FILTER REGULAR SEQUENCES AND GENERALIZED LOCAL COHOMOLOGY MODULES

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ABSTRACT. Let $\mathfrak{a}$, $\mathfrak{b}$ be ideals of a commutative Noetherian ring R and let M, N be finite R-modules. The concept of an $\mathfrak{a}$-filter grade of $\mathfrak{b}$ on M is introduced and several characterizations and properties of this notion are given. Then, using the above characterizations, we obtain some results on generalized local cohomology modules $\mathrm{H}^i_{\mathfrak{a}}(M,N)$. In particular, first we determine the least integer i for which $\mathrm{H}^i_{\mathfrak{a}}(M,N)$ is not Artinian. Then we prove that $\mathrm{H}^i_{\mathfrak{a}}(M,N)$ is Artinian for all $i \in \mathbb{N}_0$ if and only if $\dim R/(\mathfrak{a} + \operatorname{Ann} M + \operatorname{Ann} N) = 0$. Also, we establish the Nagel-Schenzel formula for generalized local cohomology modules. Finally, in a certain case, the set of attached primes of $\mathrm{H}^i_{\mathfrak{a}}(M,N)$ is determined and a comparison between this set and the set of attached primes of $\mathrm{H}^i_{\mathfrak{a}}(N)$ is given.

1. Introduction

Throughout this paper, R is a commutative Noetherian ring with nonzero identity, $\mathfrak{a}$, $\mathfrak{b}$ are ideals of R and M, N ,L are finite R-modules. We will use $\mathbb{N}$ (respectively $\mathbb{N}_0$) to denote the set of positive (respectively non-negative) integers.

The theory of local cohomology, which was introduced by Grothendieck, is a useful tool for attacking problems in commutative algebra and algebraic geometry. The reader is referred to [6] for basic facts concerning local cohomology modules. It is an interesting problem [12, third problem] to determine if a given local cohomology module is Artinian. In recent years there have appeared many papers in this area (see [2, 23, 28], for example). In this paper, among other things, we provide some results on Artinianness of local cohomology modules and, furthermore, we identify the least integer i such that the i-th local cohomology module is not Artinian.

Indeed, in this paper, we consider the concept of generalized local cohomology functor which was first introduced, in the local case, by Herzog [11] and, in the general case, by Bijan-Zadeh [4]. The i-th generalized local cohomology functor $\mathrm{H}^i_{\mathfrak{a}}(\cdot\,,\cdot)$ is defined by

$$\mathrm{H}^i_{\mathfrak{a}}(X,Y) = \varinjlim_n \operatorname{Ext}^i_R(X/\mathfrak{a}^n X, Y)$$

for all R-modules X, Y and $i \in \mathbb{N}_0$. Clearly, this notion is a natural generalization of the ordinary local cohomology functor.

There is a lot of current interest in the theory of filter regular sequences in commutative algebra; and, in recent years, there have appeared many papers concerned with the role of these sequences in the theory of local cohomology. In particular case, when one works on a local ring, the concept of a filter regular sequence has

2010 *Mathematics Subject Classification.* 13D45, 13E10.

Key words and phrases. generalized local cohomology module, filter regular sequence, Nagel-Schenzel formula, Artinianness, Attached prime.

been studied in [26, 29] and has led to some interesting results. We will denote the supremum of all numbers $n \in \mathbb{N}_0$ for which there exists an $\mathfrak{a}$-filter regular M-sequence of length n in $\mathfrak{b}$ by f-grad$(\mathfrak{a}, \mathfrak{b}, M)$. In a local ring $(R, \mathfrak{m})$, f-grad$(\mathfrak{m}, \mathfrak{a}, M)$ is known as f-depth$(\mathfrak{a}, M)$. Lü and Tang [15] proved that

$$\text{f-depth}(\mathfrak{a}, M) = \inf\{i \in \mathbb{N}_0 | \dim \operatorname{Ext}^i_R(R/\mathfrak{a}, M) > 0\}$$

and that f-depth$(\mathfrak{a}, M)$ is the least integer i such that $\mathrm{H}^i_{\mathfrak{a}}(M)$ is not Artinian. As a theorem, we generalize their results and characterize f-grad$(\mathfrak{a}, \mathfrak{b}, M)$ to non local cases as follows.

$$\begin{aligned} \text{f-grad}(\mathfrak{a}, \mathfrak{b}, M) &= \inf\{i \in \mathbb{N}_0 | \operatorname{Supp} \operatorname{Ext}^i_R(R/\mathfrak{b}, M) \nsubseteq V(\mathfrak{a})\} \\ &= \inf\{i \in \mathbb{N}_0 | \operatorname{Supp} \mathrm{H}^i_{\mathfrak{b}}(M) \nsubseteq V(\mathfrak{a})\}, \\ \text{f-grad}(\mathfrak{a}, \mathfrak{b} + \operatorname{Ann} N, M) &= \inf\{i \in \mathbb{N}_0 | \operatorname{Supp} \mathrm{H}^i_{\mathfrak{b}}(N, M) \nsubseteq V(\mathfrak{a})\}, \end{aligned}$$

and

$$\begin{aligned} &\sup_{A \in \mathcal{M}} \text{f-grad}\Big(\bigcap_{\mathfrak{m} \in A} \mathfrak{m}, \mathfrak{a} + \operatorname{Ann} M, N\Big) \\ &\qquad = \inf\{i \in \mathbb{N}_0 | \mathrm{H}^i_{\mathfrak{a}}(M, N) \text{ is not Artinian}\} \\ &\qquad = \inf\{i \in \mathbb{N}_0 | \operatorname{Supp} \mathrm{H}^i_{\mathfrak{a}}(M, N) \nsubseteq \max(R)\} \\ &\qquad = \inf\{i \in \mathbb{N}_0 | \operatorname{Supp} \mathrm{H}^i_{\mathfrak{a}}(M, N) \nsubseteq A \text{ for all } A \in \mathcal{M}\} \\ &\qquad = \inf\{i \in \mathbb{N}_0 | \dim \operatorname{Ext}^i_R(M/\mathfrak{a}M, N) > 0\}, \end{aligned}$$

where $\mathcal{M}$ is the set of all finite subsets of $\max(R)$.

As an application of this theorem, we show that, if $n \in \mathbb{N}$, then $\mathrm{H}^i_{\mathfrak{a}}(M, N)$ is Artinian for all $i < n$ if and only if $\mathrm{H}^i_{\mathfrak{a}R_{\mathfrak{p}}}(M_{\mathfrak{p}}, N_{\mathfrak{p}})$ is Artinian for all $i < n$ and all prime ideals $\mathfrak{p}$. Also, we prove that $\mathrm{H}^i_{\mathfrak{a}}(M, N)$ is an Artinian R-module for all $i \in \mathbb{N}_0$ if and only if $\dim R/(\mathfrak{a} + \operatorname{Ann} M + \operatorname{Ann} N) = 0$. In particular, $\operatorname{Ext}^i_R(M, N)$ has finite length for all $i \in \mathbb{N}_0$ if and only if $\dim R/(\operatorname{Ann} M + \operatorname{Ann} N) = 0$.

Let $x_1, \ldots, x_n$ be an $\mathfrak{a}$-filter regular N-sequence in $\mathfrak{a}$. Then the formula

$$\mathrm{H}^i_{\mathfrak{a}}(N) = \begin{cases} \mathrm{H}^i_{(x_1,\ldots,x_n)}(N) & \text{if } i < n\,, \\ \mathrm{H}^{i-n}_{\mathfrak{a}}(\mathrm{H}^n_{(x_1,\ldots,x_n)}(N)) & \text{if } i \geq n\,, \end{cases}$$

is known as Nagel-Schenzel formula (see [22] and [13]). We generalize the above formula for the generalized local cohomology modules. Indeed, we prove that:

(i) $\mathrm{H}^i_{\mathfrak{a}}(M, N) \cong \mathrm{H}^i_{(x_1,\ldots,x_n)}(M, N)$ for all $i < n$;

(ii) if $\operatorname{proj}\dim M = d$ and L is projective, then

$$\mathrm{H}^{i+n}_{\mathfrak{a}}(M \otimes_R L, N) \cong \mathrm{H}^i_{\mathfrak{a}}(M, \mathrm{H}^n_{(x_1,\ldots,x_n)}(L, N))$$

for all $i \geq d$.

Assume that $\bar{R} = R/(\mathfrak{a} + \operatorname{Ann} M + \operatorname{Ann} N)$ and that the ideal $\mathfrak{r}$ is the inverse image of the Jacobson radical of $\bar{R}$ in R. If $\bar{R}$ is semi local, then, by using the isomorphisms described in (i) and Theorem 4.2, we prove that

$$\begin{aligned} \text{f-grad}(\mathfrak{r}, \mathfrak{a} + \operatorname{Ann} M, N) &= \inf\{i \in \mathbb{N}_0 | \mathrm{H}^i_{\mathfrak{a}}(M, N) \text{ is not Artinian}\} \\ &= \inf\{i \in \mathbb{N}_0 | \mathrm{H}^i_{\mathfrak{a}}(M, N) \ncong \mathrm{H}^i_{\mathfrak{r}}(M, N)\}. \end{aligned}$$

Let $(R,\mathfrak{m})$ be a local ring and $\dim N = n$. Macdonald and Sharp [17, Theorem 2.2] show that

$$\operatorname{Att}\mathrm{H}^n_{\mathfrak{m}}(N) = \{\mathfrak{p} \in \operatorname{Ass} N \mid \dim R/\mathfrak{p} = n\}.$$

As an extension of this result, Dibaei and Yassemi [9, Theorem A] proved

$$\operatorname{Att}\mathrm{H}^n_{\mathfrak{a}}(N) = \{\mathfrak{p} \in \operatorname{Ass} N \mid \operatorname{cd}_{\mathfrak{a}}(R/\mathfrak{p}) = n\},$$

where $\operatorname{cd}_{\mathfrak{a}}(M)$ is the greatest integer i such that $\mathrm{H}^i_{\mathfrak{a}}(M) \neq 0$. Finally, if $d = \operatorname{proj\,dim} M < \infty$, then Gu and Chu [10, Theorem 2.3] proved that $\mathrm{H}^{n+d}_{\mathfrak{a}}(M,N)$ is Artinian and

$$\operatorname{Att}\mathrm{H}^{n+d}_{\mathfrak{a}}(M,N) = \{\mathfrak{p} \in \operatorname{Ass} N \mid \operatorname{cd}_{\mathfrak{a}}(M,R/\mathfrak{p}) = n+d\},$$

where, for an R-module Y, $\operatorname{cd}_{\mathfrak{a}}(M,Y)$ is the greatest integer i such that $\mathrm{H}^i_{\mathfrak{a}}(M,Y) \neq 0$. Notice that $\operatorname{cd}_{\mathfrak{a}}(M,N) \leq d+n$ [4, Lemma 5.1]. We prove the above result in general case where R is not necessarily local. As a corollary we deduce that $\operatorname{Att}\mathrm{H}^{n+d}_{\mathfrak{a}}(M,N) \subseteq \operatorname{Att}\mathrm{H}^n_{\mathfrak{a}}(N)$. Also,we give an example to show that this inclusion may be strict. Indeed, our example not only show that the Theorem 2.1 of [19] is not true, but it also rejects all of the following conclusions in [19].

Finally, Let $\operatorname{proj\,dim} M = d < \infty$ and $\dim N = n < \infty$ and $\mathfrak{b} = Ann\,\mathrm{H}^n_{\mathfrak{a}}(N)$. We prove that, if $R/\mathfrak{b}$ is a complete semilocal ring, then

$$\operatorname{Att}\mathrm{H}^{n+d}_{\mathfrak{a}}(M,N) = \operatorname{Supp}\operatorname{Ext}^d_R(M,R) \cap \operatorname{Att}\mathrm{H}^n_{\mathfrak{a}}(N).$$

In particular, if in addition, $\operatorname{proj\,dim}_{R_{\mathfrak{p}}} M_{\mathfrak{p}} = \operatorname{proj\,dim} M$ for all $\mathfrak{p} \in \operatorname{Supp} M$, then

$$\operatorname{Att}\mathrm{H}^{n+d}_{\mathfrak{a}}(M,N) = \operatorname{Supp} M \cap \operatorname{Att}\mathrm{H}^n_{\mathfrak{a}}(N).$$

2. Filter regular sequences

We say that a sequence $x_1,\dots,x_n$ of elements of R is an $\mathfrak{a}$-filter regular M-sequence, if $x_i \notin \mathfrak{p}$ for all $\mathfrak{p} \in \operatorname{Ass} M/(x_1,\dots,x_{i-1})M \setminus V(\mathfrak{a})$ and for all $i = 1,\dots,n$. In addition, if $x_1,\dots,x_n$ belong to $\mathfrak{b}$, then we say that $x_1,\dots,x_n$ is an $\mathfrak{a}$-filter regular M-sequence in $\mathfrak{b}$. Note that $x_1,\dots,x_n$ is an R-filter regular M-sequence if and only if it is a weak M-sequence in the sense of [7, Definition 1.1.1].

Some parts of the next elementary proposition are included in [22, Proposition 2.2] in the case where $(R,\mathfrak{m})$ is local and $\mathfrak{a} = \mathfrak{m}$.

Proposition 2.1. *Let $x_1,\dots,x_n$ be a sequence of elements of R and $n \in \mathbb{N}$. Then the following statements are equivalent.*

- (i) *$x_1,\dots,x_n$ is an $\mathfrak{a}$-filter regular M-sequence.*
- (ii) *$\operatorname{Supp}((x_1,\dots,x_{i-1})M :_M x_i)/(x_1,\dots,x_{i-1})M \subseteq V(\mathfrak{a})$ for all $i = 1,\dots,n$.*
- (iii) *$x_1/1,\dots,x_n/1$ is a weak $M_{\mathfrak{p}}$-sequence for all $\mathfrak{p} \in \operatorname{Supp} M \setminus V(\mathfrak{a})$.*
- (iv) *$x_1^{\alpha_1},\dots,x_n^{\alpha_n}$ is an $\mathfrak{a}$-filter regular M-sequence for all positive integers $\alpha_1,\dots,\alpha_n$.*
- (v) *x_i is a weak $(M/(x_1,\dots,x_{i-1})M)/\Gamma_{\mathfrak{a}}(M/(x_1,\dots,x_{i-1})M)$-sequence for all $i = 1,\dots,n$.*
- (vi) *$(x_1,\dots,x_{i-1})M :_M x_i \subseteq (x_1,\dots,x_{i-1})M :_M \langle\mathfrak{a}\rangle$ for all $i = 1,\dots,n$, where $N :_M \langle\mathfrak{a}\rangle = \{x \in M \mid \mathfrak{a}^m x \subseteq N \text{ for some } m \in \mathbb{N}\}$ for any submodule N of M.*

It is clear from definition, that, for a given $n \in \mathbb{N}$, one can find an $\mathfrak{a}$-filter regular M-sequence of length n. The following theorem characterizes the existence of an $\mathfrak{a}$-filter regular M-sequence of length n in $\mathfrak{b}$.

Theorem 2.2. *Let $n \in \mathbb{N}$. Then the following statements are equivalent.*

(i) $\mathfrak{b}$ *contains an* $\mathfrak{a}$*-filter regular* M*-sequence of length* n.
(ii) *Any* $\mathfrak{a}$*-filter regular* M*-sequence in* $\mathfrak{b}$ *of length less than* n *can be extended to an* $\mathfrak{a}$*-filter regular* M*-sequence of length* n *in* $\mathfrak{b}$.
(iii) $\operatorname{Supp}\operatorname{Ext}^i_R(R/\mathfrak{b}, M) \subseteq V(\mathfrak{a})$ *for all* $i < n$.
(iv) *If* $\operatorname{Supp} N = V(\mathfrak{b})$*, then* $\operatorname{Supp}\operatorname{Ext}^i_R(N, M) \subseteq V(\mathfrak{a})$ *for all* $i < n$.
(v) $\operatorname{Supp}\operatorname{H}^i_{\mathfrak{b}}(M) \subseteq V(\mathfrak{a})$ *for all* $i < n$.
(vi) *If* $\operatorname{Ann} N \subseteq \mathfrak{b}$*, then* $\operatorname{Supp}\operatorname{H}^i_{\mathfrak{b}}(N, M) \subseteq V(\mathfrak{a})$ *for all* $i < n$.

Proof. The implications (ii)⇒ (i), (iv)⇒ (iii) and (vi)⇒ (v) are clear.

(i)⇒ (ii). Assume the contrary that $x_1, \ldots, x_t$ is an $\mathfrak{a}$-filter regular M-sequence in $\mathfrak{b}$ such that $t < n$ and that it can not be extended to an $\mathfrak{a}$-filter regular M-sequence of length n in $\mathfrak{b}$. Then $\mathfrak{b} \subseteq \mathfrak{p}$ for some $\mathfrak{p} \in \operatorname{Ass} M/(x_1, \ldots, x_t)M \setminus V(\mathfrak{a})$. So that $\mathfrak{b}R_{\mathfrak{p}} \subseteq \mathfrak{p}R_{\mathfrak{p}} \in \operatorname{Ass}_{R_{\mathfrak{p}}} M_{\mathfrak{p}}/(x_1/1, \ldots, x_t/1)M_{\mathfrak{p}}$. It follows that $x_1/1, \ldots, x_t/1$ is a maximal $M_{\mathfrak{p}}$-sequence in $\mathfrak{b}R_{\mathfrak{p}}$, which is a contradiction in view of the hypothesis, Proposition 2.1 and [7, Theorem 1.2.5].

(i)⇒ (iv) Suppose that $x_1, \ldots, x_n$ is an $\mathfrak{a}$-filter regular M-sequence in $\mathfrak{b}$. Let $t \in \mathbb{N}$ be such that $x_i^t \in \operatorname{Ann} N$ for all $i = 1, \ldots, n$. By Proposition 2.1, for any $\mathfrak{p} \in \operatorname{Supp} M \setminus V(\mathfrak{a})$, $x_1^t/1, \ldots, x_n^t/1$ is a weak $M_{\mathfrak{p}}$-sequence in $\operatorname{Ann}_{R_{\mathfrak{p}}} N_{\mathfrak{p}}$. So that, for all $i < n$, we have $\operatorname{Ext}^i_{R_{\mathfrak{p}}}(N_{\mathfrak{p}}, M_{\mathfrak{p}}) = 0$. Therefore (iv) holds.

(i)⇒(vi) Suppose that $x_1, \ldots, x_n$ is an $\mathfrak{a}$-filter regular M-sequence in $\mathfrak{b}$. For any $\mathfrak{p} \in \operatorname{Supp} M \setminus V(\mathfrak{a})$, $x_1/1, \ldots, x_n/1$ is a weak $M_{\mathfrak{p}}$-sequence in $\mathfrak{b}R_{\mathfrak{p}}$. So that, by [4, Proposition 5.5], $\operatorname{H}^i_{\mathfrak{b}R_{\mathfrak{p}}}(N_{\mathfrak{p}}, M_{\mathfrak{p}}) = 0$ for all $i < n$. This proves the implication (i)⇒(vi).

Next we prove the implications (iii)⇒(i) and (v)⇒(i) by induction on n. Let $n = 1$. In either cases $\operatorname{Supp}\operatorname{Hom}_R(R/\mathfrak{b}, M) \subseteq V(\mathfrak{a})$. Therefore (i) holds. Suppose that, for all $i \in \mathbb{N}_0$, $T^i(\cdot)$ is either $\operatorname{Ext}^i_R(R/\mathfrak{b}, \cdot)$ or $\operatorname{H}^i_{\mathfrak{b}}(\cdot)$. Assume that $n > 1$ and $\operatorname{Supp} T^i(M) \subseteq V(\mathfrak{a})$ for all $i < n$. Then $\mathfrak{b}$ contains an $\mathfrak{a}$-filter regular M-sequence, say x_1. The exact sequences

$$0 \longrightarrow 0 :_M x_1 \longrightarrow M \xrightarrow{x_1} x_1 M \longrightarrow 0$$

and

$$0 \longrightarrow x_1 M \longrightarrow M \longrightarrow M/x_1 M \longrightarrow 0$$

induce the long exact sequences

$$\cdots \longrightarrow T^i(0 :_M x_1) \longrightarrow T^i(M) \longrightarrow T^i(x_1 M) \longrightarrow T^{i+1}(0 :_M x_1) \longrightarrow \cdots$$

and

$$\cdots \longrightarrow T^i(x_1 M) \longrightarrow T^i(M) \longrightarrow T^i(M/x_1 M) \longrightarrow T^{i+1}(x_1 M) \longrightarrow \cdots .$$

Since $\operatorname{Supp} 0 :_M x_1 \subseteq V(\mathfrak{a})$, by Proposition 2.1, it follows that $\operatorname{Supp} T^i(0 :_M x_1) \subseteq V(\mathfrak{a})$ for all $i \in \mathbb{N}_0$. Therefore, using the above long exact sequences, we have $\operatorname{Supp} T^i(M/x_1 M) \subseteq V(\mathfrak{a})$ for all $i < n - 1$. Hence, by inductive hypothesis, $\mathfrak{b}$ contains an $\mathfrak{a}$-filter regular $M/x_1 M$-sequence of length $n - 1$ such as $x_2, \ldots, x_n$. This completes the inductive step, since $x_1, \ldots, x_n$ is an $\mathfrak{a}$-filter regular M-sequence in $\mathfrak{b}$. □

Remark 2.3. One may use Theorem 2.2 (iii)⇒(ii) and Proposition 2.1 to see that $\operatorname{Supp} M/\mathfrak{b}M \subseteq V(\mathfrak{a})$ if and only if, for each $n \in \mathbb{N}$, there exists an $\mathfrak{a}$-filter regular M-sequence of length n in $\mathfrak{b}$. Moreover, if $\operatorname{Supp} M/\mathfrak{b}M \nsubseteq V(\mathfrak{a})$,then it follows from

Theorem 2.2 that any two maximal $\mathfrak{a}$-filter regular M-sequences in $\mathfrak{b}$ have the same length. Therefore, we may give the following.

Definition 1. Let $\operatorname{Supp} M/\mathfrak{b}M \nsubseteq V(\mathfrak{a})$. Then the common length of all maximal $\mathfrak{a}$-filter regular M-sequences in $\mathfrak{b}$ is denoted by $\text{f-grad}(\mathfrak{a},\mathfrak{b},M)$ and is called the $\mathfrak{a}$-filter grade of $\mathfrak{b}$ on M. We set $\text{f-grad}(\mathfrak{a},\mathfrak{b},M)=\infty$ whenever $\operatorname{Supp} M/\mathfrak{b}M \subseteq V(\mathfrak{a})$.

Let $(R,\mathfrak{m})$ be a local ring. Then the $\mathfrak{m}$-filter grade of $\mathfrak{b}$ on M is called the filter depth of $\mathfrak{b}$ on M and is denoted by $\text{f-depth}(\mathfrak{b},M)$. Notice that, by Remark 2.3, $\text{f-depth}(\mathfrak{b},M)<\infty$ if and only if $M/\mathfrak{b}M$ has finite length.

Remark 2.4. The following equalities follows immediately from Theorem 2.2.

$$\text{f-grad}(\mathfrak{a},\operatorname{Ann} N,M)=\inf\{i\in\mathbb{N}_0 |\ \operatorname{Supp}\operatorname{Ext}^i_R(N,M)\nsubseteq V(\mathfrak{a})\},$$
$$\text{f-grad}(\mathfrak{a},\mathfrak{b}+\operatorname{Ann} N,M)=\inf\{i\in\mathbb{N}_0 |\ \operatorname{Supp}\operatorname{H}^i_{\mathfrak{b}}(N,M)\nsubseteq V(\mathfrak{a})\}.$$

In particular,

$$\begin{aligned}\text{f-grad}(\mathfrak{a},\mathfrak{b},M)&=\inf\{i\in\mathbb{N}_0 |\ \operatorname{Supp}\operatorname{Ext}^i_R(R/\mathfrak{b},M)\nsubseteq V(\mathfrak{a})\}\\&=\inf\{i\in\mathbb{N}_0 |\ \operatorname{Supp}\operatorname{H}^i_{\mathfrak{b}}(M)\nsubseteq V(\mathfrak{a})\}.\end{aligned}$$

Suppose in addition that $(R,\mathfrak{m})$ is local. Then

$$\begin{aligned}\text{f-depth}(\mathfrak{b},M)&=\inf\{i\in\mathbb{N}_0 |\ \dim\operatorname{Ext}^i_R(R/\mathfrak{b},M)>0\}\\&=\inf\{i\in\mathbb{N}_0 |\ \operatorname{Supp}\operatorname{H}^i_{\mathfrak{b}}(M)\nsubseteq\{\mathfrak{m}\}\}.\end{aligned}$$

3. A generalization of Nagel-Schenzel formula

Let $x_1,\ldots,x_n$ be an $\mathfrak{a}$-filter regular M-sequence in $\mathfrak{a}$. Then, by [13, Proposition 1.2],

$$\operatorname{H}^i_{\mathfrak{a}}(M)=\begin{cases}\operatorname{H}^i_{(x_1,\ldots,x_n)}(M) & \text{if } i<n\,,\\ \operatorname{H}^{i-n}_{\mathfrak{a}}(\operatorname{H}^n_{(x_1,\ldots,x_n)}(M)) & \text{if } i\geq n\,.\end{cases}$$

This formula was first obtained by Nagel and Schenzel, in [22, Lemma 3.4], in the case where $(R,\mathfrak{m})$ is a local ring and $\mathfrak{a}=\mathfrak{m}$. Afterwards Khashyarmanesh, Yassi and Abbasi [14, Theorem 3.2] and Mafi [18, Lemma 2.8] generalized the second part of this formula for the generalized local cohomology modules as follows.

Suppose that M has finite projective dimension d and that $x_1,\ldots,x_n$ is an $\mathfrak{a}$-filter regular N-sequence in $\mathfrak{a}$. Then

$$\operatorname{H}^{i+n}_{\mathfrak{a}}(M,N)\cong\operatorname{H}^i_{\mathfrak{a}}(M,\operatorname{H}^n_{(x_1,\ldots,x_n)}(N))$$

for all $i\geq d$.

The following theorem establishes the Nagel-Schenzel formula for the generalized local cohomology modules. The first part of the following theorem is needed in the proof of the Corollary 4.5.

Theorem 3.1. *Let $x_1,\ldots,x_n$ be an $\mathfrak{a}$-filter regular N-sequence in $\mathfrak{a}$. Then the following statements hold.*

(i) $\operatorname{H}^i_{\mathfrak{a}}(M,N)\cong\operatorname{H}^i_{(x_1,\ldots,x_n)}(M,N)$ *for all* $i<n$.

(ii) *If* $\operatorname{proj\,dim} M=d<\infty$ *and* L *is projective, then*

$$\operatorname{H}^{i+n}_{\mathfrak{a}}(M\otimes_R L,N)\cong\operatorname{H}^i_{\mathfrak{a}}(M,\operatorname{H}^n_{(x_1,\ldots,x_n)}(L,N))$$

for all $i\geq d$.

Proof. (i) Set $\boldsymbol{x} = x_1, \dots, x_n$. Since $\Gamma_{\mathfrak{a}}(N) \subseteq \Gamma_{(\boldsymbol{x})}(N)$ we have a natural monomorphism $\varphi_{M,N} : \mathrm{H}^0_{\mathfrak{a}}(M,N) \to \mathrm{H}^0_{(\boldsymbol{x})}(M,N)$. Now, let $\mu_i(\mathfrak{p}, N)$ be the i-th Bass number of N with respect to a prime ideal $\mathfrak{p}$ and let $0 \longrightarrow E^0 \xrightarrow{d^0} E^1 \xrightarrow{d^1} E^2 \longrightarrow \cdots$ be the minimal injective resolution of N. Then, by Proposition 2.1, $\mu_i(\mathfrak{p}, N) = 0$ for all $\mathfrak{p} \in \operatorname{Supp} N \cap V(\boldsymbol{x}) \setminus V(\mathfrak{a})$ and all $i < n$. So

$$\begin{aligned}\Gamma_{\mathfrak{a}}(E^i) &= \bigoplus_{\mathfrak{p} \in \operatorname{Supp} N \cap V(\mathfrak{a})} E(R/\mathfrak{p})^{\mu_i(\mathfrak{p},N)} \\ &= \bigoplus_{\mathfrak{p} \in \operatorname{Supp} N \cap V(\boldsymbol{x})} E(R/\mathfrak{p})^{\mu_i(\mathfrak{p},N)} = \Gamma_{(\boldsymbol{x})}(E^i)\end{aligned}$$

for all $i < n$. Therefore φ_{M,E^i} is an isomorphism for all $i < n$. Now let $i < n$. Since $\varphi_{M,E^{i-1}}$ and φ_{M,E^i} are isomorphisms and $\varphi_{M,E^{i+1}}$ is a monomorphism, one can use the following commutative diagram

$$\begin{array}{ccccc} \mathrm{H}^0_{\mathfrak{a}}(M, E^{i-1}) & \longrightarrow & \mathrm{H}^0_{\mathfrak{a}}(M, E^{i}) & \longrightarrow & \mathrm{H}^0_{\mathfrak{a}}(M, E^{i+1}) \\ \downarrow{\scriptstyle \varphi_{M,E^{i-1}}} & & \downarrow{\scriptstyle \varphi_{M,E^{i}}} & & \downarrow{\scriptstyle \varphi_{M,E^{i+1}}} \\ \mathrm{H}^0_{(\boldsymbol{x})}(M, E^{i-1}) & \longrightarrow & \mathrm{H}^0_{(\boldsymbol{x})}(M, E^{i}) & \longrightarrow & \mathrm{H}^0_{(\boldsymbol{x})}(M, E^{i+1}) \end{array}$$

to see that the induced homomorphism

$$\bar{\varphi}_{M,E^i} : \mathrm{H}^i_{\mathfrak{a}}(M,N) = \frac{\ker \mathrm{H}^0_{\mathfrak{a}}(M, d^i)}{\operatorname{im} \mathrm{H}^0_{\mathfrak{a}}(M, d^{i-1})} \to \frac{\ker \mathrm{H}^0_{(\boldsymbol{x})}(M, d^i)}{\operatorname{im} \mathrm{H}^0_{(\boldsymbol{x})}(M, d^{i-1})} = \mathrm{H}^i_{(\boldsymbol{x})}(M,N)\,,$$

is an isomorphism.

(ii) Set $F(\cdot) = \mathrm{H}^0_{\mathfrak{a}}(M, \cdot)$ and $G(\cdot) = \mathrm{H}^0_{(\boldsymbol{x})}(L, \cdot)$. Then F and G are left exact functors and $FG(\cdot) \cong \mathrm{H}^0_{\mathfrak{a}}(M \otimes_R L, \cdot)$. Furthermore if E is an injective R-module and $\mathsf{R}^p F$ $(p \in \mathbb{N}_0)$ is the p -th right derived functor of F, then it follows from [30, Lemma 1.1] and(i) that

$$\begin{aligned}\mathsf{R}^p F(G(E)) = \mathrm{H}^p_{\mathfrak{a}}(M, \mathrm{H}^0_{(\boldsymbol{x})}(L, E)) &\cong \mathrm{H}^p_{\mathfrak{a}}(M, \mathrm{H}^0_{\mathfrak{a}}(L, E)) \\ &\cong \operatorname{Ext}^p_R(M, \operatorname{Hom}_R(L, \Gamma_{\mathfrak{a}}(E))) = 0\end{aligned}$$

for all $p \geq 1$. This yields the following spectral sequence

$$E_2^{p,q} = \mathrm{H}^p_{\mathfrak{a}}(M, \mathrm{H}^q_{(\boldsymbol{x})}(L,N)) \underset{p}{\Longrightarrow} \mathrm{H}^{p+q}_{\mathfrak{a}}(M \otimes_R L, N)$$

(see for example [24, Theorem 11.38]). Let $t = p + q \geq d + n$. If $q > n$, then $\mathrm{H}^q_{(\boldsymbol{x})}(N) = 0$ by [6, Corollary 3.3.3]. Since L is projective, it therefore follows that $\mathrm{H}^q_{(\boldsymbol{x})}(L,N) = 0$. On the other hand if $q < n$, then $p > d = \operatorname{proj\,dim} M$. Hence

$$E_2^{p,q} = \mathrm{H}^p_{\mathfrak{a}}(M, \mathrm{H}^q_{(\boldsymbol{x})}(L,N)) \cong \mathrm{H}^p_{\mathfrak{a}}(M, \mathrm{H}^q_{\mathfrak{a}}(L,N)) \cong \operatorname{Ext}^p_R(M, \mathrm{H}^q_{\mathfrak{a}}(L,N)) = 0.$$

Therefore, for $t \geq n + d$, there is a collapsing on the line $q = n$. Thus, there are isomorphisms

$$\mathrm{H}^{t-n}_{\mathfrak{a}}(M, \mathrm{H}^n_{(\boldsymbol{x})}(L,N)) \cong \mathrm{H}^t_{\mathfrak{a}}(M \otimes_R L, N)$$

for all $t \geq n + d$. □

4. Artinianness of generalized local cohomology modules

Let $(R,\mathfrak{m})$ be a Noetherian local ring. In view of [21, Theorem 3.1] and [15, Theorem 3.10], one can see that f-depth$(\mathfrak{a}, M)$ is the least integer i for which $\mathrm{H}^i_{\mathfrak{a}}(M)$ is not Artinian. Also, as a main result, it was proved in [8, Theorem 2.2] that f-depth$(\mathfrak{a} + \operatorname{Ann} M, N)$ is the least integer i such that $\mathrm{H}^i_{\mathfrak{a}}(M,N)$ is not Artinian. We use rather a short argument to generalize this to the case in which R is not necessarily a local ring. The following lemma is elementary.

Lemma 4.1 ([25] Exercise 8.49)**.** *Let X be an Artinian R-module, then* $\operatorname{Ass} X = \operatorname{Supp} X$ *is a finite subset of* $\max(R)$.

Theorem 4.2. *Let $\mathcal{M}$ be the set of all finite subsets of* $\max(R)$*. Then*

$$\begin{aligned}\sup_{A\in\mathcal{M}} \text{f-grad}(\cap_{\mathfrak{m}\in A}\mathfrak{m},\mathfrak{a} + \operatorname{Ann} M, N)& \\ &= \inf\{i \in \mathbb{N}_0 |\ \mathrm{H}^i_{\mathfrak{a}}(M,N) \textit{ is not Artinian}\}\\ &= \inf\{i \in \mathbb{N}_0 |\ \operatorname{Supp} \mathrm{H}^i_{\mathfrak{a}}(M,N) \nsubseteq \max(R)\}\\ &= \inf\{i \in \mathbb{N}_0 |\ \operatorname{Supp} \mathrm{H}^i_{\mathfrak{a}}(M,N) \nsubseteq A \ \textit{ for all } A \in \mathcal{M}\}\end{aligned}$$

Proof. Since $\mathrm{H}^i_{\mathfrak{a}}(M,N) \cong \mathrm{H}^i_{\mathfrak{a}+\operatorname{Ann} M}(M,N)$, we can assume that $\operatorname{Ann} M \subseteq \mathfrak{a}$. It is clear that

$$\sup_{A\in\mathcal{M}} \text{f-grad}(\cap_{\mathfrak{m}\in A}\mathfrak{m}, \mathfrak{a}, N) = \inf\{i \in \mathbb{N}_0 |\ \operatorname{Supp} \mathrm{H}^i_{\mathfrak{a}}(M,N) \nsubseteq A \ \text{ for all } A \in \mathcal{M}\}.$$

Let $\mathcal{S}$ be either $\{X \in \mathcal{C}_R |\ \operatorname{Supp} X \subseteq \max(R)\}$ or $\{X \in \mathcal{C}_R |\ \operatorname{Supp} X \subseteq A \text{ for some } A \in \mathcal{M}\}$, where $\mathcal{C}_R$ is the category of R-modules. Set $r = \inf\{i \in \mathbb{N}_0 |\ \mathrm{H}^i_{\mathfrak{a}}(M,N)$ is not Artinian$\}$ and $s = \inf\{i \in \mathbb{N}_0 |\ \mathrm{H}^i_{\mathfrak{a}}(M,N) \notin \mathcal{S}\}$. By Lemma 4.1, $r \leq s$. If $r = \infty$, there is noting to prove. Assume that $r < \infty$. We show by induction on r, that $\mathrm{H}^r_{\mathfrak{a}}(M,N) \notin \mathcal{S}$.

If $r = 0$, then $\mathrm{H}^0_{\mathfrak{a}}(M,N) \notin \mathcal{S}$. Now suppose, inductively, that $r > 0$ and that the result has been proved for smaller values of r. In view of [30, Lemma 1.1] the exact sequence

$$0 \longrightarrow \Gamma_{\mathfrak{a}}(N) \longrightarrow N \longrightarrow N/\Gamma_{\mathfrak{a}}(N) \longrightarrow 0$$

induces the following long exact sequence

$$\begin{aligned}\cdots \longrightarrow \operatorname{Ext}^i_R(M,\Gamma_{\mathfrak{a}}(N)) \longrightarrow \mathrm{H}^i_{\mathfrak{a}}(M,N) \longrightarrow \mathrm{H}^i_{\mathfrak{a}}(M,N/\Gamma_{\mathfrak{a}}(N))\\ \longrightarrow \operatorname{Ext}^{i+1}_R(M,\Gamma_{\mathfrak{a}}(N)) \longrightarrow \cdots .\end{aligned}$$

Since $\mathrm{H}^0_{\mathfrak{a}}(M,N)$ has finite length, we have

$$\operatorname{Supp} \mathrm{H}^0_{\mathfrak{a}}(M,N) = \operatorname{Ass} \operatorname{Hom}_R(M,\Gamma_{\mathfrak{a}}(N)) = \operatorname{Ass} \Gamma_{\mathfrak{a}}(N);$$

so that $\Gamma_{\mathfrak{a}}(N) \in \mathcal{S}$. Thus $\operatorname{Ext}^i_R(M,\Gamma_{\mathfrak{a}}(N)) \in \mathcal{S}$ for all $i \in \mathbb{N}_0$. It follows that for each $i \in \mathbb{N}_0$, $\mathrm{H}^i_{\mathfrak{a}}(M,N) \in \mathcal{S}$ if and only if $\mathrm{H}^i_{\mathfrak{a}}(M,N/\Gamma_{\mathfrak{a}}(N)) \in \mathcal{S}$. Also we have $\mathrm{H}^i_{\mathfrak{a}}(M,N)$ is Artinian if and only if $\mathrm{H}^i_{\mathfrak{a}}(M,N/\Gamma_{\mathfrak{a}}(N))$ is Artinian. Hence we can replace N by $N/\Gamma_{\mathfrak{a}}(N)$ and assume that N is an $\mathfrak{a}$-torsion free R-module. Thus there exists an element $x \in \mathfrak{a}$ which is a non-zero divisor on N. The exact sequence

$$0 \longrightarrow N \xrightarrow{x} N \longrightarrow N/xN \longrightarrow 0$$

induces the long exact sequence

$$\cdots \longrightarrow \mathrm{H}^i_{\mathfrak{a}}(M,N) \xrightarrow{x} \mathrm{H}^i_{\mathfrak{a}}(M,N) \xrightarrow{f_i} \mathrm{H}^i_{\mathfrak{a}}(M,N/xN) \longrightarrow \mathrm{H}^{i+1}_R(M,N) \longrightarrow \cdots .$$

Since $\mathrm{H}^i_{\mathfrak{a}}(M,N)$ is Artinian for all $i<r$, we may use the above sequence to see that $\mathrm{H}^i_{\mathfrak{a}}(M,N/xN)$ is Artinian for all $i<r-1$. On the other hand, $\mathrm{H}^r_{\mathfrak{a}}(M,N)$ is not Artinian. Hence, using the above exact sequence and [6, Theorem 7.1.2], we see that $0:_{\mathrm{H}^r_{\mathfrak{a}}(M,N)} x \cong \mathrm{H}^{r-1}_{\mathfrak{a}}(M,N/xN)/\operatorname{im} f_{r-1}$ is not Artinian. Thus $\mathrm{H}^{r-1}_{\mathfrak{a}}(M,N/xN)$ is not Artinian; and hence, by inductive hypothesis, $\mathrm{H}^{r-1}_{\mathfrak{a}}(M,N/xN)\notin\mathcal{S}$. So, again by using the above sequence, we get $\mathrm{H}^r_{\mathfrak{a}}(M,N)\notin\mathcal{S}$. This completes the inductive step. □

Corollary 4.3. *Suppose that* $\operatorname{Supp} L=\operatorname{Supp} M/\mathfrak{a}M$. *Then*

$$\inf\{i\in\mathbb{N}_0|\ \mathrm{H}^i_{\mathfrak{a}}(M,N) \textit{ is not Artinian}\}=\inf\{i\in\mathbb{N}_0|\ \dim\operatorname{Ext}^i_R(L,N)>0\}.$$

Proof. Let $n\in\mathbb{N}_0$. Then, by the Theorem 4.2, $\mathrm{H}^i_{\mathfrak{a}}(M,N)$ is an Artinian R-module for all $i\leq n$ if and only if $n<\text{f-grad}(\mathfrak{m}_1\cap\ldots\cap\mathfrak{m}_t,\mathfrak{a}+\operatorname{Ann}M,N)$ for some maximal ideals $\mathfrak{m}_1,\ldots,\mathfrak{m}_t$ of R. By the Remark 2.4, it is equivalent to $\operatorname{Supp}\operatorname{Ext}^i_R(L,N)\subseteq\{\mathfrak{m}_1,\ldots,\mathfrak{m}_t\}$ for some maximal ideals $\mathfrak{m}_1,\ldots,\mathfrak{m}_t$ of R and for all $i\leq n$. This proves the assertion. □

The following corollary extend the main result of [28] to the generalized local cohomology modules.

Corollary 4.4. *Let* $n\in\mathbb{N}$. *Then* $\mathrm{H}^i_{\mathfrak{a}}(M,N)$ *is Artinian for all* $i<n$ *if and only if* $\mathrm{H}^i_{\mathfrak{a}R_{\mathfrak{p}}}(M_{\mathfrak{p}},N_{\mathfrak{p}})$ *is Artinian for all* $i<n$ *and all prime ideal* $\mathfrak{p}$.

Proof. This is immediate by the Corollary 4.3. □

Corollary 4.5. *Let* $\overline{R}=R/(\mathfrak{a}+\operatorname{Ann}M+\operatorname{Ann}N)$ *be a semi local ring and let* $\mathfrak{r}$ *be the inverse image of the Jacobson radical of* $\overline{R}$ *in* R. *Then we have*

$$\begin{aligned}\text{f-grad}(\mathfrak{r},\mathfrak{a}+\operatorname{Ann}M,N)&=\inf\{i\in\mathbb{N}_0|\ \mathrm{H}^i_{\mathfrak{a}}(M,N) \textit{ is not Artinian}\}\\&=\inf\{i\in\mathbb{N}_0|\ \mathrm{H}^i_{\mathfrak{a}}(M,N)\ncong\mathrm{H}^i_{\mathfrak{r}}(M,N)\}\end{aligned}$$

Proof. The first equality is immediate by Theorem 4.2. To prove the second equality, let $n\leq\text{f-grad}(\mathfrak{r},\mathfrak{a}+\operatorname{Ann}M,N)$ and let $x_1,\ldots,x_n$ be an $\mathfrak{r}$-filter regular N-sequence in $\mathfrak{a}+\operatorname{Ann}M$. Then $x_1,\ldots,x_n$ is an $\mathfrak{a}+\operatorname{Ann}M$-filter regular N-sequence. So by Theorem 3.1(i),

$$\mathrm{H}^i_{\mathfrak{a}}(M,N)\cong\mathrm{H}^i_{\mathfrak{a}+\operatorname{Ann}M}(M,N)\cong\mathrm{H}^i_{(x_1,\ldots,x_n)}(M,N)\cong\mathrm{H}^i_{\mathfrak{r}}(M,N)$$

for all $i<n$. If $\text{f-grad}(\mathfrak{r},\mathfrak{a}+\operatorname{Ann}M,N)=\infty$, then the above argument shows that, $\inf\{i\in\mathbb{N}_0|\ \mathrm{H}^i_{\mathfrak{a}}(M,N)\ncong\mathrm{H}^i_{\mathfrak{r}}(M,N)\}=\infty$ and therefore the required equality holds. Therefore, we may assume that $\text{f-grad}(\mathfrak{r},\mathfrak{a}+\operatorname{Ann}M,N)=n<\infty$. By the first equality, $\mathrm{H}^n_{\mathfrak{a}}(M,N)$ is not Artinian while $\mathrm{H}^n_{\mathfrak{r}}(M,N)$ is Artinian. Hence the second equality holds. □

It was shown in [31, Theorem 2.2] that if $\dim R/\mathfrak{a}=0$, then $\mathrm{H}^i_{\mathfrak{a}}(M,N)$ is Artinian for all $i\in\mathbb{N}_0$. The following corollary is a generalization of this.

Corollary 4.6. *Let* $\overline{R}=R/(\mathfrak{a}+\operatorname{Ann}M+\operatorname{Ann}N)$. *Then* $\mathrm{H}^i_{\mathfrak{a}}(M,N)$ *is an Artinian* R*-module for all* $i\in\mathbb{N}_0$ *if and only if* $\dim\overline{R}=0$. *In particular,* $\operatorname{Ext}^i_R(M,N)$ *has finite length for all* $i\in\mathbb{N}_0$ *if and only if* $\dim R/(\operatorname{Ann}M+\operatorname{Ann}N)=0$.

Proof. Assume that $\mathfrak{p}$ is a prime ideal of R. By the Corollary 4.5, $\mathrm{H}^i_{\mathfrak{a}R_{\mathfrak{p}}}(M_{\mathfrak{p}},N_{\mathfrak{p}})$ is Artinian for all i if and only if $\text{f-depth}((\mathfrak{a}+\operatorname{Ann}M)R_{\mathfrak{p}},N_{\mathfrak{p}})=\infty$ or equivalently $\dim_{R_{\mathfrak{p}}}N_{\mathfrak{p}}/(\mathfrak{a}R_{\mathfrak{p}}+(\operatorname{Ann}M)R_{\mathfrak{p}})N_{\mathfrak{p}}=0$ (Remark 2.3). Now, the result follows by Corollary 4.4. □

5. Attached primes of the top generalized local cohomology modules

Let $X \neq 0$ be an R-module. If, for every $x \in R$, the endomorphism on X given by multiplication by x is either nilpotent or surjective, then $\mathfrak{p} = \sqrt{\operatorname{Ann} X}$ is prime and X is called a $\mathfrak{p}$-secondary R-module. If for some secondary submodules $X_1, \dots, X_n$ of X we have $X = X_1 + \dots + X_n$, then we say that X has a secondary representation. One may assume that the prime ideals $\mathfrak{p}_i = \sqrt{\operatorname{Ann} X_i}$, $i = 1, \dots, n$, are distinct and, by omitting redundant summands, that the representation is minimal. Then the set $\operatorname{Att} X = \{\mathfrak{p}_1, \dots, \mathfrak{p}_n\}$ does not depend on the choice of a minimal secondary representation of X. Every element of $\operatorname{Att} X$ is called an attached prime ideal of X. It is well known that an Artinian R-module has a secondary representation. The reader is referred to [16] for more information about the theory of secondary representation.

Let $(R, \mathfrak{m})$ be a local ring and $n = \dim N < \infty$ and $d = \operatorname{proj\,dim} M < \infty$. It was proved in [10, Theorem 2.3] that $\mathrm{H}^{n+d}_{\mathfrak{a}}(M,N)$ is Artinian and that

$$\operatorname{Att} \mathrm{H}^{n+d}_{\mathfrak{a}}(M,N) = \{\mathfrak{p} \in \operatorname{Ass} N \,|\, \operatorname{cd}_{\mathfrak{a}}(M, R/\mathfrak{p}) = n + d\},$$

where, for an R-module Y, $\operatorname{cd}_{\mathfrak{a}}(M,Y)$ is the greatest integer i such that $\mathrm{H}^i_{\mathfrak{a}}(M,Y) \neq 0$. Notice that $\operatorname{cd}_{\mathfrak{a}}(M,N) \leq d+n$ [4, Lemma 5.1]. Next, we prove the above result without the local assumption on R. The following lemmas are needed.

Lemma 5.1 ([1] Theorem A and B)**.** *Let* $\operatorname{proj\,dim} M < \infty$*. Then*

(i) $\operatorname{cd}_{\mathfrak{a}}(M,N) \leq \operatorname{cd}_{\mathfrak{a}}(M,L)$ *whenever* $\operatorname{Supp} N \subseteq \operatorname{Supp} L$.

(ii) $\operatorname{cd}_{\mathfrak{a}}(M,L) = \max\{\operatorname{cd}_{\mathfrak{a}}(M,N), \operatorname{cd}_{\mathfrak{a}}(M,K)\}$ *whenever* $0 \to N \to L \to K \to 0$ *is an exact sequence.*

Lemma 5.2. *Let* $\operatorname{proj\,dim} M < \infty$*,* $\dim N < \infty$*,* $t = \operatorname{cd}_{\mathfrak{a}}(M,N) \geq 0$ *and*

$$\Sigma = \{L \subsetneqq N \,|\, \operatorname{cd}_{\mathfrak{a}}(M,L) < t\}.$$

Then Σ has the largest element with respect to inclusion, L say, and the following statements hold.

(i) *If K is a non-zero submodule of N/L, then* $\operatorname{cd}_{\mathfrak{a}}(M,K) = t$.

(ii) $\mathrm{H}^t_{\mathfrak{a}}(M,N) \cong \mathrm{H}^t_{\mathfrak{a}}(M,N/L)$.

(iii) *If* $t = \operatorname{proj\,dim} M + \dim N$*, then*

$$\operatorname{Ass} N/L = \{\mathfrak{p} \in \operatorname{Ass} N \,|\, \operatorname{cd}_{\mathfrak{a}}(M, R/\mathfrak{p}) = t\}.$$

Proof. Since N is Noetherian, Σ has a maximal element, say L. Now assume that L_1, L_2 are elements of Σ. Using the exact sequence

$$0 \to L_1 \cap L_2 \to L_1 \oplus L_2 \to L_1 + L_2 \to 0$$

and Lemma 5.1 we see that $t > \operatorname{cd}_{\mathfrak{a}}(M, L_1 + L_2)$. Hence the sum of any two elements of Σ is again in Σ. It follows that L contains every element of Σ; and so it is the largest one.

(i) Let $K = K'/L$ be a non-zero submodule of N/L. Since L is the largest element of Σ, by applying Lemma 5.1 to the exact sequence

$$0 \to L \to K' \to K \to 0$$

we see that $t = \operatorname{cd}_{\mathfrak{a}}(M,K)$.

(ii) The exact sequence $0 \to L \to N \to N/L \to 0$ induces the exact sequence

$$0 = \mathrm{H}^t_{\mathfrak{a}}(M,L) \to \mathrm{H}^t_{\mathfrak{a}}(M,N) \to \mathrm{H}^t_{\mathfrak{a}}(M,N/L) \to \mathrm{H}^{t+1}_{\mathfrak{a}}(M,L) = 0.$$

Thus $\mathrm{H}^t_{\mathfrak{a}}(M,N) \cong \mathrm{H}^t_{\mathfrak{a}}(M,N/L)$.

(iii) Assume that $\mathrm{cd}_{\mathfrak{a}}(M,N) = \operatorname{proj\,dim} M + \dim N$. For each $\mathfrak{p}$ in $\operatorname{Ass} L$, we have $\mathrm{cd}_{\mathfrak{a}}(M,R/\mathfrak{p}) < t$; so that

$$\{\mathfrak{p} \in \operatorname{Ass} N \mid \mathrm{cd}_{\mathfrak{a}}(M,R/\mathfrak{p}) = t\} \subseteq \operatorname{Ass} N/L.$$

To establish the reverse inclusion, let $\mathfrak{p} \in \operatorname{Ass} N/L$. Then by (i) and [4, Lemma 5.1] $t = \operatorname{proj\,dim} M + \dim R/\mathfrak{p}$. Therefore $\mathfrak{p} \in \operatorname{Ass} N$ and equality holds. □

Theorem 5.3. *Let $d = \operatorname{proj\,dim} M < \infty$ and $n = \dim N < \infty$. Then the R-module $\mathrm{H}^{n+d}_{\mathfrak{a}}(M,N)$ is Artinian and*

$$\operatorname{Att} \mathrm{H}^{n+d}_{\mathfrak{a}}(M,N) = \{\mathfrak{p} \in \operatorname{Ass} N \mid \mathrm{cd}_{\mathfrak{a}}(M,R/\mathfrak{p}) = n+d\}.$$

Proof. Let $\boldsymbol{x} = x_1,\dots,x_n$ be an $\mathfrak{a}$-filter regular N-sequence in $\mathfrak{a}$ and let $E^\bullet$ be the minimal injective resolution of $\mathrm{H}^n_{(\boldsymbol{x})}(N)$. Since, by [6, Exercise 7.1.7], $\mathrm{H}^n_{(\boldsymbol{x})}(N)$ is Artinian, every component of $E^\bullet$ is Artinian. On the other hand by 3.1

$$\mathrm{H}^{n+d}_{\mathfrak{a}}(M,N) \cong \mathrm{H}^d_{\mathfrak{a}}(M,\mathrm{H}^n_{(\boldsymbol{x})}(N)) \cong H^d(\operatorname{Hom}_R(M,\Gamma_{\mathfrak{a}}(E^\bullet))).$$

It follows that $\mathrm{H}^{n+d}_{\mathfrak{a}}(M,N)$ is Artinian.

Now we prove that $\operatorname{Att} \mathrm{H}^{n+d}_{\mathfrak{a}}(M,N) = \{\mathfrak{p} \in \operatorname{Ass} N \mid \mathrm{cd}_{\mathfrak{a}}(M,R/\mathfrak{p}) = n+d\}$. If $\mathrm{cd}_{\mathfrak{a}}(M,N) < n+d$, then $\operatorname{Att} \mathrm{H}^{n+d}_{\mathfrak{a}}(M,N) = \emptyset = \{\mathfrak{p} \in \operatorname{Ass} N \mid \mathrm{cd}_{\mathfrak{a}}(M,R/\mathfrak{p}) = n+d\}$. So one can assume that $t = \mathrm{cd}_{\mathfrak{a}}(M,N) = n+d$. Let L be the largest submodule of N such that $\mathrm{cd}_{\mathfrak{a}}(M,L) < t$. By Lemma 5.2, there is no non-zero submodule K of N/L such that $\mathrm{cd}_{\mathfrak{a}}(M,K) < t$. Also we have $\mathrm{H}^t_{\mathfrak{a}}(M,N) \cong \mathrm{H}^t_{\mathfrak{a}}(M,N/L)$ and $\operatorname{Ass} N/L = \{\mathfrak{p} \in \operatorname{Ass} N \mid \mathrm{cd}_{\mathfrak{a}}(M,R/\mathfrak{p}) = t\}$. Moreover $t = \mathrm{cd}_{\mathfrak{a}}(M,N/L) = \operatorname{proj\,dim} M + \dim N/L$. Thus we may replace N by N/L and prove that $\operatorname{Att} \mathrm{H}^t_{\mathfrak{a}}(M,N) = \operatorname{Ass} N$. Now, for any non-zero submodule K of N, $\mathrm{cd}_{\mathfrak{a}}(M,K) = t$ and $\dim K = n$.

Assume that $\mathfrak{p} \in \operatorname{Att} \mathrm{H}^t_{\mathfrak{a}}(M,N)$. We have $\mathfrak{p} \supseteq \operatorname{Ann} \mathrm{H}^t_{\mathfrak{a}}(M,N) \supseteq \operatorname{Ann} N$. Hence $\mathfrak{p} \in \operatorname{Supp} N$. Now Let $x \in R \setminus \bigcup_{\mathfrak{p} \in \operatorname{Ass} N} \mathfrak{p}$. The exact sequence

$$0 \longrightarrow N \xrightarrow{x} N \longrightarrow N/xN \longrightarrow 0$$

induces the exact sequence

$$\mathrm{H}^t_{\mathfrak{a}}(M,N) \xrightarrow{x} \mathrm{H}^t_{\mathfrak{a}}(M,N) \to \mathrm{H}^t_{\mathfrak{a}}(M,N/xN) = 0.$$

Therefore $x \notin \bigcup_{\mathfrak{p} \in \operatorname{Att} \mathrm{H}^t_{\mathfrak{a}}(M,N)} \mathfrak{p}$. So $\bigcup_{\mathfrak{p} \in \operatorname{Att} \mathrm{H}^t_{\mathfrak{a}}(M,N)} \mathfrak{p} \subseteq \bigcup_{\mathfrak{p} \in \operatorname{Ass} N} \mathfrak{p}$. Thus $\mathfrak{p} \subseteq \mathfrak{q}$ for some $\mathfrak{q} \in \operatorname{Ass} N$. Hence $\mathfrak{p} = \mathfrak{q}$ and $\operatorname{Att} \mathrm{H}^t_{\mathfrak{a}}(M,N) \subseteq \operatorname{Ass} N$. Next we prove the reverse inclusion. Let $\mathfrak{p} \in \operatorname{Ass} N$ and let T be a $\mathfrak{p}$-primary submodule of N. We have $t = \mathrm{cd}_{\mathfrak{a}}(M,R/\mathfrak{p}) = \mathrm{cd}_{\mathfrak{a}}(M,N/T)$. Moreover N/T has no non-zero submodule K such that $\mathrm{cd}_{\mathfrak{a}}(M,K) < t$. Hence, using the above argument, one can show that $\operatorname{Att} \mathrm{H}^t_{\mathfrak{a}}(M,N/T) \subseteq \operatorname{Ass} N/T = \{\mathfrak{p}\}$. It follows that

$$\{\mathfrak{p}\} = \operatorname{Att} \mathrm{H}^t_{\mathfrak{a}}(M,N/T) \subseteq \operatorname{Att} \mathrm{H}^t_{\mathfrak{a}}(M,N).$$

This completes the proof. □

Corollary 5.4. *Let $d = \operatorname{proj\,dim} M < \infty$ and $n = \dim N < \infty$. Then*

$$\operatorname{Att} \mathrm{H}^{n+d}_{\mathfrak{a}}(M,N) \subseteq \operatorname{Supp} M \cap \operatorname{Att} \mathrm{H}^n_{\mathfrak{a}}(N).$$

Proof. If $\operatorname{Att} \mathrm{H}^{n+d}_{\mathfrak{a}}(M,N) = \emptyset$, there is nothing to prove. Assume that $\mathfrak{p} \in \operatorname{Att} \mathrm{H}^{n+d}_{\mathfrak{a}}(M,N)$. Then, by 5.3, $\mathfrak{p} \in \operatorname{Ass} N$ and $\mathrm{H}^{n+d}_{\mathfrak{a}}(M,R/\mathfrak{p}) \neq 0$. Next one can use the spectral sequence

$$E_2^{p,q} = \operatorname{Ext}^p_R(M, \mathrm{H}^q_{\mathfrak{a}}(R/\mathfrak{p})) \underset{p}{\Longrightarrow} \mathrm{H}^{p+q}_{\mathfrak{a}}(M, R/\mathfrak{p})$$

to see that $\mathrm{H}^{n+d}_{\mathfrak{a}}(M,R/\mathfrak{p}) \cong \operatorname{Ext}^d_R(M, \mathrm{H}^n_{\mathfrak{a}}(R/\mathfrak{p}))$. Therefore $\mathrm{H}^n_{\mathfrak{a}}(R/\mathfrak{p}) \neq 0$; and hence $\operatorname{cd}_{\mathfrak{a}}(R/\mathfrak{p}) = n$. Thus, again by 5.3, $\mathfrak{p} \in \operatorname{Att} \mathrm{H}^n_{\mathfrak{a}}(N)$. Also, we have $\mathfrak{p} \supseteq \operatorname{Ann} \operatorname{Ext}^d_R(M, \mathrm{H}^n_{\mathfrak{a}}(N)) \supseteq \operatorname{Ann} M$, which completes the proof. □

Let X be an R-module. Set $E = \bigoplus_{\mathfrak{m} \in \max R} E(R/\mathfrak{m})$ (minimal injective cogenerator of R) and $D = \operatorname{Hom}_R(\cdot, E)$. We note that the canonical map $X \to DDX$ is an injection. If this map is an isomorphism we say that X is (Matlis) reflexive. The following lemma yields a classification of modules which are reflexive with respect to E.

Lemma 5.5 ([3] Theorem 12)**.** *An R-module X is reflexive if and only if it has a finite submodule S such that X/S is artinian and that $R/AnnX$ is a complete semilocal ring.*

Assume that $\mathfrak{a} \subseteq \mathfrak{b}$ and $R/\mathfrak{a}$ is a complete semilocal ring. By above lemma $R/\mathfrak{a}$ is reflexive as an R-module. On the other hand, the category of reflexive R-modules is a Serre subcategory of the category of R-modules. Therefore $R/\mathfrak{b}$ is reflexive as an R-module and hence, by the above lemma, $R/\mathfrak{b}$ is a complete semilocal ring. We shall use the conclusion of this discussion in the proof of the next theorem.

Theorem 5.6. *Let M, N be two finite R-modules with* $\operatorname{proj\,dim} M = d < \infty$ *and* $\dim N = n < \infty$. *Let $\mathfrak{b} = Ann\, \mathrm{H}^n_{\mathfrak{a}}(N)$. If $R/\mathfrak{b}$ is a complete semilocal ring, then*

$$\operatorname{Att} \mathrm{H}^{n+d}_{\mathfrak{a}}(M,N) = \operatorname{Supp} \operatorname{Ext}^d_R(M,R) \cap \operatorname{Att} \mathrm{H}^n_{\mathfrak{a}}(N).$$

In particular, if in addition, $\operatorname{proj\,dim}_{R_{\mathfrak{p}}} M_{\mathfrak{p}} = \operatorname{proj\,dim} M$ *for all* $\mathfrak{p} \in \operatorname{Supp} M$*, then*

$$\operatorname{Att} \mathrm{H}^{n+d}_{\mathfrak{a}}(M,N) = \operatorname{Supp} M \cap \operatorname{Att} \mathrm{H}^n_{\mathfrak{a}}(N).$$

Proof. Since $\operatorname{Ext}^d_R(M,\cdot)$ is a right exact R-linear covariant functor,we have

$$\mathrm{H}^{n+d}_{\mathfrak{a}}(M,N) \cong \operatorname{Ext}^d_R(M, \mathrm{H}^n_{\mathfrak{a}}(N)) \cong \operatorname{Ext}^d_R(M,R) \otimes_R \mathrm{H}^n_{\mathfrak{a}}(N).$$

Set $\mathfrak{c} = \operatorname{Ann} \mathrm{H}^{n+d}_{\mathfrak{a}}(M,N)$. It is clear that $\mathfrak{b} \subseteq \mathfrak{c}$. Therefore $R/\mathfrak{c}$ is a complete semilocal ring. Now, by Lemma 5.5, [6, Exercise 7.2.10] and [5, VI.1.4 Proposition 10] we have

$$\begin{aligned}
\operatorname{Att} \mathrm{H}^{n+d}_{\mathfrak{a}}(M,N) &= \operatorname{Att} DD\, \mathrm{H}^{n+d}_{\mathfrak{a}}(M,N) \\
&= \operatorname{Ass} D\, \mathrm{H}^{n+d}_{\mathfrak{a}}(M,N) \\
&= \operatorname{Ass} D(\operatorname{Ext}^d_R(M,R) \otimes_R \mathrm{H}^n_{\mathfrak{a}}(N)) \\
&= \operatorname{Ass} \operatorname{Hom}_R(\operatorname{Ext}^d_R(M,R), D\, \mathrm{H}^n_{\mathfrak{a}}(N)) \\
&= \operatorname{Supp} \operatorname{Ext}^d_R(M,R) \cap \operatorname{Ass} D\, \mathrm{H}^n_{\mathfrak{a}}(N) \\
&= \operatorname{Supp} \operatorname{Ext}^d_R(M,R) \cap \operatorname{Att} DD\, \mathrm{H}^n_{\mathfrak{a}}(N) \\
&= \operatorname{Supp} \operatorname{Ext}^d_R(M,R) \cap \operatorname{Att} \mathrm{H}^n_{\mathfrak{a}}(N)
\end{aligned}$$

The final assertion follows immediately from the first equality, [20, Lemma 19.1(iii)] and the fact that $\operatorname{Supp} \operatorname{Ext}^d_R(M,R) \subseteq \operatorname{Supp} M$. □

By Corollary 5.4 $\operatorname{Att} \mathrm{H}^{n+d}_{\mathfrak{a}}(M,N) \subseteq \operatorname{Att} \mathrm{H}^{n}_{\mathfrak{a}}(N)$. Next, we give an example to show that this inclusion may be strict even if $(R,\mathfrak{m})$ is a complete regular local ring and $\mathfrak{a} = \mathfrak{m}$. Also, this example shows that the following theorem of Mafi is not true.

[19, Theorem 2.1]: Let $(R,\mathfrak{m})$ be a commutative Notherian local ring and $n = \dim N$, $d = \operatorname{proj}\dim M < \infty$. If $\mathrm{H}^{n+d}_{\mathfrak{m}}(M,N) \neq 0$, then

$$\operatorname{Att} \mathrm{H}^{n+d}_{\mathfrak{m}}(M,N) = \operatorname{Att} \mathrm{H}^{n}_{\mathfrak{m}}(N).$$

Example 5.7. *Let $(R,\mathfrak{m})$ be a complete regular local ring of a dimension $n \geq 2$ and assume that R has two distinct prime ideals $\mathfrak{p},\mathfrak{q}$ such that $\dim R/\mathfrak{p} = \dim R/\mathfrak{q} = 1$. Set $M = R/\mathfrak{p}$ and $N = R/\mathfrak{p} \oplus R/\mathfrak{q}$. Then, by Theorem 5.3,*

$$\operatorname{Att} \mathrm{H}^{1}_{\mathfrak{m}}(N) = \{\mathfrak{p},\mathfrak{q}\}.$$

On the other hand, $\operatorname{proj}\dim M = \dim R - \operatorname{depth} M = n-1$ *and* $\dim N = 1$. *Now, by Theorem 5.6,*

$$\operatorname{Att} \mathrm{H}^{n}_{\mathfrak{m}}(M,N) = \operatorname{Supp} M \cap \operatorname{Att} \mathrm{H}^{1}_{\mathfrak{m}}(N) = \{\mathfrak{p}\}.$$

Therefore [19, Theorem 2.1] *is not true. Also, by* [6, Proposition 7.2.11],

$$\sqrt{(Ann\, \mathrm{H}^{n}_{\mathfrak{m}}(M,N))} = \bigcap_{\mathfrak{p} \in \operatorname{Att} \mathrm{H}^{n}_{\mathfrak{m}}(M,N)} \mathfrak{p} = \mathfrak{p}$$

and

$$\sqrt{(Ann\, \mathrm{H}^{1}_{\mathfrak{m}}(N))} = \bigcap_{\mathfrak{p} \in \operatorname{Att} \mathrm{H}^{1}_{\mathfrak{m}}(N)} \mathfrak{p} = \mathfrak{p} \cap \mathfrak{q}.$$

Hence, again, Corollary 2.2 and Corollary 2.3 of [19] *are not true. We note that, the other results of* [19] *are concluded from* [19, Theorem 2.1, Corollary 2.2 and Corollary 2.3].

It is known that if $(R,\mathfrak{m})$ is a local ring and $\dim M = n > 0$, then $\mathrm{H}^{n}_{\mathfrak{m}}(M)$ is not finite [6, Corollary 7.3.3]. It was proved in [10, Proposition 2.6] that if $d = \operatorname{proj}\dim M < \infty$ and $0 < n = \dim N$, then $\mathrm{H}^{n+d}_{\mathfrak{m}}(M,N)$ is not finite whenever it is non-zero. Next, we provide a generalization of this result. The following lemma, which is needed in the proof of the next proposition, is elementary.

Lemma 5.8. *Let X be an R-module. Then X has finite length if and only if X is Artinian and $\operatorname{Att} X \subseteq \max R$. Moreover if X has finite length, then $\operatorname{Att} X = \operatorname{Supp} X = \operatorname{Ass} X$.*

Proposition 5.9. *Let $d = \operatorname{proj}\dim M < \infty$, $0 < n = \dim N < \infty$. If $\mathrm{H}^{n+d}_{\mathfrak{a}}(M,N) \neq 0$, then it is not finite.*

Proof. Assume that $\mathfrak{p} \in \operatorname{Att} \mathrm{H}^{n+d}_{\mathfrak{a}}(M,N)$. By 5.3, $\mathrm{H}^{n+d}_{\mathfrak{a}}(M,N)$ is an Artinian R-module and $n+d = \operatorname{cd}_{\mathfrak{a}}(M,R/\mathfrak{p}) = \operatorname{proj}\dim M + \dim R/\mathfrak{p}$. Therefore $\dim R/\mathfrak{p} = n > 0$; So that $\operatorname{Att} \mathrm{H}^{n+d}_{\mathfrak{a}}(M,N) \nsubseteq \max R$. It follows that, in view of 5.8, $\mathrm{H}^{n+d}_{\mathfrak{a}}(M,N)$ is not finite. □

Acknowledgment

The authors would like to thank the referees for careful reading of the manuscript and for helpful suggestions.

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