# ON THE MAXIMAL SPECTRUM OF A MODULE AND ZARISKI TOPOLOGY

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Abstract. For any module M over a commutative ring R, $Spec_R(M)$ (resp. $Max_R(M)$) of $M$ is the collection of all prime (resp. maximal) submodules. In this article we investigate the interplay between the topological properties of $Max_R(M)$ and module theoretic properties of $M$. Also, for various types of modules M, we obtain some conditions under which $Max_R(M)$ is homeomorphic with the maximal ideal space of some ring.

## 1. Introduction

Throughout this article, $R$ is a commutative ring with non zero identity and $M$ is a unitary R-module. For any ideal $I$ of $R$ containing $Ann_R(M)$, $\bar{R}$ and $\bar{I}$ denote $R/Ann(M)$ and $I/Ann(M)$ respectively. Also $\mathbb{N}$, $\mathbb{Z}$, and $\mathbb{Q}$ denote the set of positive integers, the ring of integers, and the field of rational numbers respectively. Moreover the notation "$\subset$" will denote the strict inclusion.

For $M$ as an $R$-module and $N$ a submodule, we recall the *colon ideal of $M$ into $N$*, $(N : M) = \{r \in R | rM \subseteq N\}$.

A submodule $P$ of $M$ is said to be a *prime submodule* if $P \neq M$ and whenever $r \in R$ and $e \in M$ satisfy $re \in P$, then $r \in P$ or $e \in (P : M)$. If $P$ is a prime submodule, then $(P : M)$ is a prime ideal of $R$. Moreover if $Q$ is a maximal submodule of $M$, then $Q$ is a prime submodule and $(Q : M)$ is a maximal ideal of $R$ [9] and [10].

If $Spec_R(M) \neq \emptyset$ (resp. $Max_R(M) \neq \emptyset$), the mapping $\psi : Spec_R(M) \to Spec(\bar{R})$ (resp. $\phi : Max_R(M) \to Max(\bar{R})$) such that $\psi(P) = \overline{(P : M)}$ (resp. $\phi(Q) = \overline{(Q : M)}$) for every $P \in Spec_R(M)$ (resp. $Q \in Max_R(M)$), is called the *natural map* of $Spec_R(M)$ (resp. $Max_R(M)$) [11] (resp. [1]).

$M$ is said to be *primeful* (resp. *$Max$-surjective*) if either $M = (0)$ or $M \neq (0)$ and the natural map of $Spec_R(M)$ (resp. $Max_R(M)$) is surjective [12] (resp. [1]).

$M$ is said to be *$X$-injective* if either $Spec_R(M) = \emptyset$ or $Spec_R(M) \neq \emptyset$ and the natural map of $Spec_R(M)$ is injective [2].

The *Zariski topology* on $X = Spec_R(M)$ is the topology $\tau_M$ described by taking the set $Z(M) = \{V_M(N) | N$ is a submodule of $M\}$ as the set of closed sets of $X$, where

$$V_M(N) = \{P \in X | (P : M) \supseteq (N : M)\}.$$

When $M = R$, $\tau_M = \tau_R$ is the well known Zariski topology on $Spec(R)$ [11].

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There exists a topology on $Max_R(M)$ having $Z^m(M) = \{V^m_M(N) | N$ is a submodule of $M\}$ as the set of closed sets of $Max_R(M)$, where

$$V^m_M(N) = \{Q \in Max_R(M) | (Q : M) \supseteq (N : M)\}.$$

We denote this topology by $\tau^m_M$. In fact $\tau^m_M$ is the same as the subspace topology induced by $\tau_M$ on $Max_R(M)$. When $M = R$, this topology is denoted by $\tau^m_R$ and for every ideal $I$ of $R$, we have

$$V^m_R(I) = \{q \in Max(R) | q \supseteq I\}.$$

In the rest of this article $Spec_R(M)$ (resp. $Max_R(M)$) is always equipped with the Zariski topology $\tau_M$ (resp. $\tau^m_M$) and $Max_R(M)$ is assumed to be a non-empty subset of $Spec_R(M)$.

The present authors introduced the concept of $Max$-injective modules and investigated some important properties of this family of modules. An $R$-modules $M$ is called $Max$-injective if the natural map of $Max_R(M)$ is injective [4]. Clearly, every $X$-injective module is $Max$-injective.

A topological space $W$ is said to be $Max$-spectral if it is homeomorphic with the maximal ideal space of some ring (see Definition 3.17). $Max$-spectral spaces have been characterized by Hochster in [8, p. 57, Proposition 11].

In this article, we investigate the interplay between the topological properties of $Max_R(M)$ and module theoretic properties of $M$ (see Proposition 3.2, Theorem 3.6, Theorem 3.13, Corollary 3.15, Proposition 3.19, and Theorem 3.24). Theorem 3.14 provides useful information about the relationship between topological properties of $Max_R(M)$ and $Max(\bar{R})$. Also we consider the conditions under which $Max_R(M)$ is a Noetherian topological space (see Proposition 3.2, Theorem 3.6, Theorem 3.14, and Corollary 3.15). Moreover, we study the topological space $Max_R(M)$ from the point of view of $Max$-spectral spaces (see Theorem 3.24). It is shown that if $M$ is a $Max$-injective module over a PID, then $Max_R(M)$ is a $Max$-spectral topological space (see Theorem 3.24 (g)). These results enable us to provide a large family of modules such that their maximal submodules are $Max$-spectral.

## 2. Preliminaries

In this section we review some preliminary results which will be needed in next section.

**Definition 2.1.** For a topological space $X$, we recall

(a) $X$ is quasi compact if it satisfies one of the following two equivalent conditions.
  (1) Every collection of open subsets whose union is $X$ contains a finite subcollection whose union is $X$.
  (2) Every collection of closed subsets whose intersection is empty set contains a finite subcollection whose intersection is empty set (see [15, Definition 2.135]).

(b) $X$ is said to be Noetherian if the open subsets of $X$ satisfy the ascending chain condition (or maximal condition). (see [6, Chap. 6, Example 5]).

(c) $X$ is said to be connected if it is not the union $X = X_0 \cup X_1$ of two disjoint closed non-empty subsets $X_0$ and $X_1$ (see [15, Definition 2.105]).

(d) $X$ is said to be irreducible if $X$ is not the union of two proper closed subsets. For $X' \subseteq X$, $X'$ is irreducible if it is irreducible as a space with the relative topology. This is equivalent to say that, if $F$, $G$ are closed subsets of $X$ such that $X' \subseteq F \cup G$, then $X' \subseteq F$ or $X' \subseteq G$ (see [7, Ch. II, p. 119]).

(e) A maximal irreducible subset of $X$ is called an irreducible component of $X$. It is well known that every irreducible component of $X$ is closed in $X$ (see [7, Ch. II]).

**Remark 2.2.** Let $X$ and $Y$ be two topological spaces.

(a) Let $f$ be a continuous mapping from $X$ to $Y$.
  (1) If $X$ is a connected (resp. quasi compact) topological space, then $f(X)$ is a connected (resp. quasi compact) topological space (see [15, Theorem 2.107 and Theorem 2.138]).
  (2) For every irreducible subset $E$ of $X$, $f(E)$ is an irreducible subset of $Y$ (see [7, Ch. II]).

(b) If $X$ is a Noetherian topological space, then every subspace of $X$ is a Noetherian topological space, and $X$ is a quasi compact topological space (see [6, Chap. 6, Exc. 5]).

(c) Every Noetherian topological space has only finitely many irreducible components (see [7, p. 124, Proposition 10]).

(d) Closed subspaces of quasi compact topological spaces are quasi compact (see [15, Theorem 2.137]).

(e) Every finite topological space is quasi compact (see [15, p. 51]).

(f) Closure of any connected (resp. irreducible) subspace is connected (resp. irreducible) (see [15, Corollary 2.112] and [7, Ch. II]).

(g) Let $A$ and $B$ be subsets of $X$ such that $A \subseteq B \subseteq X$, where $B$ is closed in $X$ and equipped with the relative topology. Then $A$ is an irreducible closed subset of $B$ if and only if $A$ is an irreducible closed subset of $X$ (see Definition 2.1 (d)).

## 3. Main results

As it was mentioned before, $Spec_R(M)$ (resp. $Max_R(M)$) is always equipped with Zariski topology $\tau_M$ (resp. $\tau_M^m$).

**Lemma 3.1.** Let $M$ be an $R$-module and let $\phi : Max_R(M) \to Max(\bar{R})$ be the natural map of $Max_R(M)$. Then the following hold.

(a) $\phi$ is a continuous map.

(b) If $M$ is $Max$-surjective, then $\phi$ is closed and open mapping.

*Proof.* (a) This follows from the fact that $\phi^{-1}(V_{\bar{R}}^m(\bar{I})) = V_M^m(IM)$ for every ideal $I$ of $R$ containing $Ann(M)$.

(b) Let $N$ be a submodule of $M$ and let $V_M^m(N)$ be a closed subset of $Max_R(M)$. Then as in the proof part (a), we have

$$\phi^{-1}(V_{\bar{R}}^m(\overline{(N:M)})) = V_M^m((N:M)M) = V_M^m(N).$$

Hence $\phi(V_M^m(N)) = V_{\bar{R}}^m(\overline{(N:M)})$ because $\phi$ is surjective. Also $\phi$ is open by similar arguments and the proof is completed.

□

A topological space $W$ is a cofinite topological space when its open sets are empty and $W$ and all subsets with a finite complement. This topology is denoted by $\tau^{fc}$.

**Proposition 3.2.** Let $R$ be a ring such that the intersection of every infinite collection of maximal ideals of $R$ is zero (for example, when $R$ is PID or one dimensional Noetherian domain) and let $M$ be an $R$-module. Then $Max_R(M)$ is a Noetherian topological space.

*Proof.* Let $V^m_M(N)$ be a closed subset of $Max_R(M)$ for some submodule $N$ of $M$. If $V^m_M(N)$ is infinite, then $(N : M)$ is contained in an infinite number of maximal ideals of $R$. Since the intersection of every infinite collection of maximal ideals of $R$ is zero, $(N : M) = (0)$ so that $V^m_M(N) = Max_R(M)$. It follows that $\tau^m_M \subseteq \tau^{fc}$ and hence $Max_R(M)$ is a Noetherian topological space because every cofinite topological space is Noetherian. □

**Notation 3.3.** Let $M$ be an $R$-module and W be a subset of $Max_R(M)$. We will denote the intersection of all elements in $W$ by $\Im(W)$ and the closure of $W$ in $Max_R(M)$ (resp. $Spec_R(M)$) by $Cl^m(W)$ (resp. $Cl(W)$).

**Lemma 3.4.** Let M be an R-module and $W$ be a subset of $Max_R(M)$. Then $Cl^m(W) = V^m_M(\Im(W))$. Hence, W is closed if and only if $V^m_M(\Im(W)) = W$.

*Proof.* Let $W$ be a subset of $Max_R(M)$. It is well known that

$$Cl^m(W) = Cl(W) \cap Max_R(M).$$

But $Cl(W) = V(\Im(W))$ by [11, Proposition 5.1]. It follows that $Cl^m(W) = V^m_M(\Im(W))$. □

For a proper ideal $I$ of $R$, we recall that the J-radical $I$, denoted by $J^m_R(I)$, is the intersection of all maximal ideals containing $I$. An ideal $I$ of $R$ is a J-radical ideal if $I = J^m_R(I)$.

**Definition 3.5.** Let $M$ be an $R$-module. The J-radical of a submodule $N$ of $M$, denoted by $J^m_M(N)$, is the intersection of all members of $V^m_M(N)$. In case that $V^m_M(N) = \emptyset$, we define $J^m_M(N) = M$. A submodule $N$ of $M$ is said to be a J-radical submodule if $N = J^m_M(N)$.

**Theorem 3.6.** *Let $M$ be an $R$-module. Then the following are equivalent.*

(a) *$Max_R(M)$ is a Noetherian topological space.*
(b) *The ascending chain condition for J-radical submodules of $M$ holds.*

*Proof.* (a)⇒(b) Straightforward.

(b)⇒(a) Let

$$V^m_M(N_1) \supseteq V^m_M(N_2) \supseteq \cdots \supseteq V^m_M(N_i) \supseteq \cdots$$

be a descending chain of closed sets $V^m_M(N_i)$ of $Max_R(M)$, where $N_i$ is a submodule of $M$. Hence

$$J^m_M(N_1) \subseteq J^m_M(N_2) \subseteq \cdots \subseteq J^m_M(N_i) \subseteq \cdots$$

is an ascending chain of J-radical submodules of $M$. So by hypothesis, there exists a $k \in \mathbb{N}$ such that for all $n > k$, we have $J^m_M(N_{k+n}) = J^m_M(N_k)$. Now by using Lemma 3.4, for all $n > k$, $V^m_M(N_{k+n}) = V^m_M(N_k)$ and the proof is completed. □

**Corollary 3.7.** Let $M$ be a Noetherian $R$-module. Then $Max_R(M)$ is a Noetherian topological space.

We recall that if $I$ is an ideal of $R$, then the J-components of $I$ are the minimal members of the family of J-radical prime ideals containing $I$ (see [16, p. 631]).

**Definition 3.8.** Let $M$ be an $R$-module and $L$ a submodule of $M$. A submodule $P$ of $M$ is a J-component of $L$, if $(P : M)$ is a J-component of $(L : M)$. Clearly, this definition is the generalization of J-component of an ideal in rings.

**Definition 3.9.** A module $M$ is said to have property (JFC) if every closed subset of $Max_R(M)$ has a finite number of irreducible components.

**Example 3.10.** Let $M$ be an $R$-module. Then $M$ has property (JFC) in each of the following cases:

(a) $Max_R(M)$ is a Noetherian topological space (see parts (b) and (c) of Remark 2.2);
(b) $R$ is PID (see Proposition 3.2 and part (a));
(c) $M$ is Noetherian (see Corollary 3.7 and part (a));
(d) $M$ is semi local (see Remark 2.2 (e) and part (a)).

When $M$ is the $R$-module $R$, then $R$ has property (JFC) if and only if every ideal of $R$ has a finite number of J-components (see [16, p. 632]). Theorem 3.13 (d) extends the this property for modules.

The proof of the following lemma is easy and is omitted.

**Lemma 3.11.** Let $M$ be a $Max$-surjective $R$-module. Then the following hold.

(a) If $N$ is a submodule of $M$, then
$$J^m_R((N : M)) = (J^m_M(N) : M).$$
(b) If $q$ is a J-radical ideal of $R$ containing $Ann_R(M)$, then there exists a submodule $Q$ of $M$ such that $(Q : M) = q$.

**Remark 3.12.** If $S$ is a commutative ring with non zero identity, then there exists a one-to-one correspondence between the J-radical prime ideals of ring $S$ and irreducible closed subsets of $Max(S)$ (see [16, p. 631]).

**Theorem 3.13.** *Let $M$ be a $Max$-surjective $R$-module. Then the following hold.*

(a) *If $Y \subseteq Max_R(M)$, then $Y$ is an irreducible closed subset of $Max_R(M)$ if and only if $Y = V^m_M(N)$ for some submodule $N$ of $M$ such that $(N : M)$ is a J-radical prime ideal of $R$.*
(b) *If $W \subseteq Max_R(M)$ and $L$ is submodule of $M$, then $W$ is an irreducible component of $V^m_M(L)$ if and only if $W = V^m_M(N')$ for some J-component $N'$ of $L$.*
(c) *If $Z \subseteq Max_R(M)$, then $Z$ is an irreducible component of $Max_R(M)$ if and only if $Z = V^m_M(pM)$ for some J-component ideal $p$ of $Ann_R(M)$.*
(d) *$M$ has property (JFC) if and only if every submodule of $M$ has a finite number of J-components.*

*Proof.* (a) ($\Rightarrow$) Let $Y$ be an irreducible closed subset of $Max_R(M)$. Since $Y$ is closed, $Y = V^m_M(N)$ for some submodule $N$ of $M$. It turns out that $\phi(V^m_M(N)) = V^m_{\bar{R}}(\overline{(N : M)})$ is an irreducible closed subset of $Max(\bar{R})$ by Lemma 3.1 and Remark 2.2 (a). Now by Remark 3.12, $\overline{(N : M)}$ is a J-radical prime ideal of $\bar{R}$ so that $(N : M)$ is a J-radical prime ideal of $R$. Conversely, let $V^m_M(K)$ be a closed subset of $Max_R(M)$, where $K$ is a submodule of $M$ such that $(K : M)$ is a J-radical prime ideal of $R$. We show that $V^m_M(K)$ is irreducible. To see this, let $E$ and $E'$ be submodules of $M$ with
$$V^m_M(K) \subseteq V^m_M(E) \cup V^m_M(E').$$

Hence as in the proof of Lemma 3.1 (b), we have

$$V^m_{\bar{R}}(\overline{(K:M)}) \subseteq V^m_{\bar{R}}(\overline{(E:M)}) \cup V^m_{\bar{R}}(\overline{(E':M)}).$$

Since $(K:M)$ is a J-radical prime ideal of $R$, it is easy to check that $\overline{(K:M)}$ is a J-radical prime ideal of $\bar{R}$. Therefore $V^m_{\bar{R}}(\overline{(K:M)})$ is an irreducible closed subset of $Max(\bar{R})$ by Remark 3.12. Hence by Definition 2.1 (d),

$$V^m_{\bar{R}}(\overline{(K:M)}) \subseteq V^m_{\bar{R}}(\overline{(E:M)}) \ or \ V^m_{\bar{R}}(\overline{(K:M)}) \subseteq V^m_{\bar{R}}(\overline{(E':M)}).$$

Suppose that $V^m_{\bar{R}}(\overline{(K:M)}) \subseteq V^m_{\bar{R}}(\overline{(E:M)})$. This implies that $V^m_M(K) \subseteq V^m_M(E)$. By similar arguments, $V^m_M(K) \subseteq V^m_M(E')$ when $V^m_{\bar{R}}(\overline{(K:M)}) \subseteq V^m_{\bar{R}}(\overline{(E':M)})$.

(b) ($\Rightarrow$) Let $W$ be an irreducible component of $V^m_M(L)$. By Definition 2.1 (e) and Remark 2.2 (g), $W$ is an irreducible closed subset of $Max_R(M)$. So by part (a), $W = V^m_M(N'_1)$ for some submodule $N'_1$ of $M$ such that $(N'_1:M)$ is a J-radical prime ideal of $R$. We claim that $N'_1$ is a J-component of $L$ or equivalently, $(N'_1:M)$ is a J-component of $(L:M)$. Clearly $(N'_1:M) \supseteq (L:M)$ by using Lemma 3.11 (a). So by the above arguments, it is enough to show that $(N'_1:M)$ is a minimal member of the family of J-radical prime ideals containing $(L:M)$. To see this, let $q$ be a J-radical prime ideal of $R$ with

$$(L:M) \subseteq q \subseteq (N'_1:M).$$

Since $M$ is $Max$-surjective, there exists a submodule $Q$ of $M$ such that $q = (Q:M)$ by Lemma 3.11 (b). Hence

$$V^m_M(L) \supseteq V^m_M(Q) \supseteq V^m_M(N'_1).$$

Also $V^m_M(Q)$ is an irreducible closed subset of $V^m_M(L)$ by part (a), and Remark 2.2 (g). Since $W = V^m_M(N'_1)$ is an irreducible component of $V^m_M(L)$, by the above arguments, we have $V^m_M(Q) = V^m_M(N'_1)$. Now by using Lemma 3.11 (a), $q = (N'_1 : M)$ as desired.

($\Leftarrow$) Let $N''_2$ be a J-component of $L$. Then $V^m_M(N''_2)$ is an irreducible closed subset of $V^m_M(L)$ by part (a) and Remark 2.2 (g). Let $L'$ be a submodule of $M$ such that $(L':M)$ is a J-radical prime ideal of $R$ and

$$V^m_M(N''_2) \subseteq V^m_M(L') \subseteq V^m_M(L).$$

Since $N''_2$ be a J-component of $L$, by using Lemma 3.11 (a), we have $V^m_M(N''_2) = V^m_M(L')$ as required.

(c) This follows from part (b) and Lemma 3.11 (b) and the fact that if $N$ is a submodule of $M$, then

$$V^m_M((N:M)M) = V^m_M(N).$$

(d) Follows from part (b).

□

Let $X$ be a topological space. We consider strictly decreasing chain $Z_0$, $Z_1$,..., $Z_r$ of length $r$ of irreducible closed subsets $Z_i$ of $X$. The supremum of the lengths, taken over all such chains, is called the combinatorial dimension of $X$ and denoted by $dim(X)$. For the empty set, $\emptyset$, the combinatorial dimension of $\emptyset$ is defined to be $-1$.

**Theorem 3.14.** *Let $M$ be a $Max$-surjective $R$-module. Then the following hold.*

(a) *$Max_R(M)$ is a Noetherian topological space if and only if $Max(\bar{R})$ is a Noetherian topological space.*

(b) *$Max_R(M)$ is a connected topological space if and only if $Max(\bar{R})$ is a connected topological space.*
(c) *$Max_R(M)$ is an irreducible topological space if and only if $Max(\bar{R})$ is an irreducible topological space.*
(d) *$Max_R(M)$ is a quasi compact topological space if and only if $Max(\bar{R})$ is a quasi compact topological space.*
(e) *$dim(Max_R(M)) = dim(Max(\bar{R}))$.*

*Proof.* Let $\phi : Max_R(M) \to Max(\bar{R})$ be the natural map of $Max_R(M)$.

(a) ($\Rightarrow$) Let $V^m_{\bar{R}}(\bar{I}_1) \supseteq V^m_{\bar{R}}(\bar{I}_2) \supseteq ... \supseteq V^m_{\bar{R}}(\bar{I}_i) \supseteq ...$ be a descending chain of closed sets in $Max(\bar{R})$, where each $\bar{I}_i$ is an ideal of $\bar{R}$. Since $\phi$ is continuous by Lemma 3.1 (a),

$$\phi^{-1}(V^m_{\bar{R}}(\bar{I}_1)) \supseteq \phi^{-1}(V^m_{\bar{R}}(\bar{I}_2)) \supseteq ... \supseteq \phi^{-1}(V^m_{\bar{R}}(\bar{I}_i)) \supseteq ...$$

is a descending chain of closed sets in $Max_R(M)$. By hypothesis, there exists a $t \in \mathbb{N}$ such that for all $n > t$, $\phi^{-1}(V^m_{\bar{R}}(\bar{I}_{t+n})) = \phi^{-1}(V^m_{\bar{R}}(\bar{I}_t))$. Hence for all $n > t$, we have $V^m_{\bar{R}}(\bar{I}_{t+n}) = V^m_{\bar{R}}(\bar{I}_t)$ because $\phi$ is surjective. Therefore, $Max(\bar{R})$ is a Noetherian topological space. To show the converse, by Theorem 3.6, it is enough to show that the ascending chain condition for J-radical submodules of $M$ holds. To see this, let

$$N_1 \subseteq N_2 \subseteq \cdots \subseteq N_i \subseteq \cdots$$

be an ascending chain of J-radical submodules of $M$. Then by Lemma 3.11 (a), one can see that

$$\overline{(N_1 : M)} \subseteq \overline{(N_2 : M)} \subseteq \cdots \subseteq \overline{(N_i : M)} \subseteq \cdots$$

is an ascending chain of J-radical ideals of $\bar{R}$. So by Theorem 3.6, there exists a $k \in \mathbb{N}$ such that for all $n > k$, $\overline{(N_{k+n} : M)} = \overline{(N_k : M)}$. Hence for all $n > k$,

$$V^m_M(N_{k+n}) = V^m_M((N_{k+n} : M)M) = V^m_M((N_k : M)M) = V^m_M(N_k).$$

So for all $n > k$, we have

$$N_{k+n} = J^m_M(N_{k+n}) = J^m_M(N_k) = N_k,$$

as desired.

(b) First assume that $Max_R(M)$ is a connected topological space. Then $Max(\bar{R}) = \phi(Max_R(M))$ is connected by Lemma 3.1 and Remark 2.2 (a). To see the reverse implication, we assume that $Max(\bar{R})$ is a connected topological space. If $Max_R(M)$ is a disconnected topological space, then there exist submodules $N$ and $K$ of $M$ such that

$$Max_R(M) = V^m_M(N) \cup V^m_M(K)$$

and

$$V^m_M(N) \cap V^m_M(K) = \emptyset,$$

where $V^m_M(N) \neq \emptyset$, and $V^m_M(K) \neq \emptyset$. Hence as in the proof of Lemma 3.1 (b), we have

$$Max(\bar{R}) = V^m_{\bar{R}}(\overline{(N : M)}) \cup V^m_{\bar{R}}(\overline{(K : M)}).$$

It is easy to cheek that

$$V^m_{\bar{R}}(\overline{(N : M)}) \cap V^m_{\bar{R}}(\overline{(K : M)}) = \emptyset, V^m_{\bar{R}}(\overline{(N : M)}) \neq \emptyset, \text{ and } V^m_{\bar{R}}(\overline{(K : M)}) \neq \emptyset.$$

Therefore $Max(\bar{R})$ is a disconnected topological space, a contradiction. Hence $Max_R(M)$ is a connected topological space.

(c) We have similar argument as in part (b).

(d) (⇒) This follows from Lemma 3.1 (a) and Remark 2.2 (a). To show the converse, let $\{V^m_M(N_\alpha) : \alpha \in \Lambda\}$ be a family of closed subset of $Max_R(M)$ such that $\cap_{\alpha\in\Lambda} V^m_M(N_\alpha) = \emptyset$, where $N_\alpha$ is a submodule of $M$ for every $\alpha \in \Lambda$. Then $\{\phi(V^m_M(N_\alpha)) : \alpha \in \Lambda\}$ is a family of closed subset of $Max(\bar{R})$ because $\phi$ is closed by Lemma 3.1 (b). Since $\phi$ is surjective, it is easy to see that $\cap_{\alpha\in\Lambda}\phi(V^m_M(N_\alpha)) = \emptyset$. As $Max(\bar{R})$ is quasi compact, there exists a finite subset $\Gamma$ of $\Lambda$ such that $\cap_{\alpha\in\Gamma}\phi(V^m_M(N_\alpha)) = \emptyset$. This implies that $\cap_{\alpha\in\Gamma} V^m_M(N_\alpha) = \emptyset$ and hence $Max_R(M)$ is quasi compact.

(e) Let $Z_0 \supset Z_1 \supset ... \supset Z_n$ be a descending chain of irreducible closed subset of $Max_R(M)$. Then by Theorem 3.13 (a), for $i$ $(1 \le i \le n)$, there exists submodule $L_i$ of $M$ such that $(L_i : M)$ is a J-radical prime ideal of $R$ and $Z_i = V^m_M(L_i)$. It follows that

$$V^m_{\bar{R}}(\overline{(L_0 : M)}) \supset V^m_{\bar{R}}(\overline{(L_1 : M)}) ... \supset V^m_{\bar{R}}(\overline{(L_n : M)})$$

is a descending chain of irreducible closed subset of $Max(\bar{R})$ by Remark 3.12. Hence

$$dim(Max_R(M)) \le dim(Max(\bar{R})).$$

Now let

$$A_0 \supset A_1 \supset ... \supset A_t$$

be a descending chain of irreducible closed subset of $Max(\bar{R})$. By Remark 3.12, for each $i$ $(1 \le i \le t)$, there exists a J-radical prime ideal $\bar{p}_i$ of $\bar{R}$ such that $A_i = V^m_{\bar{R}}(\bar{p}_i)$. This yields that

$$p_0 \subset p_1 \subset ... \subset p_t$$

is an ascending chain of J-radical prime ideal of $R$. Since $M$ is $Max$-surjective, by Lemma 3.11 (b), for every $p_i$ $(1 \le i \le t)$, there exists a submodule $Q_i$ of $M$ such that $p_i = (Q_i : M)$. Hence by Theorem 3.13 (a),

$$V^m_M(Q_0) \supset V^m_M(Q_1) \supset ... \supset V^m_M(Q_t)$$

is a descending chain of irreducible closed subset of $Max_R(M)$. It follows that $dim(Max_R(M)) \ge dim(Max(\bar{R}))$ and the proof is completed. □

**Corollary 3.15.** Let $M$ be a $Max$-surjective $R$-module. Then the following hold.

(a) If $R$ is Noetherian, then $Max_R(M)$ is a Noetherian topological space.

(b) If $\Psi$ is the family of all J-radical prime ideal of $R$, then we have

$$dim(Max_R(M)) = sup\{n | p_0 \subset p_1 \subset ... \subset p_n \textit{ is an ascending chain of } \Psi \}.$$

*Proof.* (a) Follows from Theorem 3.14 (a).

(b) Apply the technique of Theorem 3.14 (e). □

**Remark 3.16.** We recall that an $R$-module $M$ is a Hilbert module if every prime submodule in $M$ is the intersection of all the maximal submodules containing it. For example, every finitely generated divisible module over an integral domain is a Hilbert module (see [13, p. 2]). Let $M$ be a Hilbert $R$-module. If $Max_R(M)$ is connected (resp. irreducible) topological space, then $Spec_R(M)$ is connected (resp. irreducible) topological space. Since if $M$ is Hilbert, by [11, Proposition 5.1] it is easy to see that $Cl(Max_R(M)) = Spec_R(M)$. Now the result follows from the Remark 2.2 (f).

**Definition 3.17.** We say that a topological space $W$ is a *Max-spectral space* if $W$ is homeomorphic with the maximal ideal space of some ring $S$ (with the topology inherited from $Spec(S)$).

**Remark 3.18.** *Max*-spectral spaces have been characterized by Hochster [8, p.57, Proposition 11] as the topological spaces W which satisfy the following conditions:

(a) W is a $T_1$ space;
(b) W is quasi-compact.

**Proposition 3.19.** Let $M$ be an $R$-module. Then the following are equivalent.

(a) $M$ is *Max*-injective.
(b) $Max_R(M)$ is a $T_0$ space.
(c) $Max_R(M)$ is a $T_1$ space.
(d) $Max_R(M)$ is a $T_2$ space.

*Proof.* Straightforward. □

**Corollary 3.20.** Let $M$ be an $R$-module.

(a) If $Max_R(M)$ is a *Max*-spectral topological space, then $M$ is *Max*-injective.
(b) If $M$ is primeful and $Spec_R(M)$ is a *Max*-spectral topological space, then $Spec_R(M) = Max_R(M)$.

*Proof.* This follows from Remark 3.18, Proposition 3.19, and [2, Theorem 4.3]. □

We recall that a topological space $X$ is spectral if it is homeomorphic to $Spec(S)$ with the Zariski topology for some ring $S$ (see [8]).

**Remark 3.21.** Let $M = \mathbb{Z}(p^\infty) \oplus \mathbb{Z}$. Then $M$ is a primeful $\mathbb{Z}$-module and $Spec_R(M)$ is a spectral topological space but $Spec_R(M) \neq Max_R(M)$ by [3, Table of examples 3.1 and 3.2]. This shows that part (b) in Corollary 3.20 is not valid in general if the word "*Max*-spectral" is replaced with "spectral".

**Example 3.22.**

(a) $Max_{\mathbb{Z}}(\mathbb{Z}_2 \oplus \mathbb{Z}_3)$ is a *Max*-spectral topological space by [3, Table of examples 3.1 and 3.2] and Remark 3.18.
(b) $Max_{\mathbb{Q}}(\mathbb{Q} \oplus \mathbb{Q})$ is not a *Max*-spectral topological space because $0 \oplus \mathbb{Q}$ and $\mathbb{Q} \oplus 0$ are maximal submodules of the $\mathbb{Q}$-module $\mathbb{Q} \oplus \mathbb{Q}$ with $(0 \oplus \mathbb{Q} : \mathbb{Q} \oplus \mathbb{Q}) = (\mathbb{Q} \oplus 0 : \mathbb{Q} \oplus \mathbb{Q})$, while $0 \oplus \mathbb{Q} \neq \mathbb{Q} \oplus 0$. Thus $Max_{\mathbb{Q}}(\mathbb{Q} \oplus \mathbb{Q})$ is not *Max*-spectral by Corollary 3.20 (a).

Let $M$ be an $R$-module such that $Max_R(M)$ is a *Max*-spectral topological space. For a submodule $N$ of $M$, it is natural to ask the following question: Is $Max_R(M/N)$ a *Max*-spectral topological space?

In Proposition 3.23 (c), we give a positive answer to this question under some additional conditions.

**Proposition 3.23.** Let $M$ be an $R$ module and let $N$ be a submodule of $M$. Then the following hold.

(a) If $Max_R(M)$ is a $T_1$ topological space, then so is $Max_R(M/N)$.
(b) If $Max_R(M)$ is a Noetherian topological space, then so is $Max_R(M/N)$.
(c) Let $Max_R(M)$ be a *Max*-spectral space. Then $Max_R(M/N)$ is a *Max*-spectral space in the following cases:
    (i) The subspace $H := \{Q \in Max_R(M) | Q \supseteq N\}$ of $Max_R(M)$ is closed;
    (ii) $R$ is a ring such that the intersection of every infinite collection of maximal ideals of $R$ is zero (for example, when $R$ is PID or one dimensional Noetherian domain).

*Proof.* (a) Follows from Proposition 3.19 and the fact that if $N$ is a submodule of $M$, then

$$Max_R(M/N) = \{Q/N | Q \in Max_R(M), Q \supseteq N\}.$$

(b) We define the map $f : Max_R(M/N) \to H$, where $H := \{Q \in Max_R(M) | Q \supseteq N\}$ and $f(Q/N) = Q$ for every $Q/N \in Max_R(M/N)$. Clearly $f$ is a bijection map. Now let $V^m_M(E) \cap H$ be a closed set of $H$, where $E$ is a submodule of $M$. Then

$$f^{-1}(V^m_M(E) \cap H) = f^{-1}(V^m_M(E)) \cap f^{-1}(H) = f^{-1}(V^m_M(E)) \cap Max_R(M/N)$$
$$= f^{-1}(V^m_M(E)) = V^m_M(K/N),$$

where $K = (E : M)M + N$. So $f : Max_R(M/N) \to H$ is a continuous map. It is easy to check that

$$f(V^m_M(L/N)) = V^m_M(L) \cap H$$

for every submodule $L$ of $M$ containing $N$. Hence $f : Max_R(M/N) \to H$ is a closed map so that $Max_R(M/N)$ is homeomorphic with $H$. Now since $Max_R(M)$ is Noetherian, $H$ is Noetherian by Remak 2.2 (b). Hence $Max_R(M/N)$ is a Noetherian space as desired.

(c)(i) As in the proof part (b), we see that $Max_R(M/N)$ is homeomorphic with $H$. Now the result follows by part (a), Remark 3.18, and Remark 2.2 (d).

(c)(ii) This follows from Proposition 3.2, Remark 3.18, Remark 2.2 (b), and part (a). □

The next theorem is an important result about an $R$-module $M$ for which $Max_R(M)$ is $Max$-spectral. This result is obtained by combining Lemma 3.1, Proposition 3.2, Theorem 3.6, Proposition 3.19, Remark 2.2 (e), and Remark 3.18.

**Theorem 3.24.** *Let $M$ be a $Max$-injective $R$-module. Then $Max_R(M)$ is a $Max$-spectral topological space in each of the following cases:*

(a) *$M$ is $Max$-surjective;*
(b) *$Im(\phi)$ is quasi compact, where $\phi : Max_R(M) \to Max(\bar{R})$ is the natural map of $Max_R(M)$;*
(c) *$Ann_R(M)$ is a maximal ideal of $R$;*
(d) *$Max_R(M)$ is a finite set;*
(e) *$Max(R)$ is a finite set;*
(f) *$Max(\bar{R})$ is Noetherian, in particular when $R$ is Noetherian;*
(g) *The intersection of every infinite of maximal ideals of $R$ is zero, in particular when $R$ is PID or one dimensional Noetherian domain;*
(h) *The ascending chain condition for J-radical submodules of $M$ holds.*

**Remark 3.25.** Let $M = \oplus_{i\in\mathbb{N}}\mathbb{Z}/p_i\mathbb{Z}$, where $p_i$ is a prime integer for each $i \in \mathbb{N}$. Then $M$ is an $X$-injective module over $\mathbb{Z}$ but $Spec_{\mathbb{Z}}(M)$ is not a spectral topological space by [3, Table of examples 3.1 and 3.2]. This shows that the words "$Max$-injective", "$Max_R(M)$", and "$Max$-spectral" in part (g) of Theorem 3.24, can not be replaced with "$X$-injective", "$Spec_R(M)$", and "spectral" respectively.

An $R$-module $M$ is multiplication if for every submodule $N$ of $M$, there exits an ideal $I$ of $R$ such that $N = IM$ (see [14]).

**Corollary 3.26.** Let $M$ be an $R$-module. Then $Max_R(M)$ is a $Max$-spectral topological space in each of the following cases:

(a) $M$ is finitely generated and multiplication;

(b) $M$ is primeful and top; (We refer the reader to [14] and [5] for the concept and properties of top modules.
(c) $M$ is primeful and $X$-injective;
(d) $M$ is $X$-injective and $R$ is PID.

*Proof.* This follows from parts (a) and (g) of Theorem 3.24 and taking into account the following facts from [14, Theorem 3.5], [2, Proposition 3.3], [12, Theorem 2.2 3.3], and [1, Proposition 3.3 (c)],

Fact 1. Let denote the class of multiplication, top, $X$-injective, and $Max$-injective modules respectively by $\Gamma_1$, $\Gamma_2$, $\Gamma_3$, and $\Gamma_4$, then

$$\Gamma_1 \subseteq \Gamma_2 \subseteq \Gamma_3 \subseteq \Gamma_4.$$

Fact 2. If we denote the class of finitely generated, primeful, and $Max$-surjective modules respectively by $\Omega_1$, $\Omega_2$, and $\Omega_3$, then

$$\Omega_1 \subseteq \Omega_2 \subseteq \Omega_3.$$

□

In below, by using Corollary 3.26 (d) and [3, Table of examples 3.1], we provide further examples about of modules such that their maximal submodules are $Max$-spectral.

**Example 3.27.** Let $p$ (resp. $p_i$, $i \in \mathbb{N}$) be a prime number. Then for each of the following cases, $Max_{\mathbb{Z}}(M)$ is a $Max$-spectral topological space.

(a) $M = \mathbb{Z}_{(p)} = S^{-1}\mathbb{Z}$, where $S = \mathbb{Z} \setminus (p)$.
(b) $M = \mathbb{Z}(p^{\infty}) \oplus \mathbb{Z}$.
(c) $M = \oplus_{i\in\mathbb{N}}\mathbb{Z}/p_i\mathbb{Z}$.
(d) $M = \prod_{i\in\mathbb{N}} \mathbb{Z}/p_i\mathbb{Z}$.
(e) $M = \mathbb{Q} \oplus (\oplus_{i\in\mathbb{N}}\mathbb{Z}/p_i\mathbb{Z})$.

Unfortunately, we have not bee able to find a $Max$-injective $R$-module $M$ such that $Max_R(M)$ is not $Max$-spectral. This motivates the following question.

**Question 3.28.** Let $M$ be a $Max$-injective $R$-module. Is $Max_R(M)$ a $Max$-spectral topological space?

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## References

[1] H. Ansari-Toroghy and R. Ovlyaee-Sarmazdeh, *Modules for which the natural map of the maximal spectrum is surjective*, Colloq. Math, **119** (2010), 217–227.
[2] ———, *On the prime spectrum of X-injective modules*, Comm. Algebra, **38** (2010), 2606–2621.
[3] ———, *On the prime spectrum of a module and Zariski topologies*, Comm. Algebra, **38** (2010), 4461-4475.
[4] H. Ansari-Toroghy and S. Keyvani, *Some new classes of modules*, (2011), submitted.
[5] H. Ansari-Toroghy and S. Keyvani, *Strongly top modules*, Bull. Malays. Math. Sci. Soc. (2), accepted.
[6] M.F. Atiyah and I.G. Macdonald, *Introduction to commutative algebra*, Addison-Wesley, 1969.
[7] N. Bourbaki, *Algebra Commutative*, Chap. 1,2, Paris: Hermann, 1961.

[8] M. Hochster, *Prime ideal structure in commutative rings*, Trans. Amer. Math. Soc, **142** (1969), 43–60.

[9] Chin-Pi Lu, *Prime submodules of modules*, Comment. Math. Univ. St. Pauli, **33** (1984), no. 1, 61–69.

[10] ———, *Spectra of modules*, Comm. Algebra, **23** (1995), no. 10, 3741–3752.

[11] ———, *The zariski topology on the prime spectrum of a module*, Houston J. Math, **25** (1999), no. 3, 417–432.

[12] ———, *A module whose prime spectrum has the surjective natural map*, Houston J. Math, **33** (1) (2007), 125–143.

[13] M. Maani Shirazi and H. Sharif, *Hilbert modules*, International Journal of Pure and Applied Mathematics, **20** (2005), no. 1, 1–7.

[14] R.L. McCasland, M.E. Moore, and P.F. Smith, *On the spectrum of a module over a commutative ring*, Comm. Algebra, **25** (1997), no. 1, 79–103.

[15] Jesper M. Moller, *General Topology*, Matematisk Institut, Universitetsparken 5, DK-2100 Kobenhavn.

[16] J. Ohm and R.L. Pendleton, *Ring with Noetherian spectrum*, Duke Math. J, **35** (1968), 631–639.

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