\
&= \left(\frac{p}{p-r}\right)^{\frac{1}{r}} \|f\|_{L_{(p,\infty)}}.
\end{aligned}$$

On the other hand by definition

$$\big[\mu(E)\big]^{-\frac{1}{r}+\frac{1}{p}} \left(\int_E |f|^r \, d\mu\right)^{\frac{1}{r}} \leq |||f|||_{L_{(p,\infty)}},$$

for all $E \in \mathcal{A}$ such that $\mu(E) < \infty$ now, let us consider $A = \{x : |f(x)| > \alpha\}$ for $f \in L_{(p,\infty)}$. Observe that $\mu(A) < \infty$. Then

$$\begin{aligned}
|||f|||^p_{L_{(p,\infty)}} &\geq \left(\big[\mu(A)\big]^{-\frac{1}{r}+\frac{1}{p}} \left(\int_A |f|^r \, d\mu\right)^{\frac{1}{r}}\right)^p \\
&\geq \big[D_f(\alpha)\big]^{-\frac{p}{r}+1} \left(\int_A \alpha^r \, d\mu\right)^{\frac{p}{r}} \\
&= \big[D_f(\alpha)\big]^{-\frac{p}{r}+1} \alpha^p \big[D_f(\alpha)\big]^{\frac{p}{r}} = \alpha^p D_f(\alpha).
\end{aligned}$$

That is

$$\alpha^p D_f(\alpha) \leq |||f|||_{L_{(p,\infty)}},$$

and thus

$$\sup_{\alpha>0} \alpha^p D_f(\alpha) \leq |||f|||_{L_{(p,\infty)}}.$$

□

Lemma 5.1 (Fatou for $L_{(p,\infty)}$)**.** *For all measurable function g_n on X we have*

$$\left|\left| \liminf_{n\to\infty} |g_n| \right|\right|_{L_{(p,\infty)}} \leq C_p \liminf_{n\to\infty} \left|\left| g_n \right|\right|_{L_{(p,\infty)}}.$$

for some constant C_p that depends only on $p \in (0,\infty)$.

Proof.

$$\begin{aligned}
\left|\left| \liminf_{n\to\infty} |g_n| \right|\right|_{L_{(p,\infty)}} &\leq \left|\left|\left| \liminf_{n\to\infty} |g_n| \right|\right|\right|_{L_{(p,\infty)}} \\
&= \sup_{0<\mu(E)<\infty} \left[\mu(E)\right]^{-\frac{1}{r}+\frac{1}{p}} \left(\int_E \left(\liminf_{n\to\infty} |g_n| \right)^r d\mu \right)^{\frac{1}{r}} \\
&\leq \sup_{0<\mu(E)<\infty} \left[\mu(E)\right]^{-\frac{1}{r}+\frac{1}{p}} \left(\int_E \liminf_{n\to\infty} |g_n|^r \, d\mu \right)^{\frac{1}{r}}.
\end{aligned}$$

By Fatou's lemma

$$\begin{aligned}
&\leq \sup_{0<\mu(E)<\infty} \left[\mu(E)\right]^{-\frac{1}{r}+\frac{1}{p}} \left(\liminf_{n\to\infty} \int_E |g_n|^r \, d\mu \right)^{\frac{1}{r}} \\
&\leq \liminf_{n\to\infty} \sup_{0<\mu(E)<\infty} \left[\mu(E)\right]^{-\frac{1}{r}+\frac{1}{p}} \left(\int_E |g_n|^r \, d\mu \right)^{\frac{1}{r}} \\
&\leq \liminf_{n\to\infty} \left(\frac{p}{p-r} \right)^{\frac{1}{r}} \left\| g_n \right\|_{L_{(p,\infty)}} \\
&= \left(\frac{p}{p-r} \right)^{\frac{1}{r}} \liminf_{n\to\infty} \left\| g_n \right\|_{L_{(p,\infty)}}.
\end{aligned}$$

Finally

$$\left\| \liminf_{n\to\infty} |g_n| \right\|_{L_{(p,\infty)}} \leq \left(\frac{p}{p-r} \right)^{\frac{1}{r}} \liminf_{n\to\infty} \left\| g_n \right\|_{L_{(p,\infty)}}.$$

□

The following result is an improvement of lemma 5.1.

Lemma 5.2. *For all measurable functions g_n on X we have*

$$\big\| \liminf_{n\to\infty} |g_n| \big\|_{L_{(p,\infty)}} \leq \liminf_{n\to\infty} \big\| g_n \big\|_{L_{(p,\infty)}}.$$

Proof. Since

$$D_{\liminf_{n\to\infty} |g_n|}(\lambda) \leq \liminf_{n\to\infty} D_{g_n}(\lambda),$$

Then

$$\left\{ C > 0 : \liminf_{n\to\infty} D_{g_n}(\lambda) \leq \frac{C^p}{\lambda^p} \right\} \subseteq \left\{ C > 0 : D_{\liminf_{n\to\infty} |g_n|}(\lambda) \leq \frac{C^p}{\lambda^p} \right\},$$

and thus

$$\begin{aligned} \big\| \liminf_{n\to\infty} |g_n| \big\|_{L_{(p,\infty)}} &= \inf \left\{ C > 0 : D_{\liminf_{n\to\infty} |g_n|}(\lambda) \leq \frac{C^p}{\lambda^p} \right\} \\ &\leq \inf \left\{ C > 0 : \liminf_{n\to\infty} D_{g_n}(\lambda) \leq \frac{C^p}{\lambda^p} \right\} \\ &= \liminf_{n\to\infty} \left(\inf \left\{ C > 0 : D_{g_n}(\lambda) \leq \frac{C^p}{\lambda^p} \right\} \right) \\ &= \liminf_{n\to\infty} \big\| g_n \big\|_{L_{(p,\infty)}}. \end{aligned}$$

□

Proposition 5.2. *Let $0 < p < 1$, $0 < s < \infty$ and $(X, \mathcal{A}, \mu)$ be a measurable space*

a) Let f be a measurable function on X. Then

$$\int_{\{|f| \leq s\}} |f| \, d\mu \leq \frac{s^{1-p}}{1-p} \big\| f \big\|^p_{L_{(p,\infty)}}.$$

b) Let f_j, $1 \leq j \leq m$, be measurable functions on X. Then

$$\big\| \max_{1\leq j\leq m} |f_j| \big\|^p_{L_{(p,\infty)}} \leq \sum_{j=1}^{m} \big\| f_j \big\|^p_{L_{(p,\infty)}}.$$

And also

c)

$$\|f_1 + ... + f_m\|^p_{L_{(p,\infty)}} \le m \frac{2-p}{1-p} \sum_{j=1}^{m} \|f_j\|^p_{L_{(p,\infty)}}.$$

The latter estimate is refereed to as the p−normability of Weak L_p for $p < 1$.

Proof. By proposition 4.4 (b) with $p = 1$, we have

$$\begin{aligned}
\int\limits_{\{|f|\le s\}} |f|\, d\mu = \int\limits_X |f|\chi_{\{|f|\le s\}}\, d\mu &= \int\limits_X |f^s|\, d\mu \\
&= \int\limits_0^s D_f(\alpha)\, d\alpha - sD_f(s) \\
&\le \int\limits_0^s \frac{\alpha^p D_f(\alpha)}{\alpha^p}\, d\alpha \\
&\le \|f\|^p_{L_{(p,\infty)}} \int\limits_0^s \frac{d\alpha}{\alpha^p} = \frac{s^{1-p}}{1-p} \|f\|^p_{L_{(p,\infty)}}.
\end{aligned}$$

(b) Let $\max\limits_{1\le j\le k} |f_j(x)| = f_k(x)$ for some $1 \le k \le m$. Then

$$\begin{aligned}
D_{\max|f_j|}(\alpha) &= \mu\left(\{x : \max_{1\le j\le m} |f_j(x)| > \alpha\}\right) \\
&= \mu\Big(\{x : f_k(x) > \lambda\}\Big) = D_{f_k}(\alpha) \quad \text{for some} \quad 1 \le k \le m \\
&\le \sum_{j=1}^{m} D_{f_j}(\alpha).
\end{aligned}$$

Then

$$\alpha^p D_{\max|f_j|}(\alpha) \le \sum_{j=1}^{m} \sup \alpha^p D_{f_j}(\alpha),$$

and thus

$$\|\max_{1\le j\le m} |f_j|\|^p_{L_{(p,\infty)}} \le \sum_{j=1}^{m} \|f_j\|^p_{L_{(p,\infty)}}.$$

(c) Observe that

$$\max_{1\leq j\leq m} |f_j| \leq |f_1| + |f_2| + ... + |f_m|,$$

from this we have

$$\Big\{x : \max_{1\leq j\leq m} |f_j(x)| > \alpha\Big\} \subset \Big\{x : |f_1| + ... + |f_m| > \alpha\Big\},$$

then

$$\begin{aligned}&\Big\{x : |f_1| + ... + |f_m| > \alpha\Big\}\\ &= \left(\Big\{x : |f_1| + ... + |f_m| > \alpha\Big\} \cap \Big\{x : \max_{1\leq j\leq m} |f_j(x)| \leq \alpha\Big\}\right)\\ &\quad \cup \Big\{x : \max_{1\leq j\leq m} |f_j(x)| > \alpha\Big\}.\end{aligned}$$

And thus

$$\begin{aligned}D_{f_1+...+f_m}(\alpha) &= \mu\big(\{x : |f_1 + ... + f_m| > \alpha\}\big)\\ &\leq \mu\big(\{x : |f_1| + ... + |f_m| > \alpha\}\big)\\ &\leq \mu\left(\{x : |f_1| + ... + |f_m| > \alpha\} \cap \{x : \max_{1\leq j\leq m} |f_j(x)| \leq \alpha\}\right)\\ &\quad + \mu\left(\{x : \max_{1\leq j\leq m} |f_j(x)| > \alpha\}\right)\\ &= \mu\left(\Big\{x \in \{x : \max_{1\leq j\leq m} |f_j(x)| \leq \alpha\} : |f_1| + ... + |f_m| > \alpha \Big\}\right)\\ &\quad + \mu\left(\{x : \max_{1\leq j\leq m} |f_j(x)| > \alpha\}\right)\\ &= \mu\left(\Big\{x : \big(|f_1| + ... + |f_m|\big)\chi_{\{x:\max|f_j|\leq\alpha\}} > \alpha\Big\}\right)\\ &\quad + \mu\left(\{x : \max_{1\leq j\leq m} |f_j(x)| > \alpha\}\right).\end{aligned}$$

By Chebyshev's inequality

$$
\begin{aligned}
D_{f_1+\ldots+f_m}(\alpha) &\leq \frac{1}{\alpha} \int\limits_{\{x:\max|f_j|\leq\alpha\}} \Big(|f_1| + \ldots + |f_m|\Big)\, d\mu + D_{\max|f_j|}(\alpha) \\
&= \sum_{j=1}^{m} \frac{1}{\alpha} \int\limits_{\{x:\max|f_j|\leq\alpha\}} |f_j|\, d\mu + D_{\max|f_j|}(\alpha) \\
&\leq \sum_{j=1}^{m} \frac{1}{\alpha} \int\limits_{\{x:\max|f_j|\leq\alpha\}} \max_{1\leq j\leq m} |f_j|\, d\mu + D_{\max|f_j|}(\alpha).
\end{aligned}
$$

By part (a) we have

$$
\begin{aligned}
&\sum_{j=1}^{m} \frac{1}{\alpha} \int\limits_{\{x:\max|f_j|\leq\alpha\}} \max_{1\leq j\leq m} |f_j|\, d\mu + D_{\max|f_j|}(\alpha) \\
&\leq \sum_{j=1}^{m} \frac{1}{\alpha} \frac{\alpha^{1-p}}{1-p} \Big\| \max_{1\leq j\leq m} |f_j| \Big\|^p_{L_{(p,\infty)}} + \frac{\big\| \max_{1\leq j\leq m} |f_j| \big\|^p_{L_{(p,\infty)}}}{\alpha^p} \\
&\leq \sum_{j=1}^{m} \frac{\alpha^{-p}}{1-p} \Big\| \max_{1\leq j\leq m} |f_j| \Big\|^p_{L_{(p,\infty)}} + \sum_{j=1}^{m} \frac{\big\| \max_{1\leq j\leq m} |f_j| \big\|^p_{L_{(p,\infty)}}}{\alpha^p}.
\end{aligned}
$$

Finally by part (b) we obtain

$$
\begin{aligned}
\alpha^p D_{f_1+\ldots+f_m}(\alpha) &\leq \sum_{j=1}^{m} \Big(\frac{1}{1-p} + 1\Big) \Big\| \max_{1\leq j\leq m} |f_j| \Big\|^p_{L_{(p,\infty)}} \\
&= \sum_{j=1}^{m} \left(\frac{2-p}{1-p}\right) \Big\| \max_{1\leq j\leq m} |f_j| \Big\|^p_{L_{(p,\infty)}} \\
&\leq \frac{2-p}{1-p} \sum_{j=1}^{m} \sum_{j=1}^{m} \big\| f_j \big\|^p_{L_{(p,\infty)}} \\
&= m \frac{2-p}{1-p} \sum_{j=1}^{m} \big\| f_j \big\|^p_{L_{(p,\infty)}}.
\end{aligned}
$$

□

Proposition 5.3 (Lyapunov's inequality for Weak L_p)**.** *Let (X,μ) be measurable space. Suppose that $0<p_0<p<p_1<\infty$ and $\frac{1}{p}=\frac{1-\theta}{p_0}+\frac{\theta}{p_1}$ for some $\theta\in[0,1]$. If $f\in L_{(p_0,\infty)}\cap L_{(p_1,\infty)}$ then $f\in L_{(p,\infty)}$ and*

$$\big\|f\big\|_{L_{(p,\infty)}} \le \big\|f\big\|^{1-\theta}_{L_{(p_0,\infty)}}\big\|f\big\|^{\theta}_{L_{(p_1,\infty)}}.$$

Proof. Observe that

$$\begin{aligned}\alpha^p D_f(\alpha) &= \alpha^{p(1-\theta+\theta)}\big[D_f(\alpha)\big]^{p\left(\frac{1}{p}\right)}\\
&= \alpha^{p(1-\theta)}\alpha^{p\theta}\big[D_f(\alpha)\big]^{p\left(\frac{1-\theta}{p_0}+\frac{\theta}{p_1}\right)}\\
&= \alpha^{p(1-\theta)}\big[D_f(\alpha)\big]^{p\left(\frac{1-\theta}{p_0}\right)}\alpha^{p\theta}\big[D_f(\alpha)\big]^{\frac{p\theta}{p_1}}\\
&= \big[\alpha^{p_0}D_f(\alpha)\big]^{p\left(\frac{1-\theta}{p_0}\right)}\big[\alpha^{p_1}D_f(\alpha)\big]^{\frac{p\theta}{p_1}}.\end{aligned}$$

Thus

$$\alpha^p D_f(\alpha) \le \left[\sup_{\alpha>0}\alpha^{p_0}D_f(\alpha)\right]^{p\left(\frac{1-\theta}{p_0}\right)}\left[\sup_{\alpha>0}\alpha^{p_1}D_f(\alpha)\right]^{\frac{p\theta}{p_1}}$$

$$\alpha^p D_f(\alpha) \le \left[\big\|f\big\|^{p_0}_{L_{(p_0,\infty)}}\right]^{p\left(\frac{1-\theta}{p_0}\right)}\left[\big\|f\big\|^{p_1}_{L_{(p_1,\infty)}}\right]^{\frac{p\theta}{p_1}},$$

finally

$$\sup_{\alpha>0}\alpha^p D_f(\alpha) \le \left[\big\|f\big\|^{p_0}_{L_{(p_0,\infty)}}\right]^{p\left(\frac{1-\theta}{p_0}\right)}\left[\big\|f\big\|^{p_1}_{L_{(p_1,\infty)}}\right]^{\frac{p\theta}{p_1}}$$

$$\big\|f\big\|^p_{L_{(p,\infty)}} \le \left[\big\|f\big\|^{1-\theta}_{L_{(p_0,\infty)}}\big\|f\big\|^{\theta}_{L_{(p_1,\infty)}}\right]^p$$

$$\big\|f\big\|_{L_{(p,\infty)}} \le \big\|f\big\|^{1-\theta}_{L_{(p_0,\infty)}}\big\|f\big\|^{\theta}_{L_{(p_1,\infty)}}.$$

□

Theorem 5.1 (Hölder's inequality for Weak spaces). *Let f_j be in $L_{(p_j,\infty)}$ where $0 < p_j < \infty$ and $1 \leq j \leq k$. Let*

$$\frac{1}{p} = \frac{1}{p_1} + ... + \frac{1}{p_k}.$$

Then

$$\left\|f_1...f_k\right\|_{L_{(p,\infty)}} \leq p^{-\frac{1}{p}} \prod_{j=1}^{k} p_j^{\frac{1}{p_j}} \prod_{j=1}^{k} \left\|f_j\right\|_{L_{(p_j,\infty)}}.$$

Proof. Let us consider $\left\|f_j\right\|_{L_{(p_j,\infty)}} = 1$, $1 \leq j \leq k$. And let $x_1, \ldots, x_n$ be a positive real numbers such that

$$\frac{1}{x_1} \cdots \frac{1}{x_k} = \alpha,$$

then

$$D_{f_1...f_k}(\alpha) = D_{f_1...f_k}\left(\frac{1}{x_1} \cdots \frac{1}{x_k}\right)$$

$$\leq D_{f_1}\left(\frac{1}{x_1}\right) + D_{f_2}\left(\frac{1}{x_2}\right) + \ldots + D_{f_k}\left(\frac{1}{x_k}\right), \tag{22}$$

since

$$1 = \left\|f_j\right\|^{p_j}_{L_{(p_j,\infty)}} \geq \sup_j \left(\frac{1}{x_j}\right)^{p_j} D_{f_j}\left(\frac{1}{x_j}\right),$$

then

$$\left(\frac{1}{x_j}\right)^{p_j} D_{f_j}\left(\frac{1}{x_j}\right) \leq 1,$$

thus

$$D_{f_j}\left(\frac{1}{x_j}\right) \leq x_j^{p_j} \quad \text{for} \quad 1 \leq j \leq k.$$

Hence, we can write (22) as follows

$$D_{f_1...f_k}\left(\frac{1}{x_1} \cdots \frac{1}{x_k}\right) \leq x_1^{p_1} + x_2^{p_2} + \ldots + x_k^{p_k}.$$

Next, let us define

$$F(x_1, \ldots, x_k) = x_1^{p_1} + x_2^{p_2} + \ldots + x_k^{p_k}.$$

In what follows, we will use the Lagrange multipliers in order to obtain the minimum value of F subject to the constrain

$$\frac{1}{x_1} \ldots \frac{1}{x_k} = \alpha.$$

That is

$$f(x_1, x_2, \ldots, x_k) = x_1^{p_1} + x_2^{p_2} + \ldots + x_k^{p_k}$$
$$g(x_1, x_2, \ldots, x_k) = x_1 x_2 \ldots x_k - \frac{1}{\alpha}.$$

Then, next

$$\nabla F = \lambda \nabla g.$$

And thus

$$p_1 x_1^{p_1 - 1} = \lambda(x_2 x_3 \ldots x_k)$$
$$p_2 x_2^{p_2 - 1} = \lambda(x_1 x_3 \ldots x_k)$$
$$\vdots$$
$$p_j x_j^{p_j - 1} = \lambda(x_1 x_3 \ldots x_k),$$

thus

$$p_1 x_1^{p_1} = \lambda(x_1 x_2 \ldots x_k)$$
$$p_2 x_2^{p_2} = \lambda(x_1 x_2 \ldots x_k)$$
$$\vdots$$
$$p_j x_j^{p_j} = \lambda(x_1 x_2 \ldots x_k).$$

Observe that

$$x_1 x_2 \ldots x_k = \frac{1}{\alpha}. \tag{23}$$

On the other hand note that

$$p_1 x_1^{p_1} = p_j x_j^{p_j} \quad \text{for} \quad 2 \leq j \leq k. \tag{24}$$

Now replacing (24) into (23) we have

$$x_1 \left(\frac{p_1}{p_2}\right)^{\frac{1}{p_2}} x_1^{\frac{p_1}{p_2}} \left(\frac{p_1}{p_3}\right)^{\frac{1}{p_3}} x^{\frac{p_1}{p_3}} \ldots \left(\frac{p_1}{p_k}\right)^{\frac{1}{p_k}} x_1^{\frac{p_1}{p_k}}$$

$$= x_1 \left(\frac{p_1}{p_2}\right)^{\frac{1}{p_2}} \left(\frac{p_1}{p_3}\right)^{\frac{1}{p_3}} \ldots \left(\frac{p_1}{p_k}\right)^{\frac{1}{p_k}} x_1^{\frac{p_1}{p_2}+\frac{p_1}{p_3}+\ldots+\frac{p_1}{p_k}} = \frac{1}{\alpha}, \tag{25}$$

but

$$\left(\frac{p_1}{p_1}\right)^{\frac{1}{p_1}} = 1,$$

then we can write (25) as follows

$$\left(\frac{p_1}{p_1}\right)^{\frac{1}{p_1}} \left(\frac{p_1}{p_2}\right)^{\frac{1}{p_2}} \left(\frac{p_1}{p_3}\right)^{\frac{1}{p_3}} \ldots \left(\frac{p_1}{p_k}\right)^{\frac{1}{p_k}} x_1^{\frac{p_1}{p_1}+\frac{p_1}{p_2}+\frac{p_1}{p_3}+\ldots+\frac{p_1}{p_k}} = \frac{1}{\alpha}.$$

And, thus

$$\frac{p_1^{\frac{1}{p_1}+\frac{1}{p_2}+\ldots+\frac{1}{p_k}}}{\prod_{j=1}^{k} p_j^{\frac{1}{p_j}}} x_1^{p_1\left(\frac{1}{p_1}+\frac{1}{p_2}+\ldots+\frac{1}{p_k}\right)} = \frac{1}{\alpha}.$$

Then

$$p_1^{\frac{1}{p}} x^{\frac{p_1}{p}} = \frac{\prod_{j=1}^{k} p_j^{\frac{1}{p_j}}}{\alpha},$$

hence

$$x_1^{p_1} = \frac{1}{p_1 \alpha^p} \left[\prod_{j=1}^{k} p_j^{\frac{1}{p_j}}\right]^p.$$

Therefore the $x_1 \ldots x_k$ such that

$$\begin{cases} x_1^{p_1} = \dfrac{1}{p_1 \alpha^p} \left[\prod_{j=1}^{k} p_j^{\frac{1}{p_j}}\right]^p \\ \\ x_j^{p_j} = \dfrac{p_1}{p_j} x_1^{p_1} \end{cases} \tag{26}$$

are the unique critical real point.

For this critical real point, using (26) we have

$$x_1^{p_1} + x_2^{p_2} + \ldots + x_k^{p_k} = x_1^{p_1} + \frac{p_1}{p_2} x_1^{p_1} + \ldots + \frac{p_1}{p_k} x_1^{p_1}$$

$$= p_1 x_1^{p_1} \left[\frac{1}{p_1} + \ldots + \frac{1}{p_k} \right]$$

$$= \frac{1}{\alpha^p} \left[\prod_{j=1}^{k} p_j^{\frac{1}{p_j}} \right]^p \frac{1}{p}.$$

On the other hand observe that one can make the function

$$F(x_1, \ldots, x_k) = x_1^{p_1} + \ldots + x_k^{p_k},$$

subject to the constrain

$$x_1 x_2 \ldots x_k = \frac{1}{\alpha},$$

as big as one wish. Indeed if $x_1 = \frac{M}{\alpha}$, $x_2 = \frac{1}{M}$ and $x_j = 1$ for $3 \leq j \leq k$. Then

$$F(x_1, \ldots, x_k) = x_1^{p_1} + x_2^{p_2} + \ldots + x_k^{p_k}$$

$$= \left(\frac{M}{\alpha} \right)^{p_1} + \left(\frac{1}{M} \right)^{p_1} + 1 + \ldots + 1$$

$$= \left(\frac{M}{\alpha} \right)^{p_1} + \left(\frac{1}{M} \right)^{p_1} + k - 2 \to \infty,$$

as $M \to \infty$, therefore the critical part (26) is a minimum. Then

$$D_{f_1 \ldots f_k}(\alpha) \leq \frac{1}{p\alpha^p} \left[\prod_{j=1}^{k} p_j^{\frac{1}{p_j}} \right]^p$$

$$\alpha^p D_{f_1 \ldots f_k}(\alpha) \leq \frac{1}{p} \left[\prod_{j=1}^{k} p_j^{\frac{1}{p_j}} \right]^p,$$

thus, we have

$$\alpha^p D_{f_1 \dots f_k}(\alpha) \leq \frac{1}{p} \left[\prod_{j=1}^{k} p_j^{\frac{1}{p_j}} \right]^p$$

$$\left\| f_1 \dots f_k \right\|_{L_{(p,\infty)}} \leq \left(\frac{1}{p} \right)^{\frac{1}{p}} \left(\prod_{j=1}^{k} p_j^{\frac{1}{p_j}} \right) \prod_{j=1}^{k} \left\| f_j \right\|_{L_{(p_j,\infty)}}, \tag{27}$$

since $\left\| f_j \right\|_{L_{(p_j,\infty)}} = 1$.

In general, if $\left\| f_j \right\|_{L_{(p_j,\infty)}} \neq 1$, $1 \leq j \leq k$ choose $g_j = \dfrac{f_j}{\|f_j\|_{L_{(p_j,\infty)}}}$ and use (27)

□

Theorem 5.2 (Completeness)**.** *Weak L_p with the quasi-norm $\|.\|_{L_{(p,\infty)}}$ is complete for all $0 < p < \infty$.*

Proof. Let $\{f_n\}_{n\in\mathbb{N}}$ be a Cauchy sequence in $\Big(\text{Weak } L_p, \|.\|_{L_{(p,\infty)}}\Big)$. Then for every $\epsilon > 0$ there exists an $n_0 \in \mathbb{N}$ such that

$$\|f_n - f_m\|_{L_{(p,\infty)}} < \epsilon^{\frac{1}{p}+1}$$

if $m, n \geq n_0$, that is,

$$\left(\sup_{\lambda > 0} \lambda^p D_{f_n - f_m}(\lambda) \right)^{1/p} = \|f_n - f_m\|_{L_{(p,\infty)}} < \epsilon^{\frac{1}{p}+1},$$

taking $\lambda = \epsilon$ we have

$$\epsilon^p \mu\Big(\{ x \in X : |f_n(x) - f_m(x)| > \epsilon \} \Big) < \epsilon^{p+1},$$

for $m, n \geq n_0$. Hence

$$\mu\Big(\{ x \in X : |f_n(x) - f_m(x)| > \epsilon \} \Big) < \epsilon,$$

for $m, n \geq n_0$. This means that $\{f_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence in the measure μ. We therefore apply theorem 3.2 and conclude that there exists an $\mathcal{A}-$measurable function f such that some subsequence of $\{f_n\}_{n\in\mathbb{N}}$ converges to f $\mu-$a.e. Let $\{f_{n_k}\}_{k\in\mathbb{N}}$ be such subsequence of $\{f_n\}_{n\in\mathbb{N}}$ of $\{f_n\}_{n\in\mathbb{N}}$ then $f_{n_k} \to f$ μ-a.e as $k \to \infty$. If we apply twice lemma 5.2 we obtain firstly

$$\begin{aligned}\|f\|_{L_{(p,\infty)}} &= \|\liminf |f_{n_k}|\|_{L_{(p,\infty)}} \\ &\leq \liminf \|f_{n_k}\|_{L_{(p,\infty)}} < \infty,\end{aligned}$$

thus $f \in$ Weak L_p.
Secondly

$$\begin{aligned}\|f - f_n\|_{L_{(p,\infty)}} &= \|\liminf |f_{n_k} - f_n|\|_{L_{(p,\infty)}} \\ &\leq \liminf \|f_{n_k} - f_n\|_{L_{(p,\infty)}} < \epsilon^{\frac{1}{p}+1},\end{aligned}$$

if $n_k, n \geq n_0$.
This prove that Weak L_p is complete for $0 < p < \infty$. $\square$

6 Multiplication Operators

Let $F(X)$ be a function space on non-empty set X. Let $u : X \to \mathbb{C}$ be a function such that $u.f$ on $F(X)$ whenever $f \in F(X)$.
Then the transformation $f \to u.f$ on $F(X)$ is denoted by M_u. In case $F(X)$ is a topological space and M_u is continuous, we call it a multiplication operator induced by u.

In this section boundedness and invertibility of the multiplication M_u are characterized in terms of the boundedness and invertibility of the complex-valued measurable function u respectively.

Theorem 6.1. *The linear transformation $M_u : f \to u.f$ on the Weak L_p spaces is bounded if only if u is essentially bounded. Moreover*

$$\big\|M_u\big\| = \|u\|_\infty.$$

Proof. Let $u \in L_\infty(u)$, then we find

$$\begin{aligned}
\big\|M_u f\big\|_{L_{(p,\infty)}} &= \sup_{\lambda>0} \lambda \Big[D_{M_u f}(\lambda)\Big]^{1/p} \\
&= \sup_{\lambda>0} \lambda \Big[\mu\big(\{x \in X : |M_u f(x)| > \lambda\}\big)\Big]^{1/p} \\
&= \sup_{\lambda>0} \lambda \Big[\mu\big(\{x \in X : |(u.f)(x)| > \lambda\}\big)\Big]^{1/p} \\
&\leq \sup_{\lambda>0} \lambda \Big[\mu\Big(\Big\{x \in X : |f(x)| > \tfrac{\lambda}{\|u\|_\infty}\Big\}\Big)\Big]^{1/p}.
\end{aligned}$$

$$\text{Since} \quad \big\{x \in X : |(u.f)(x)| > \lambda\big\} \subset \big\{x \in X : \|u\|_\infty |f(x)| > \lambda\big\} = \Big\{x \in X : |f(x)| > \tfrac{\lambda}{\|u\|_\infty}\Big\}.$$

Putting $\alpha = \dfrac{\lambda}{\|u\|_\infty}$ we have

$$\begin{aligned}
&\sup_{\lambda>0} \lambda\Big[\mu\Big(\Big\{x \in X : |f(x)| > \tfrac{\lambda}{\|u\|_\infty}\Big\}\Big)\Big]^{1/p} \\
&= \sup_{\alpha>0} \alpha\|u\|_\infty \Big[\mu\big(\{x \in X : |f(x)| > \alpha\}\big)\Big]^{1/p} \\
&= \|u\|_\infty \sup_{\alpha>0} \alpha \Big[\mu\big(\{x \in X : |f(x)| > \alpha\}\big)\Big]^{1/p} \\
&= \|u\|_\infty \big\|f\big\|_{L_{(p,\infty)}}.
\end{aligned}$$

Hence, we have proved that

$$\big\|M_u f\big\|_{L_{(p,\infty)}} \leq \|u\|_\infty \big\|f\big\|_{L_{(p,\infty)}}. \tag{28}$$

Conversely, suppose M_u is a bounded operator. If u is not essentially bounded function, then for every $n \in \mathbb{N}$, the set $E_n = \big\{x \in X : |u(x)| > n\big\}$ has positive measure and note that

$$\big\{x \in X : n\chi_{E_n}(x) > \lambda\big\} \subset \big\{x \in X : |u\chi_{E_n}(x)| > \lambda\big\},$$

then

$$\begin{aligned}
&\sup_{\lambda>0} \lambda\Big[\mu\Big(\{x \in X : n\chi_{E_n}(x) > \lambda\}\Big)\Big]^{1/p} \\
&\leq \sup_{\lambda>0} \lambda\Big[\mu\Big(\{x \in X : |u\chi_{E_n}(x)| > \lambda\}\Big)\Big]^{1/p}.
\end{aligned}$$

Thus

$$n\big\|\chi_{E_n}\big\|_{L_{(p,\infty)}} \leq \big\|M_u\chi_{E_n}\big\|_{L_{(p,\infty)}}.$$

This contradicts the boundedness of M_u.

Clearly from (28) we have

$$\big\|M_u\big\| \leq \|u\|_\infty. \tag{29}$$

Next, for $\epsilon > 0$, let

$$E = \Big\{x \in X : |u(x)| > \|u\|_\infty - \epsilon\Big\}.$$

Note $\mu(E) > 0$. Then

$$\Big\{x \in X : \big(\|u\|_\infty - \epsilon\big)\chi_E(x) > \lambda\Big\} \subset \Big\{x \in X : |u\chi_E(x)| > \lambda\Big\},$$

and thus

$$\sup_{\lambda>0} \lambda\Big[\mu\Big(\ \{x \in X : \big(\|u\|_\infty - \epsilon\big)\chi_E(x) > \lambda\}\ \Big)\Big]^{1/p}$$
$$\leq \sup_{\lambda>0} \lambda\Big[\mu\Big(\ \{x \in X : |u\chi_{E_n}(x)| > \lambda\}\ \Big)\Big]^{1/p}.$$

Therefore

$$\big(\|u\|_\infty - \epsilon\big)\big\|\chi_E\big\|_{L_{(p,\infty)}} \leq \big\|M_u\chi_E\big\|_{L_{(p,\infty)}}$$

$$\|u\|_\infty - \epsilon \leq \frac{\big\|M_u\chi_E\big\|_{L_{(p,\infty)}}}{\big\|\chi_E\big\|_{L_{(p,\infty)}}} \leq \big\|M_u\big\|.$$

Thus

$$\|u\|_\infty \leq \big\|M_u\big\|, \tag{30}$$

finally from (29) and (30)

$$\big\|M_u\big\| = \|u\|_\infty.$$

□

Theorem 6.2. *The set of all multiplication operator on Weak*L_p*'is an maximal-abelian subalgebra of the set* $B\,(Weak\,L_p)$*, the algebra of all bounded linear operation on Weak* L_p*.*

Proof. Let

$$\mathcal{H} = \Big\{M_u : u \in L_\infty\Big\},$$

and consider the operator product

$$M_u.M_v = M_{uv},$$

where M_u, $M_v \in \mathcal{H}$, let us check that $\mathcal{H}$ is a Banach algebra. Lea u, $v \in L_\infty$ then $|u| \leq \|u\|_\infty$ and $|v| \leq \|v\|_\infty$ therefore:

$$\|uv\|_\infty \leq \|v\|_\infty\|u\|_\infty,$$

this implies that product is an inner operation, moreover the usual function product is associative, commutative and distributive with respect to the sum and the scalar product, thus we conclude that $\mathcal{H}$ is a subalgebra of $B\,(\text{Weak } L_p)$.

Now, we like to check that $\mathcal{H}$ is a maximal subalgebra, that is, given $N \in B\,(\text{Weak } L_p)$, if N commute with $\mathcal{H}$ we have to prove that $N \in \mathcal{H}$.

Consider the unit function $e : X \to \mathbb{C}$ defined by $e(x) = 1$ for all $x \in X$ let $N \in B\,(\text{Weak } L_p)$ be an operator which commute with $\mathcal{H}$ and let χ_E be the characteristic function of a measurable set E. Then

$$\begin{aligned} N(\chi_E) &= N\left[M_{\chi_E}(e)\right] \\ &= M_{\chi_E}\left[N(e)\right] \\ &= \chi_E.N(e) \\ &= N(e).\chi_E \\ &= M_w(\chi_E), \end{aligned}$$

where $w = N(e)$. Similarly

$$N(s) = M_w(s), \tag{31}$$

for any simple function.
Now, let us check that $w \in L_\infty$. By way of contradiction assume that $w \notin L_\infty$, then the set

$$E_n = \Big\{x \in X : |w(x)| > n\Big\},$$

has a positive measure for each $n \in \mathbb{N}$. Note that:

$$M_w(\chi_{E_n})(x) = \big(w\chi_{E_n}\big)(x) \geq n\chi_{E_n}(x),$$

for all $x \in X$. By the monotonicity (Property 1) of the distribution function, we have

$$D_{w\chi_{E_n}}(\lambda) \geq D_{\chi_{E_n}}\left(\frac{\lambda}{n}\right),$$

thus

$$\sup_{\lambda>0} \lambda^p D_{w\chi_{E_n}}(\lambda) \geq \sup_{\lambda>0} \lambda^p D_{\chi_{E_n}}\left(\frac{\lambda}{n}\right),$$

Putting $\alpha = \frac{\lambda}{n}$ we have

$$\begin{aligned}\big\|w\chi_{E_n}\big\|^p_{L_{(p,\infty)}} = \sup_{\lambda>0} \lambda^p D_{w\chi_{E_n}}(\lambda) &\geq n^p \sup_{\alpha>0} \alpha^p D_{\chi_{E_n}}(\alpha) \\ &= n^p \big\|\chi_{E_n}\big\|^p_{L_{(p,\infty)}},\end{aligned}$$

since χ_{E_n} is a simple function then by (31) we have

$$M_w(\chi_{E_n}) = N(\chi_{E_n}),$$

Hence

$$\big\|N(\chi_{E_n})\big\|_{L_{(p,\infty)}} \geq n \big\|\chi_{E_n}\big\|_{L_{(p,\infty)}},$$

Therefore N is a unbounded operator. This is a contradiction to the fact N is bounded.
So then $w \in L_\infty$ and by theorem 6.1 M_w is bounded.

Next, given $f \in \text{Weak}\, L_p$ there exists a nondecreasing sequence $\{s_n\}_{n\in\mathbb{N}}$ of a measurable simple functions such that $\lim\limits_{n\to\infty} s_n = f$, then by (31) we have

$$\begin{aligned}N(f) = N\left(\lim_{n\to\infty} s_n\right) = \lim_{n\to\infty} N(s_n) &= \lim_{n\to\infty} M_w(s_n) \\ &= M_w\left(\lim_{n\to\infty} s_n\right) \\ &= M_w(f).\end{aligned}$$

Therefore, $N(f) = M_w(f)$ for all $f \in \text{Weak}\, L_p$ and thus we conclude that $N \in \mathcal{H}$.

□

Corolario 6.1. *The multiplication operator M_u is invertible if only if u is invertible on L_∞.*

Proof. Let M_u be invertible, the there exists $N \in B\big(\text{Weak}\, L_p\big)$ such that:

$$M_u.N = N.M_u = I, \tag{32}$$

where I represent the identity operator. Let us check that N commute with $\mathcal{H}$. Let $M_w \in \mathcal{H}$, then:

$$M_w.M_u = M_u.M_w. \tag{33}$$

Applying N to (33) and by (32) we obtain:

$$N.M_w.M_u.N = N.M_u.M_w.N,$$
$$N.M_w.I = I.M_w.N,$$
$$N.M_w = M_w.N,$$

and thus we concluded that N commute with $\mathcal{H}$, by theorem 6.2 $N \in \mathcal{H}$ then there exists $g \in L_\infty$ such that $N = M_g$, hence

$$M_u.M_g = M_g.M_u = I,$$

this implies that $ug = gu = 1$, a.e$[\mu]$ this means that u is invertible on L_∞.

On the other hand, assume u is invertible on L_∞ that is, $\dfrac{1}{u} \in L_\infty$, then:

$$\begin{aligned} M_u.M_{1/u} = M_{1/u}.M_u &= M_{(1/u)u} \\ &= M_1 = I, \end{aligned}$$

which means that M_u is invertible on $B\,(\text{Weak } L_p)$.

□

Lemma 6.1. *Let M_u be a compact operator, for $\epsilon > 0$ define*

$$A_\epsilon(u) = \big\{x \in X : |u(x)| \geq \epsilon\big\},$$

and

$$Weak\, L_p\big[A_\epsilon(u)\big] = \Big\{f\chi_{A_\epsilon(u)} : f \in Weak\, L_p\Big\}.$$

Then $Weak\, L_p\big[A_\epsilon(u)\big]$ is a closed invariant subspace of $Weak\, L_p$ under M_u. Moreover

$$M_u\Big|_{Weak\, L_p[A_\epsilon(u)]},$$

is a compact operator.

Proof. Let $h,\ s \in \text{Weak } L_p\big[A_\epsilon(u)\big]$ and $\alpha,\ \beta \in \mathbb{R}$. Then $h = f\chi_{A_\epsilon(u)}$ and $s = g\chi_{A_\epsilon(u)}$ where $f, g \in \text{Weak } L_p$, thus

$$\begin{aligned} \alpha h + \beta s &= \alpha\big(f\chi_{A_\epsilon(u)}\big) + \beta\big(g\chi_{A_\epsilon(u)}\big) \\ &= (\alpha f + \beta g)\chi_{A_\epsilon(u)} \in \text{Weak } L_p\big[A_\epsilon(u)\big]. \end{aligned}$$

Which mean that Weak $L_p\big[A_\epsilon(u)\big]$ is a subspace of Weak L_p.
Next, for all $h \in$ Weak $L_p\big[A_\epsilon(u)\big]$ we have

$$\begin{aligned} M_u h = uh &= uf\chi_{A_\epsilon(u)} \\ &= (uf)\chi_{A_\epsilon(u)}, \end{aligned}$$

where $uf \in$ Weak L_p.
Therefore, $M_u h \in$ Weak $L_p\big[A_\epsilon(u)\big]$, which means that Weak $L_p\big[A_\epsilon(u)\big]$ is an invariant subspace of Weak L_p under M_u.

Now, let us show that Weak $L_p\big[A_\epsilon(u)\big]$ is a closed set. Indeed, let g be a function belonging to the closure of Weak $L_p\big[A_\epsilon(u)\big]$ then there exists a sequence $\{g_n\}_{n\in\mathbb{N}}$ in Weak $L_p\big[A_\epsilon(u)\big]$ such that

$$g_n \to g,$$

in Weak L_p. Just remain to exhibit that g belong to Weak $L_p\big[A_\epsilon(u)\big]$. Note that

$$g = g\chi_{A_\epsilon(u)} + g\chi_{A^c_\epsilon(u)}.$$

Next, we want to show that $g\chi_{A^c_\epsilon(u)} = 0$. In fact, given $\epsilon_1 > 0$ there exists $n_0 \in \mathbb{N}$ such that

$$\begin{aligned} \big\|g\chi_{A^c_\epsilon(u)}\big\|_{L_{(p,\infty)}} &= \big\|(g - g_{n_0} + g_{n_0})\chi_{A^c_\epsilon(u)}\big\|_{L_{(p,\infty)}} \\ &= \big\|(g - g_{n_0})\chi_{A^c_\epsilon(u)}\big\|_{L_{(p,\infty)}} \\ &\le \big\|g - g_{n_0}\big\|_{L_{(p,\infty)}} < \epsilon_1. \end{aligned}$$

Thus $g\chi_{A^c_\epsilon(u)} = 0$, which mean that $g = g\chi_{A_\epsilon(u)}$ that is $g \in$ Weak $L_p\big[A_\epsilon(u)\big]$.
And the proof is now complete. □

Theorem 6.3. *Let $M_u \in B\,(\text{Weak}\, L_p)$. Then M_u is compact if and only if Weak $L_p\big[A_\epsilon(u)\big]$ is finite dimensional for each $\epsilon > 0$.*

Proof. If $|u(x)| \ge \epsilon$, we should note that

$$\big|uf\chi_{A_\epsilon(u)}(x)\big| \ge \epsilon f\chi_{A_\epsilon(u)}(x),$$

and thus

$$\Big\{x \in X : \epsilon f\chi_{A_\epsilon(u)}(x) > \lambda\Big\} \subset \Big\{x \in X : \big|uf\chi_{A_\epsilon(u)}(x)\big| > \lambda\Big\},$$

then

$$\begin{aligned}
D_{uf\chi_{A_\epsilon(u)}}(\lambda) &\geq D_{\epsilon f\chi_{A_\epsilon(u)}}(\lambda) \\
\lambda\big[D_{uf\chi_{A_\epsilon(u)}}(\lambda)\big]^{1/p} &\geq \lambda\big[D_{\epsilon f\chi_{A_\epsilon(u)}}(\lambda)\big]^{1/p} \\
\sup_{\lambda>0}\lambda\big[D_{uf\chi_{A_\epsilon(u)}}(\lambda)\big]^{1/p} &\geq \sup_{\lambda>0}\lambda\big[D_{\epsilon f\chi_{A_\epsilon(u)}}(\lambda)\big]^{1/p} \\
\big\|uf\chi_{A_\epsilon(u)}\big\|_{L_{(p,\infty)}} &\geq \big\|\epsilon f\chi_{A_\epsilon(u)}\big\|_{L_{(p,\infty)}} \\
&= \epsilon\big\|f\chi_{A_\epsilon(u)}\big\|_{L_{(p,\infty)}}.
\end{aligned}$$

thus

$$\big\|M_u f\chi_{A_\epsilon(u)}\big\|_{L_{(p,\infty)}} \geq \epsilon\big\|f\chi_{A_\epsilon(u)}\big\|_{L_{(p,\infty)}}. \tag{34}$$

Now, if M_u is a compact, then for lemma 6.1, Weak $L_p\big[A_\epsilon(u)\big]$ is closed invariant subspace of M_u and by theorem A.1 (appendix)

$$M_u\Big|_{\text{Weak } L_p[A_\epsilon(u)]},$$

is a compact operator. Then by (34) $M_u\Big|_{\text{Weak } L_p[A_\epsilon(u)]}$ has a closed range in Weak $L_p\big[A_\epsilon(u)\big]$ and it is invertible, being compact, Weak $L_p\big[A_\epsilon(u)\big]$ is finite dimensional.

Conversely, suppose that Weak $L_p\big[A_\epsilon(u)\big]$ is finite dimensional for each $\epsilon > 0$. In particular for $n \in \mathbb{N}$, Weak $L_p[A_{\frac{1}{n}}(u)]$ is finite dimensional, then for each n, define

$$u_n : X \to \mathbb{C}$$

as

$$u_n(x) = \begin{cases} u(x) & \text{if } |u(x)| \geq \frac{1}{n} \\ \\ 0 & \text{if } |u(x)| < \frac{1}{n} \end{cases}$$

Then we find that

$$M_{u_n}f - M_u f = (u_n - u).f \leq \|u_n - u\|_\infty |f|,$$

and thus

$$\Big\{x \in X : |(u_n - u).f(x)| > \lambda\Big\} \subseteq \Big\{x \in X : \|u_n - u\|_\infty |f(x)| > \lambda\Big\}.$$

From this we have

$$\left\|M_{u_n}f - M_u f\right\|_{L_{(p,\infty)}} \leq \|u_n - u\|_\infty \left\|f\right\|_{L_{(p,\infty)}},$$

consequently

$$\left\|M_{u_n}f - M_u f\right\|_{L_{(p,\infty)}} < \frac{1}{n}\|f\|_{L_{(p,\infty)}},$$

which implies that M_{u_n} converge to M_u uniformly.

As Weak $L_p\big[A_\epsilon(u)\big]$ is finite dimensional so M_{u_n} is a finite rank operator. Therefore M_{u_n} is a compact operator and hence M_u is a compact operator. □

Remark 6.1. In general, the multiplication operator on measurable space is not 1-1. Indeed, let $(X, \mathcal{A}, \mu)$ be a measure space and

$$A = X \setminus supp(u) = \big\{x \in X : u(x) = 0\big\},$$

where $supp(u)$ stand for the support of u.
If $\mu(A) \neq 0$ and $f = \chi_A$, then for any $x \in X$, we have $f(x)u(x) = 0$ which implies that $M_u(f) = 0$, therefore $Ker(M_u) \neq \{0\}$ and hence M_u is not 1-1. By contrapositive, we have M_u is 1-1, then $\mu\Big(X \setminus supp(u)\Big) = 0$. On the other hand, if $\mu\big(X \setminus supp(u)\big) = 0$ and μ is a complete measure, then $M_u(f) = 0$ implies $f(x)u(x) = 0\ \forall x \in X$, then $\big\{x \in X : f(x) \neq 0\big\} \subseteq X \setminus supp(u)$ and so $f = 0\ \mu - a.e$ on X.
Hence, if $\mu\big(X \setminus supp(u)\big) = 0$ and μ is a complete measure, then M_u is 1-1.

Proposition 6.1. *M_u is 1-1 on $Y = Weak\, L_p\big(supp(u)\big)$.*

Proof. Let $Y = \text{Weak } L_p\big(supp(u)\big) = \big\{f\chi_{supp(u)} : f \in \text{Weak } L_p\big\}$.
Indeed, if $M_u(\widetilde{f}) = 0$ with $\widetilde{f} = f\chi_{supp(u)} \in Y$, then $f(x)\chi_{supp(u)}(x) = 0$ for all $x \in X$ and so

$$f(x)u(x) = 0 \quad \forall x \in supp(u)$$

$$f(x) = 0 \quad \forall x \in supp(u),$$

$$f(x)\chi_{supp(u)} = 0 \quad \forall x \in X.$$

Then $\widetilde{f} = 0$ and the proof is complete. □

Theorem 6.4. *Let M_u : Weak $L_p\big(supp\, u\big) \to$ Weak $L_p\big(supp\, u\big)$. Then M_u has closed range if an only if hence exist a $\delta > 0$ such that $|u(x)| \geq \delta$ a.e$[\mu]$ on $S = \big\{x \in X : u(x) \neq 0\big\}$ the support of u.*

Proof. If there exists a $\delta > 0$ such that $|u(x)| \geq \delta$ a.e$[\mu]$ on S, then for $f \in$ Weak L_p we have

$$\Big\{x \in X : \big|\big(\delta f \chi_S\big)(x)\big| > \lambda\Big\} \subseteq \Big\{x \in X : \big|\big(u f \chi_S\big)(x)\big| > \lambda\Big\},$$

and thus

$$D_{uf\chi_S}(\lambda) \geq D_{\delta f \chi_S}(\lambda).$$

Hence

$$\big\|M_u f \chi_S\big\|_{L_{(p,\infty)}} \geq \delta \big\|f \chi_S\big\|_{L_{(p,\infty)}},$$

Therefore M_u has closed range.
Conversely if M_u has closed range on Weak $L_p\big(S\big)$, since M_u is 1-1 on Weak $L_p\big(S\big)$ then M_u is bounded below, and thus there exists an $\delta > 0$ such that

$$\big\|M_u f\big\|_{L_{(p,\infty)}} \geq \delta \big\|f\big\|_{L_{(p\infty)}},$$

for all $f \in$ Weak $L_p\big(S\big)$, where

$$\text{Weak } L_p\big(S\big) = \Big\{f \chi_S : f \in \text{Weak } L_p\Big\}.$$

Let $E = \big\{x \in S : |u(x)| < \epsilon/2\big\}$.
If $\mu(E) > 0$, then we can find a measurable set $F \subseteq E$ such that $\chi_F \in$ Weak $L_p\big(S\big)$.
Also, we have for $\lambda > 0$

$$\Big\{x \in X : \big|u \chi_F(x)\big| > \lambda\Big\} \subseteq \Big\{x \in X : \big|\tfrac{\epsilon}{2} \chi_F(x)\big| > \lambda\Big\}.$$

So that

$$D_{u\chi_F}(\lambda) \geq D_{\frac{\epsilon}{2}\chi_F}(\lambda).$$

Hence

$$\big\|M_u \chi_F\big\|_{L_{(p,\infty)}} \leq \tfrac{\epsilon}{2} \big\|\chi_F\big\|_{L_{(p,\infty)}},$$

which is a contradiction. Therefore $\mu(E) = 0$.
This completes the proof. □

7 Composition Operator

Let $(X, \mathcal{A}, \mu)$ be a measure spaces and $T : X \to X$ such that $T^{-1}(A) \in \mathcal{A}$ for any $A \in \mathcal{A}$. If $\mu\big(T^{-1}(A)\big) = 0$ for each $A \in \mathcal{A}$ with $\mu(A) = 0$, then T is said to be non-singular transformation.

Let Y be a measurable subset of X and $T : Y \to X$ is a measurable transformation, then we define the linear transformation C_T from Weak L_p into the spaces of all complex - valued measurable functions on X as

$$\big(C_T f\big)(x) = \begin{cases} f\big(T(x)\big) & \text{if } x \in Y \\ \\ 0 & \text{otherwise} \end{cases}$$

for all $f \in$ Weak L_p.

If C_T is bounded with range in Weak L_p we say that C_T is a composition operator on Weak L_p induced by T.

In this section a necessary and sufficient condition for the boundedness of composition mapping C_T is given.

Theorem 7.1. *Let $T : X \to X$ be a non-singular measurable transformation. Then $C_T : f \to f \circ T$ induced by T is bounded on Weak L_p if and only if there exists a constant $M > 0$ such that*

$$\mu\big(T^{-1}(E)\big) \leq M\mu(E) \quad \textit{for all} \quad E \in \mathcal{A}.$$

Moreover

$$\big\|C_T\big\| = \sup_{\substack{0<\mu(E)<\infty \\ E\in\mathcal{A}}} \left(\frac{\mu\big(T^{-1}(E)\big)}{\mu(E)}\right)^{1/p}.$$

Proof. Suppose that there exists a constant $M > 0$ such that $\mu\big(T^{-1}(E)\big) \leq$

$M\mu(E)$ for all $E \in \mathcal{A}$. Then

$$\begin{aligned}\big\|C_T(f)\big\|_{L_{(p,\infty)}} &= \sup_{\lambda>0} \lambda\Big[\mu\big(\{x \in X : \big|f\big(T(x)\big)\big| > \lambda\}\big)\Big]^{1/p} \\ &= \sup_{\lambda>0} \lambda\Big[\mu\Big(T^{-1}\big(\{x \in X : |f(x)| > \lambda\}\big)\Big)\Big]^{1/p} \\ &\leq M \sup_{\lambda>0} \lambda\Big[\mu\big(\{x \in X : |f(x)| > \lambda\}\big)\Big]^{1/p} \\ &= M\big\|f\big\|_{L_{(p,\infty)}}.\end{aligned}$$

Hence

$$\big\|C_T(f)\big\|_{L_{(p,\infty)}} \leq M\big\|f\big\|_{L_{(p,\infty)}}.$$

Conversely, suppose that C_T is bounded and let $E \in \mathcal{A}$, if $\mu(E) = \infty$ then we have result. Suppose that $\mu(E) < \infty$ and consider $\chi_{T^{-1}(E)}$, then

$$\begin{aligned}\Big[\mu\big(T^{-1}(E)\big)\Big]^{1/p} &\leq \sup_{\lambda>0} \lambda\Big[\mu\big(\{x \in X : \chi_{T^{-1}(E)}(x) > \lambda\}\big)\Big]^{1/p} \\ &= \sup_{\lambda>0} \lambda\Big[\mu\big(\{x \in X : \big(\chi_E \circ T\big)(x) > \lambda\}\big)\Big]^{1/p} \\ &= \big\|\chi_E \circ T\big\|_{L_{(p,\infty)}} \\ &= \big\|C_T(\chi_E)\big\|_{L_{(p,\infty)}},\end{aligned}$$

since C_T is bounded then there exists M such that

$$\begin{aligned}\big\|C_T(\chi_E)\big\|_{L_{(p,\infty)}} &\leq M^{1/p}\big\|\chi_E\big\|_{L_{(p,\infty)}} \\ &= M^{1/p}\big[\mu(E)\big]^{1/p},\end{aligned}$$

thus

$$\Big[\mu\big(T^{-1}(E)\big)\Big]^{1/p} \leq M^{1/p}\big[\mu(E)\big]^{1/p},$$

accordingly

$$\mu\big(T^{-1}(E)\big) \leq M\mu(E).$$

for all $E \in$ Weak L_p

Next, we like to shown that

$$\|C_T\| = \sup_{\substack{0<\mu(E)<\infty \\ E\in\mathcal{A}}} \left(\frac{\mu\big(T^{-1}(E)\big)}{\mu(E)}\right)^{1/p}.$$

Indeed, let $N = \sup_{\substack{0<\mu(E)<\infty \\ E\in\mathcal{A}}} \left(\frac{\mu\big(T^{-1}(E)\big)}{\mu(E)}\right)^{1/p}$, then

$$\left(\frac{\mu\big(T^{-1}(E)\big)}{\mu(E)}\right)^{1/p} \le N \quad \text{for all} \quad E\in\mathcal{A}, \quad \mu(E)\neq 0,$$

thus

$$\mu\big(T^{-1}(E)\big) \le N^p \mu(E) \quad \text{for all} \quad E\in\mathcal{A}.$$

Now, by the first part of this theorem, we have

$$\|C_T(f)\|_{L_{(p,\infty)}} \le N\|f\|_{L_{(p,\infty)}},$$

for all $f\in$ Weak L_p.
Hence

$$\|C_T\| = \sup_{f\neq 0}\frac{\|C_T(f)\|_{L_{(p,\infty)}}}{\|f\|_{L_{(p,\infty)}}} \le N,$$

then

$$\|C_T\| \le \sup_{\substack{0<\mu(E)<\infty \\ E\in\mathcal{A}}} \left(\frac{\mu\big(T^{-1}(E)\big)}{\mu(E)}\right)^{1/p}. \tag{35}$$

On the other hand, let

$$M = \|C_T\| = \sup_{\substack{f\in\text{Weak } L_p \\ f\neq 0}}\frac{\|C_T(f)\|_{L_{(p,\infty)}}}{\|f\|_{L_{(p,\infty)}}}.$$

Then

$$\frac{\|C_T(f)\|_{L_{(p,\infty)}}}{\|f\|_{L_{(p,\infty)}}} \le M \quad \text{for all} \quad f\in\text{Weak } L_p,\ f\neq 0.$$

In particular, for $f = \chi_E$ such that $0 < \mu(E) < \infty$, $E \in \mathcal{A}$ we have that $f = \chi_E \in \text{Weak } L_p$ and

$$\frac{\left\|C_T(\chi_E)\right\|_{L_{(p,\infty)}}}{\left\|\chi_E\right\|_{L_{(p,\infty)}}} = \left(\frac{\mu\big(T^{-1}(E)\big)}{\mu(E)}\right)^{1/p} \leq M,$$

therefore

$$\sup_{\substack{0<\mu(E)<\infty \\ E\in\mathcal{A}}} \left(\frac{\mu\big(T^{-1}(E)\big)}{\mu(E)}\right)^{1/p} \leq M = \left\|C_T\right\|. \tag{36}$$

Finally from (35) and (36) we obtain

$$\left\|C_T\right\| = \sup_{\substack{0<\mu(E)<\infty \\ E\in\mathcal{A}}} \left(\frac{\mu\big(T^{-1}(E)\big)}{\mu(E)}\right)^{1/p}.$$

□

A Appendix

Definition A.1. *Let $T : X \to X$ be an operator, a subspace V of X is said to be invariant under T (or simply $T-$invariant) whenever*

$$T(V) \subseteq V.$$

Theorem A.1. *Let $T : X \to X$ be an operator. If T is compact and M is a closed T-invariant space of X. Then $T\Big|_M$ is compact.*

Proof. Let $\{x_n\}_{n\in\mathbb{N}}$ be a sequence in $M \subseteq X$. Then $\{x_n\}_{n\in\mathbb{N}} \subseteq X$, thus there exists a subsequence $\{x_{n_k}\}_{k\in\mathbb{N}}$ of $\{x_n\}_{n\in\mathbb{N}}$ such that $T(x_{n_k})$ converges in X but $T(x_{n_k}) \subseteq T(M)$, since $\{x_{n_k}\} \subseteq M$. Then $T(x_{n_k})$ converge on $\overline{T(M)} \subseteq \overline{M} = M$.
Therefore $T(x_{n_k})$ converge on M, hence $T\Big|_M$ is compact. □

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