A regularity criterion for the harmonic heat flow

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Abstract: In this short note, a logarithmically improved regularity criterion for the harmonic heat flow is established for arbitrary dimension.

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1 Introduction

We study the regularity problem for smooth solutions to the time-dependent harmonic heat flow from $\mathbb{R}^n$ into a unit sphere $\mathbb{S}^m$:

$$u_t - \Delta u = u|\nabla u|^2, \ |u| = 1, \tag{1.1}$$

$$u|_{t=0} = u_0(x), \ |u_0| = 1, . \tag{1.2}$$

The regularity of weak solutions fails in general because of the existence of a blow-up solution for large initial data. The example for the map from $B_1(0) \subset \mathbb{R}^n$ to a sphere was shown by Coron-Ghidaglia [3] for $n \geq 3$ and Chang-Ding-Ye [1] for $n = 2$. However, some smallness condition on the initial data or integrability condition on the solution itself are sufficient to give the regularity. Ogawa [9] showed the following regularity condition:

$$\nabla u \in L^2(0, T; \dot{F}^0_{\infty,2}), \tag{1.3}$$

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where $\dot{F}^0_{\infty,2}$ denotes the homogeneous Triebel-Lizorkin space (for the definition, see [10]). Fan-Ozawa [4] obtained the following regularity condition:

$$\nabla u \in L^2(0,T;\dot{F}^0_{\infty,\infty}), \text{ with } 2 \le n \le 4, \tag{1.4}$$

or

$$\Delta u \in L^1(0,T;\dot{F}^0_{\infty,\infty}). \tag{1.5}$$

Here $\dot{F}^0_{\infty,\infty}$ is the homogeneous Triebel-Lizorkin space. Very recently, Fan-Ozawa [5] refined (1.4) to the following condition:

$$\int_0^T \frac{\|\nabla u\|^2_{\dot{F}^0_{\infty,\infty}}}{1+\log^+\|\nabla u\|^0_{\dot{F}_{\infty,\infty}}} dt < \infty, \tag{1.6}$$

when $n = 2, 3, 4$.

In this short note, we will improve (1.5) to a logarithmic form:

Theorem 1.1 *Let $\nabla u_0 \in H^{s-1}(\mathbb{R}^n)$ with $s \ge 2 + \frac{n}{2}$, $|u_0| = 1$ in $\mathbb{R}^n$ and u be a smooth solution to (1.1)-(1.2) for $0 \le t < T$. Then u is smooth at time $t = T$ provided that*

$$\int_0^T \frac{\|\Delta u\|_{\dot{F}^0_{\infty,\infty}}}{1+\log^+\|\Delta u\|_{\dot{F}^0_{\infty,\infty}}} dt < \infty, \tag{1.7}$$

here $f^+(x) = \max\{0, f(x)\}$.

Remark 1.1 *In [2], logarithmic form (1.7) was established only for $n = 2, 3, 4$. Here Theorem 1.1 holds for arbitrary dimension n.*

2 Proof of Theorem 1.1

Before going to the detailed proof, let us recall the following improved Gagliardo-Nirenberg inequality which was introduced in [7, 8] and will be used in the proof:

$$\|\nabla u\|^2_{L^\infty} \le C\|u\|_{L^\infty}\|\Delta u\|_{\dot{F}^0_{\infty,\infty}}. \tag{2.1}$$

We will also use the following product estimate due to Kato and Ponce [6]:

$$\begin{aligned} \|\Lambda^\alpha(fg)\|_{L^p} \le\ & C(\|f\|_{L^{p_1}}\|\Lambda^\alpha g\|_{L^{q_1}} \\ & + \|\Lambda^\alpha f\|_{L^{p_2}}\|g\|_{L^{q_2}}), \end{aligned} \tag{2.2}$$

for $\alpha > 0$, $1 < p < \infty$, and $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{q_1} = \frac{1}{p_2} + \frac{1}{q_2}$, where $\Lambda = (-\Delta)^{\frac{1}{2}}$.

Since we deal with the regularity condition of smooth solutions, it is sufficient to establish the a priori estimates.

First, multiplying (1.1) by u_t, after integration by parts, we see that

$$\frac{1}{2}\frac{d}{dt}\int |\nabla u|^2 dx + \int |u_t|^2 dx = 0. \tag{2.3}$$

Applying Λ^s to (1.1), then multiplying it by $\Lambda^s u$, by (2.1) and (2.2), we have

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int |\Lambda^s u|^2 dx + \int |\Lambda^{s+1} u|^2 dx \\
= &\int \Lambda^s \left(u|\nabla u|^2\right) \cdot \Lambda^s u dx \\
\leq &\; C \left(\|u\|_{L^\infty} \left\|\Lambda^s(|\nabla u|^2)\right\|_{L^2} + \|\Lambda^s u\|_{L^2} \|\nabla u\|_{L^\infty}^2\right) \|\Lambda^s u\|_{L^2} \\
\leq &\; C \left(\left\|\Lambda^s(|\nabla u|^2)\right\|_{L^2} + \|\nabla u\|_{L^\infty}^2 \|\Lambda^s u\|_{L^2}\right) \|\Lambda^s u\|_{L^2} \\
\leq &\; C \left(\|\nabla u\|_{L^\infty} \left\|\Lambda^{s+1} u\right\|_{L^2} + \|\nabla u\|_{L^\infty}^2 \|\Lambda^s u\|_{L^2}\right) \|\Lambda^s u\|_{L^2} \\
\leq &\; \frac{1}{2} \left\|\Lambda^{s+1} u\right\|_{L^2}^2 + C\|\nabla u\|_{L^\infty}^2 \|\Lambda^s u\|_{L^2}^2 \\
\leq &\; \frac{1}{2} \left\|\Lambda^{s+1} u\right\|_{L^2}^2 + C\|\Delta u\|_{\dot{F}^0_{\infty,\infty}} \|\Lambda^s u\|_{L^2}^2 \\
\leq &\; C \frac{\|\Delta u\|_{\dot{F}^0_{\infty,\infty}}}{1 + \log^+ \|\Delta u\|_{\dot{F}^0_{\infty,\infty}}} \|\Lambda^s u\|_{L^2}^2 \left(1 + \log^+ \|\Delta u\|_{\dot{F}^0_{\infty,\infty}}\right) \\
&+ \frac{1}{2} \left\|\Lambda^{s+1} u\right\|_{L^2}^2 \\
\leq &\; C \frac{\|\Delta u\|_{\dot{F}^0_{\infty,\infty}}}{1 + \log^+ \|\Delta u\|_{\dot{F}^0_{\infty,\infty}}} \|\Lambda^s u\|_{L^2}^2 \left(1 + \log^+ \|\Lambda^s u\|_{L^2}\right) \\
&+ \frac{1}{2} \left\|\Lambda^{s+1} u\right\|_{L^2}^2,
\end{aligned}$$

which implies (due to Gronwall's inequality),

$$\|\nabla u\|_{L^\infty(0,T;H^{s-1})} \leq C.$$

This completes the proof.

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