Existence of Solutions for $p(x)$-Laplacian equations without Ambrosetti-Rabinowitz type condition

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Abstract. This paper investigates the existence and multiplicity of solutions for superlinear $p(x)$-Laplacian equations with Dirichlet boundary conditions. Under no Ambrosetti-Rabinowitz's superquadraticity conditions, we obtain the existence and multiplicity of solutions by using a variant Fountain theorem without Palais-Smale type assumptions.

Keywords: $p(x)$-Laplace operator; variable exponent Lebesgue-Sobolev spaces; variational approach; variant Fountain theorem

MSC: 35D05, 35J60, 35J70

1 Introduction

We consider the following superlinear elliptic problem

$$\begin{cases} -\Delta_{p(x)} u = f(x,u) + g(x,u), \text{ in } \Omega, \\ u = 0, \text{ on } \partial\Omega, \end{cases} \qquad (\mathbf{P})$$

and obtain infinitely many solutions, where Ω is a bounded smooth domain of $\mathbb{R}^N$ $(N \geq 3)$ and $p \in C(\overline{\Omega})$ with $1 < p(x) < N$ for all $x \in \overline{\Omega}$.

Generally, in order to search the existence of solutions for Dirichlet problems which is superlinear, it is essential to assume the following superquadraticity condition, which is known as Ambrosetti-Rabinowitz type condition [2]:

(AR) $\exists M > 0,\ \tau > p^+$ such that $0 < \tau F(x,s) \leq f(x,s)\,s,\ \ |s| \geq M, x \in \Omega,$

where f is nonlinear term such that $F(x,t) = \int_0^t f(x,s)ds$.

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There are many paper dealing with superlinear Dirichlet problems involving $p(x)$-Laplace operator $\Delta_{p(x)}u := \operatorname{div}(|\nabla u|^{p(x)-2}\nabla u)$, in which (AR) is the main assumption to get the existence and multiplicity of solutions [9,10]. However, as far as we are concerned, there are many functions which are superlinear but not satisfy (AR) [3,17].

It is well known that the main aim of using (AR) is to ensure the boundedness of the Palais-Smale type sequences of the corresponding functional. In the present paper we do not use (AR). Instead, we use a variant Fountain theorem not including Palais-Smale type assumptions (see Theorem 5).

The study of differential equations and variational problems involving $p(x)$-growth conditions has attracted a special interest in recent years and a lot of researchers have devoted their work to this area [5,12,14,16] since there are some physical phenomena which can be modelled by such kind of equations. In particular, we may mention some applications related to the study of elastic mechanics and electrorheological fuids [1,4,11,15,19]. The appearance of such physical models was facilitated by the development of variable exponent Lebesgue $L^{p(x)}$ and Sobolev spaces $W^{1,p(x)}$.

2 Preliminaries

At first, we shall mention some definitions and basic properties of generalized Lebesgue-Sobolev spaces $L^{p(x)}(\Omega)$, $W^{1,p(x)}(\Omega)$ and $W_0^{1,p(x)}(\Omega)$. We refer the reader to [6–8,13] for the fundamental properties of these spaces.

Set

$$C_+\left(\overline{\Omega}\right) = \left\{p; p \in C\left(\overline{\Omega}\right), \inf p(x) > 1, \forall x \in \overline{\Omega}\right\}.$$

Let $p \in C_+\left(\overline{\Omega}\right)$ and denote

$$p^- := \inf_{x\in\overline{\Omega}} p(x) \leq p(x) \leq p^+ := \sup_{x\in\overline{\Omega}} p(x) < \infty.$$

For any $p \in C_+\left(\overline{\Omega}\right)$, we define the variable exponent Lebesgue space by

$$L^{p(x)}(\Omega) = \left\{u \mid u : \Omega \to \mathbb{R} \text{ is measurable}, \int_\Omega |u(x)|^{p(x)}\,dx < \infty\right\},$$

then $L^{p(x)}(\Omega)$ endowed with the norm

$$|u|_{p(x)} = \inf\left\{\mu > 0 : \int_\Omega \left|\frac{u(x)}{\mu}\right|^{p(x)} dx \leq 1\right\},$$

becomes a Banach space.

The modular of the $L^{p(x)}(\Omega)$ space, which is the mapping $\rho : L^{p(x)}(\Omega) \to \mathbb{R}$ defined by

$$\rho(u) = \int_\Omega |u(x)|^{\ p(x)} dx,\ \forall u \in L^{p(x)}(\Omega). \tag{2.1}$$

Proposition 1 *([7,13]) If $u, u_n \in L^{p(x)}(\Omega)$ $(n = 1, 2, ...)$, we have*

$$\begin{aligned}
&(i)\ \ |u|_{p(x)} < 1\,(= 1; > 1) \Leftrightarrow \rho(u) < 1\,(= 1; > 1);\\
&(ii)\ |u|_{p(x)} > 1 \implies |u|_{p(x)}^{p^-} \leq \rho(u) \leq |u|_{p(x)}^{p^+};\\
&(iii)\ |u|_{p(x)} < 1 \implies |u|_{p(x)}^{p^+} \leq \rho(u) \leq |u|_{p(x)}^{p^-};
\end{aligned}$$

Proposition 2 *[7,13] If $u, u_n \in L^{p(x)}(\Omega)$ $(n = 1, 2, ...)$, then the following statements are equivalent:*

$$\begin{aligned}
&(i) \lim_{n\to\infty} |u_n - u|_{p(x)} = 0;\\
&(ii) \lim_{n\to\infty} \rho(u_n - u) = 0;\\
&(iii) u_n \to u \text{ in measure in } \Omega \text{ and } \lim_{n\to\infty} \rho(u_n) = \rho(u).
\end{aligned}$$

The variable exponent Sobolev space $W^{1,p(x)}(\Omega)$ is defined by

$$W^{1,p(x)}(\Omega) = \{u \in L^{p(x)}(\Omega) : |\nabla u| \in L^{p(x)}(\Omega)\},$$

with the norm

$$\|u\|_{1,p(x)} = |u|_{p(x)} + |\nabla u|_{p(x)},\ \forall u \in W^{1,p(x)}(\Omega).$$

Then $(W^{1,p(x)}(\Omega), \|\cdot\|_{1,p(x)})$ becomes a Banach space. The space $W_0^{1,p(x)}(\Omega)$ is defined as the closure of $C_0^\infty(\Omega)$ in $W^{1,p(x)}(\Omega)$ with respect to the norm $\|\cdot\|_{1,p(x)}$. For $u \in W_0^{1,p(x)}(\Omega)$, we can define an equivalent norm

$$\|u\| = |\nabla u|_{p(x)},$$

since Poincaré inequality

$$|u|_{p(x)} \leq C|\nabla u|_{p(x)},\ \forall u \in W_0^{1,p(x)}(\Omega)$$

holds, where C is a positive constant [9].

Proposition 3 *[7,13] If $1 < p^-$ and $p^+ < \infty$, then the spaces $L^{p(x)}(\Omega)$, $W^{1,p(x)}(\Omega)$ and $W_0^{1,p(x)}(\Omega)$ are separable and reflexive Banach spaces.*

Proposition 4 *[7,13] Assume that Ω is bounded, the boundary of Ω possesses the cone property and $p \in C_+(\overline{\Omega})$. If $q \in C_+(\overline{\Omega})$ and $q(x) < p^*(x) := \frac{Np(x)}{N-p(x)}$ for all $x \in \overline{\Omega}$, then the embedding $W^{1,p(x)}(\Omega) \hookrightarrow L^{q(x)}(\Omega)$ is compact and continuous.*

From**[18],** let X be a reflexive and separable Banach space, then there are $e_j \subset X$ and $e_j^* \subset X^*$ such that

$$X = \overline{span\{e_j \mid j = 1, 2, ...\}}, \qquad X^* = \overline{span\left\{e_j^* \mid j = 1, 2, ...\right\}},$$

and

$$\langle e_i^*, e_j \rangle = \begin{cases} 1 & i = j, \\ 0 & i \neq j, \end{cases}$$

where $\langle ., . \rangle$ denotes the duality product between X and X^*. For convenience, we write

$$X_j = span\{e_j\}, \qquad Y_k = \oplus_{j=1}^{k} X_j, \qquad Z_k = \overline{\oplus_{j=k}^{\infty} X_j}.$$

And let

$$B_k = \{u \in Y_k : \|u\| \leq \rho_k\},\ N_k = \{u \in Z_k : \|u\| = r_k\},\ \text{for } \rho_k > r_k > 0.$$

Let consider the C^1-functional $I_\lambda : X \to \mathbb{R}$ defined by

$$I_\lambda(u) := A(u) - \lambda B(u),\ \lambda \in [1, 2].$$

Now we give the following variant Fountain theorem (see [20], Theorem 2.2), which we use in the proof of the main results of the present paper:

Theorem 5 (Variant Fountain Theorem) *Assume the functional I_λ satisfies the followings:*

(T_1) *I_λ maps bounded sets to bounded sets uniformly for $\lambda \in [1, 2]$.*

Moreover, $I_\lambda(-u) = I_\lambda(u)$ *for all* $(\lambda, u) \in [1,2] \times X$.

(T_2) $B(u) \geq 0$; $B(u) \to \infty$ *as* $\|u\| \to \infty$ *on any finite dimensional subspace of* X.

(T_3) *There exists* $\rho_k > r_k > 0$ *such that*

$$a_k(\lambda) := \inf_{u \in Z_k, \|u\| = \rho_k} I_\lambda(u) \geq 0 > b_k(\lambda) := \max_{u \in Y_k, \|u\| = r_k} I_\lambda(u),$$

for all $\lambda \in [1,2]$ *and*

$$d_k(\lambda) := \inf_{u \in Z_k, \|u\| \leq \rho_k} I_\lambda(u) \to 0 \text{ as } k \to \infty \text{ uniformly for } \lambda \in [1,2].$$

Then there exists $\lambda_n \to 1$, $u(\lambda_n) \in Y_n$ *such that*

$$I'_{\lambda_n}|_{Y_n}(u(\lambda_n)) = 0, \; I_{\lambda_n}(u(\lambda_n)) \to c_k \in [d_k(2), b_k(1)] \text{ as } n \to \infty.$$

Particularly, if $\{u(\lambda_n)\}$ *has a convergent subsequence for every* k, *then* I_1 *has infinitely many nontrivial critical points* $\{u_k\} \in X \backslash \{0\}$ *satisfying* $I_1(u_k) \to 0^-$ *as* $k \to \infty$.

3 Main results

For problem $(\mathbf{P})$, we make the following assumptions:

$(\mathbf{P}_1)$ $f(x,-t) = -f(x,t)$ and $g(x,-t) = -g(x,t)$ for any $x \in \Omega$, $t \in \mathbb{R}$.

$(\mathbf{P}_2)$ Assume that $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function and there exist $1 < \sigma \leq \delta < p^-$ and $c_1 > 0, c_2 > 0, c_3 > 0$ such that

$$c_1 |t|^\sigma \leq f(x,t)\, t \leq c_2 |t|^\delta + c_3 |t|^\sigma, \text{ for a.e. } x \in \Omega \text{ and } t \in \mathbb{R}.$$

$(\mathbf{P}_3)$ Assume that $g : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function and $p, q \in C_+\left(\overline{\Omega}\right)$ with $p(x) \leq p^+ < q^- \leq q(x) < p^*(x)$ such that

$$|g(x,t)| \leq c\left(1 + |t|^{q(x)-1}\right), \text{ for a.e. } x \in \Omega \text{ and } t \in \mathbb{R},$$

and $g(x,t)\, t \geq 0$, for a.e. $x \in \Omega$ and $t \in \mathbb{R}$. Moreover, $\lim_{t \to 0} \frac{g(x,t)}{t^{p^- - 1}} = 0$ uniformly for $x \in \Omega$.

$(\mathbf{P}_4)$ Assume one of the following conditions holds:

$$(\mathbf{1}) \lim_{|t| \to \infty} \frac{g(x,t)}{t^{p^- - 1}} = 0 \text{ uniformly for } x \in \Omega.$$

$$(\mathbf{2})\lim_{|t|\to\infty}\frac{g(x,t)}{t^{p^{-}-1}}=-\infty \text{ uniformly for } x\in\Omega.$$

Moreover, $\frac{f(x,t)}{t^{p^{-}-1}}$ and $\frac{g(x,t)}{t^{p^{-}-1}}$ are decreasing in $t\in\mathbb{R}$ for t large enough.

$$(\mathbf{3})\liminf_{|t|\to\infty}\frac{g(x,t)\,t-\epsilon G(x,t)}{|t|^{\alpha}}\geq c>0 \text{ uniformly for } x\in\Omega,$$

where $\alpha>\delta$ and $\epsilon>0$. Moreover, $\lim\limits_{|t|\to\infty}\frac{g(x,t)}{t^{p^{-}-1}}=\infty$ uniformly for $x\in\Omega$; $\frac{g(x,t)}{t^{p^{-}-1}}$ is increasing in $t\in\mathbb{R}$ for t large enough.

Theorem 6 *Assume that* $(\mathbf{P}_1)$-$(\mathbf{P}_4)$ *hold, then problem* $(\mathbf{P})$ *has infinitely many solutions* $\{u_k\}$ *satisfying*

$$\Phi(u_k):=\int\limits_{\Omega}\frac{1}{p(x)}|\nabla u_k|^{p(x)}\,dx-\int\limits_{\Omega}G(x,u_k)dx-\int\limits_{\Omega}F(x,u_k)dx\to 0^{-}\ \ as\ k\to\infty,$$

where $\Phi:W_0^{1,p(x)}(\Omega)\to\mathbb{R}$ *is the functional corresponding to problem* $(\mathbf{P})$ *and* $G(x,t)=\int_0^t g(x,s)ds,\ F(x,t)=\int_0^t f(x,s)ds.$

Remark 7 *The conditions* $(\mathbf{P}_2)$ *and* $(\mathbf{P}_3)$ *imply the functional* Φ *is well defined and of class* C^1. *It is well known that the critical points of* Φ *are weak solutions of* $(\mathbf{P})$. *Moreover, the derivative of* Φ *is given by*

$$\langle\Phi'(u),\upsilon\rangle=\int\limits_{\Omega}|\nabla u|^{p(x)-2}\nabla u\nabla\upsilon dx-\int\limits_{\Omega}g(x,u)\upsilon dx-\int\limits_{\Omega}f(x,u)\upsilon dx,$$

for any $u,\upsilon\in W_0^{1,p(x)}(\Omega)$.

Let us consider C^1-functional $\Phi_\lambda:W_0^{1,p(x)}(\Omega)\to\mathbb{R}$ defined by

$$\Phi_\lambda(u)=\int\limits_{\Omega}\frac{1}{p(x)}|\nabla u|^{p(x)}\,dx-\int\limits_{\Omega}G(x,u)dx-\lambda\int\limits_{\Omega}F(x,u)dx:=A(u)-K(u)-\lambda B(u),$$

where $\lambda\in[1,2]$. Then $B(u)\geq 0$ and $B(u)\to\infty$ as $\|u\|\to\infty$ on any finite dimensional subspace, where $n>k>2$.

To get the proof of Theorem 6, we will apply Theorem 5. Therefore, it is enough to obtain the results of Lemma 8 and Lemma 9.

Lemma 8 *Under the assumptions of Theorem 6, there exist* $\lambda_n\to 1$, $u_n(\lambda)\in Y_n$ *such that*

$$\Phi'_{\lambda_n}|_{Y_n}(u_n(\lambda))=0,\ \Phi_{\lambda_n}(u_n(\lambda))\to c_k\in[d_k(2),b_k(1)]\ \ as\ n\to\infty.$$

PROOF. First, we prove that for some $r_k \in (0, \rho_k)$ such that

$$b_k(\lambda) := \max_{u \in Y_k, \|u\| = r_k} \Phi_\lambda(u) < 0,$$

for $\lambda \in [1,2]$, $u \in Y_k$. The norms $|\cdot|_\sigma$ and $\|\cdot\|$ is equivalent on the finite dimensional subspace Y_k. Therefore, there is a constant $c > 0$ such that

$$|u|_\sigma \geq c\,\|u\|\,, \quad \forall u \in Y_k.$$

Moreover, by ($\mathbf{P}_3$), for any $\varepsilon > 0$ there exists $C_\varepsilon > 0$ such that $|G(x,u)| \leq \varepsilon\,|u|^{p^-} + C_\varepsilon\,|u|^{q(x)}$. Then, by ($\mathbf{P}_2$) and Proposition 1, we have

$$\begin{aligned}
\Phi_\lambda(u) &\leq \frac{1}{p^-}\|u\|^{p^-} - K(u) - \lambda B(u) \\
&\leq \frac{1}{p^-}\|u\|^{p^-} - \varepsilon \int_\Omega |u|^{p^-}\,dx - C_\varepsilon \int_\Omega |u|^{q(x)}\,dx - \lambda c_1 \int_\Omega |u|^\sigma\,dx \\
&\leq \frac{1}{p^-}\|u\|^{p^-} - \varepsilon c^{p^-}\|u\|^{p^-} - C_\varepsilon\|u\|^{q^+} - c_4^\sigma\|u\|^\sigma .
\end{aligned}$$

Since $\sigma < p^- < q^+$, for $\|u\|$ small enough we get $b_k(\lambda) := \max_{u \in Y_k, \|u\| = r_k} \Phi_\lambda(u) < 0$ for all $u \in Y_k$.

Second, we shall show that for some $0 < r_k < \rho_k$ such that

$$a_k(\lambda) := \inf_{u \in Z_k, \|u\| = \rho_k} \Phi_\lambda(u) \geq 0$$

for $\lambda \in [1,2]$, and $u \in Z_k$.

Let

$$\begin{aligned}
\beta_k(q(x)) &:= \sup_{u \in Z_k, \|u\|=1} |u|_{q(x)}\,, \qquad \beta_k\left(p^-\right) := \sup_{u \in Z_k, \|u\|=1} |u|_{p^-}\,, \\
\beta_k(\delta) &:= \sup_{u \in Z_k, \|u\|=1} |u|_\delta\,, \qquad \beta_k(\sigma) := \sup_{u \in Z_k, \|u\|=1} |u|_\sigma\,.
\end{aligned}$$

Then $\beta_k(q(x)) \to 0$, $\beta_k(p^-) \to 0$, $\beta_k(\delta) \to 0$ and $\beta_k(\sigma) \to 0$ as $k \to \infty$ (see [10]). Therefore, by ($\mathbf{P}_2$) and Proposition 1, we have

$$
\begin{aligned}
\Phi_\lambda(u) &= A(u) - K(u) - \lambda B(u) \geq \frac{1}{p^+}\|u\|^{p^+} - K(u) - \lambda B(u) \\
&\geq \frac{1}{p^+}\|u\|^{p^+} - \varepsilon \int_\Omega |u|^{p^-} dx - C_\varepsilon \int_\Omega |u|^{q(x)} dx - \lambda c_2 \int_\Omega |u|^\delta dx - \lambda c_3 \int_\Omega |u|^\sigma dx \\
&\geq \frac{1}{p^+}\|u\|^{p^+} - c|u|_{p^-}^{p^-} - c|u|_{q(x)}^{q^-} - c|u|_\delta^\delta - c|u|_\sigma^\sigma \\
&\geq \frac{1}{p^+}\|u\|^{p^+} - c\beta_k^{p^-}\left(p^-\right)\|u\|^{p^-} - c\beta_k^{q^-}(q(x))\|u\|^{q^-} - c\beta_k^\delta(\delta)\|u\|^\delta - c\beta_k^\sigma(\sigma)\|u\|^\sigma
\end{aligned}
$$

where $c = \max\{\varepsilon, C_\varepsilon, 2c_2, 2c_3\}$**.** Let $\varphi \in Z_k$, $\|\varphi\| = 1$ and $0 < t < 1$, then it follows

$$
\begin{aligned}
\Phi_\lambda(t\varphi) &\geq \frac{1}{p^+}t^{p^+} - c\beta_k^{p^-}\left(p^-\right)t^{p^-} - c\beta_k^{q^-}(q(x))t^{q^-} - c\beta_k^\delta(\delta)t^\delta - c\beta_k^\sigma(\sigma)t^\sigma \\
&\geq \frac{1}{p^+}t^{q^-} - c\beta_k^{q^-}(q(x))t^{q^-} - \left(c\beta_k^{p^-}\left(p^-\right) + c\beta_k^\delta(\delta) + c\beta_k^\sigma(\sigma)\right)t^\sigma,
\end{aligned}
$$

since $\sigma < \delta < p^- < p^+ < q^-$ for sufficiently large k, by choosing $c\beta_k^{q^-}(q(x)) < \frac{1}{2p^+}$, we get

$$
\mathbf{\Phi}_\lambda(t\varphi) \geq \frac{1}{2p^+}t^{q^-} - \left(c\beta_k^{p^-}\left(p^-\right) + c\beta_k^\delta(\delta) + c\beta_k^\sigma(\sigma)\right)t^\sigma. \tag{3.1}
$$

Put $\rho_k := \left(2cp^+\beta_k^{p^-}(p^-) + 2cp^+\beta_k^\delta(\delta) + 2cp^+\boldsymbol{\beta}_k^\sigma(\sigma)\right)^{\frac{1}{q^- - \sigma}}$, then, for sufficiently large k, $\rho_k < 1$. When $t = \rho_k$, $\varphi \in Z_k$ with $\|\varphi\| = 1$, we have $\Phi_\lambda(t\varphi) \geq 0$. So, for sufficiently large k, we obtain $a_k(\lambda) := \inf_{u \in Z_k, \|u\| = \rho_k} \Phi_\lambda(u) \geq 0$.

Finally, we prove

$$
d_k(\lambda) := \inf_{u \in Z_k, \|u\| \leq \rho_k} \Phi_\lambda(u) \to 0
$$

as $k \to \infty$ uniformly. Indeed, since $Y_k \cap Z_k \neq \varnothing$ and $r_k < \rho_k$, we have

$$
d_k(\lambda) := \inf_{u \in Z_k, \|u\| \leq \rho_k} \Phi_\lambda(u) \leq b_k(\lambda) := \max_{u \in Y_k, \|u\| = r_k} \Phi_\lambda(u) < 0.
$$

By (3.1), for $\varphi \in Z_k$, $\|\varphi\| = 1$, $0 \leq t \leq \rho_k$ and $u = t\varphi$ it follows that

$$
\begin{aligned}
\Phi_\lambda(u) = \Phi_\lambda(t\varphi) &\geq \frac{1}{2p^+}t^{q^-} - \left(c\beta_k^{p^-}\left(p^-\right) + c\beta_k^\delta(\delta) + c\beta_k^\sigma(\sigma)\right)t^\sigma \\
&\geq -\left(c\beta_k^{p^-}\left(p^-\right) + c\beta_k^\delta(\delta) + c\beta_k^\sigma(\sigma)\right)t^\sigma \\
&\geq -\left(c\beta_k^{p^-}\left(p^-\right) + c\beta_k^\delta(\delta) + c\beta_k^\sigma(\sigma)\right)\rho_k^\sigma \\
&\geq -\left(c\beta_k^{p^-}\left(p^-\right) + c\beta_k^\delta(\delta) + c\beta_k^\sigma(\sigma)\right),
\end{aligned}
$$

therefore $d_k(\lambda) \to 0$ as $k \to \infty$. Hence, by Theorem 5 we can find $\lambda_n \to 1$ and $u_n(\lambda) \in Y_n$ desired as the claim. The proof is completed. $\square$

Lemma 9 *$\{u_n(\lambda)\}_{n=1}^{\infty}$ is bounded in $W_0^{1,p(x)}(\Omega)$.*

PROOF. *Since $\Phi'_{\lambda_n}|_{Y_n}(u(\lambda_n)) = 0$, then we have*

$$\Phi'_\lambda(u(\lambda_n)) = \mathbf{A}'(u(\lambda_n)) - K'(u(\lambda_n)) - \lambda_n B'(u(\lambda_n)) = o(1)\|u(\lambda_n)\|,$$

or, by Proposition 1,

$$\begin{aligned} 1 - o(1) &= \lambda_n \int_\Omega \frac{f(x, u(\lambda_n))u(\lambda_n)}{\rho(u(\lambda_n))}dx + \int_\Omega \frac{g(x, u(\lambda_n))u(\lambda_n)}{\rho(u(\lambda_n))}dx \\ &\le \lambda_n \int_\Omega \frac{f(x, u(\lambda_n))u(\lambda_n)}{\|u(\lambda_n)\|^{p^-}}dx + \int_\Omega \frac{g(x, u(\lambda_n))u(\lambda_n)}{\|u(\lambda_n)\|^{p^-}}dx \end{aligned}$$

where $\rho(u(\lambda_n))$ is defined as in (2.1)*. Passing to a subsequence, if necessary, $\|u(\lambda_n)\| \to \infty$ as $n \to \infty$, and using $(\mathbf{P}_2)$ it follows*

$$1 - o(1) \le \int_\Omega \frac{g(x, u(\lambda_n))\, u(\lambda_n)}{\|u(\lambda_n)\|^{p^-}}dx,$$

where $o(1) \to 0$ as $n \to \infty$. This is a contradiction providing that $(\mathbf{P}_4)$ $(\mathbf{1})$ holds.

Let $\{\omega_n\} \subset W_0^{1,p(x)}(\Omega)$ and put $\omega_n := \frac{u(\lambda_n)}{\|u(\lambda_n)\|}$. Since $\|\omega_n\| = 1$, up to subsequences, from Proposition 4 we get

$$\begin{aligned} \omega_n &\rightharpoonup \omega \text{ in } W_0^{1,p(x)}(\Omega), \\ \omega_n &\to \omega \text{ in } L^{\gamma(x)}(\Omega),\ p(x) \le \gamma(x) < p^*(x), \\ \omega_n(x) &\to \omega(x) \text{ a.e. } x \in \Omega. \end{aligned}$$

Then the main concern is that either $\{\omega_n\} \subset W_0^{1,p(x)}(\Omega)$ vanish or it does not vanish. We shall prove that none of these alternatives can occur and this contradiction will prove that $\{\omega_n\} \subset W_0^{1,p(x)}(\Omega)$ is bounded.

If $\omega \neq 0$, from Proposition 1, Fatou's Lemma, $(\mathbf{P}_2), (\mathbf{P}_3)$ and for n large enough, we have

$$\Phi'_\lambda(u(\lambda_n)) = \mathbf{A}'(u(\lambda_n)) - K'(u(\lambda_n)) - \lambda_n B'(u(\lambda_n)) = o(1)\|u(\lambda_n)\|,$$

or

$$
\begin{aligned}
-1+o(1) &= \lambda_n \int_\Omega \frac{-f(x,u(\lambda_n))u(\lambda_n)}{\rho(u(\lambda_n))}dx + \int_\Omega \frac{-g(x,u(\lambda_n))u(\lambda_n)}{\rho(u(\lambda_n))}dx \\
&\geq \lambda_n \int_\Omega \frac{-f(x,u(\lambda_n))u(\lambda_n)}{\|u(\lambda_n)\|^{p^-}}dx + \int_\Omega \frac{-g(x,u(\lambda_n))u(\lambda_n)}{\|u(\lambda_n)\|^{p^-}}dx.
\end{aligned}
$$

Using $\lim\limits_{|u|\to\infty} \frac{g(x,u)}{|u|^{p^- -1}} = -\infty$ in $(\mathbf{P}_4)$ $(\mathbf{2})$**,** we get

$$
\begin{aligned}
-1+o(1) &\geq \int_\Omega \frac{-g(x,u(\lambda_n))u(\lambda_n)}{\|u(\lambda_n)\|^{p^-}}dx = \int_\Omega \frac{-g(x,u(\lambda_n))u(\lambda_n)}{|u(\lambda_n)|^{p^-}} |\omega_n|^{p^-}\, dx \\
&\geq c + \int_{\{\omega\neq 0\}\cap\{|u(\lambda_n)|\geq c\}} \frac{-g(x,u(\lambda_n))u(\lambda_n)}{|u(\lambda_n)|^{p^-}} |\omega_n|^{p^-}\, dx \to \infty,
\end{aligned}
$$

which is a contradiction. Moreover, we can get the similar result if $\lim\limits_{|u|\to\infty} \frac{g(x,u)}{|u|^{p^- -1}} = \infty$ in $(\mathbf{P}_4)$ $(\mathbf{3})$.

If $\omega \equiv 0$, we can define a sequence $\{t_n\} \subset \mathbb{R}$ as in (see [17]) such that

$$
\Phi_{\lambda_n}(t_n u(\lambda_n)) := \max_{t\in[0,1]} \Phi_{\lambda_n}(t u(\lambda_n)).
$$

Let $\overline{\omega}_n := (2p^+c)^{\frac{1}{p^-}}\omega_n$ with $c>0$. Then for n large enough, we have

$$
\begin{aligned}
\Phi_{\lambda_n}(t_n u_n) &\geq \Phi_{\lambda_n}(\overline{\omega}_n) \\
&\geq A\left((2p^+c)^{\frac{1}{p^-}}\omega_n\right) - K\left((2p^+c)^{\frac{1}{p^-}}\omega_n\right) - \lambda_n B\left((2p^+c)^{\frac{1}{p^-}}\omega_n\right) \\
&\geq \frac{1}{p^+}(2p^+c)A(\omega_n) - K(\overline{\omega}_n) - \lambda_n B(\overline{\omega}_n) \geq 2c - K(\overline{\omega}_n) - \lambda_n B(\overline{\omega}_n) \\
&\geq c,
\end{aligned}
$$

which implies that $\lim\limits_{n\to\infty} \Phi_{\lambda_n}(t_n u_n) \to \infty$ by the fact $c>0$ can be large arbitrarily. Noting that $\Phi_{\lambda_n}(0)=0$ and $\Phi_{\lambda_n}(u_n)\to c$, so $0<t_n<1$ when n large enough. Hence we have $\langle \Phi'_{\lambda_n}(t_n u(\lambda_n)), t_n u(\lambda_n)\rangle = 0$. Thus, it follows

$$
\lim_{n\to\infty}[\Phi_{\lambda_n}(t_n u(\lambda_n)) - \frac{1}{\overline{p}_{t_n}} \left\langle \Phi'_{\lambda_n}(t_n u(\lambda_n)), t_n u(\lambda_n)\right\rangle] \to \infty,
$$

where $\overline{p}_{t_n} = \frac{A'(t_n u(\lambda_n))}{A(t_n u(\lambda_n))}$. Therefore,

$$
\lim_{n\to\infty}[(A(t_n u(\lambda_n)) - K(t_n u(\lambda_n)) - \lambda_n B(t_n u(\lambda_n))
$$

$$
-\frac{1}{\overline{p}_{t_n}}(A'(t_n u(\lambda_n)) + \frac{1}{\overline{p}_{t_n}}K'(t_n u(\lambda_n)) + \lambda_n \frac{1}{\overline{p}_{t_n}}B'(t_n u(\lambda_n))] \to \infty,
$$

that is,

$$\lim_{n\to\infty}[\lambda_n\frac{1}{\overline{p}_{t_n}}B'(t_nu(\lambda_n))-\lambda_nB(t_nu(\lambda_n))+\frac{1}{\overline{p}_{t_n}}K'(t_nu(\lambda_n))-K(t_nu(\lambda_n))]\to\infty.$$

Moreover, if $(\mathbf{P}_4)$ (2) holds, we have

$$\frac{1}{\overline{p}_{t_n}}f(x,su)\,su-F(x,su)+\frac{1}{\overline{p}_{t_n}}g(x,su)\,su-G(x,su)\le c,$$

for all $s>0$ and $u\in\mathbb{R}$, so we get a contradiction.

If $(\mathbf{P}_4)$ (**3**) holds, by $(\mathbf{P}_2)$, we get

$$\infty\le\frac{c_2}{\overline{p}_n}\int_\Omega|u(\lambda_n)|^\delta dx+\frac{1}{\overline{p}_n}K'(u(\lambda_n))-K(u(\lambda_n)).$$

Thus,

$$\frac{1}{\overline{p}_n}K'(u(\lambda_n))-K(u(\lambda_n))\to\infty. \tag{3.2}$$

Furthermore, using the property of $u(\lambda_n)$ (see Lemma 8), it follows that

$$\begin{aligned}
b_k(1)&\ge\lambda_n\left(\frac{1}{\overline{p}_n}B'(u(\lambda_n))-B(u(\lambda_n))\right)+\frac{1}{\overline{p}_n}K'(u(\lambda_n))-K(u(\lambda_n))\\
&\ge\frac{1}{\overline{p}_n}\left(\frac{1}{\overline{p}_n}K'(u(\lambda_n))-K(u(\lambda_n))\right)-\frac{c_2}{\overline{p}_n}\int_\Omega|u(\lambda_n)|^\delta dx\\
&\ge c\frac{1}{\overline{p}_n}K'(u(\lambda_n))-K(u(\lambda_n))-c,
\end{aligned}$$

which contradicts (3.2). Therefore $\{u(\lambda_n)\}$ is bounded. The proof is completed. □

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