A Fixed Point Approach to the Stability of Linear Differential Equations

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Abstract. In this paper, we apply the fixed point method to investigate the Hyers-Ulam-Rassias stability of the nth order linear differential equations.

1 Introduction

In 1940, Ulam [27] posed a problem concerning the stability of functional equations: "Give conditions in order for a linear function near an approximately linear function to exist."

A year later, Hyers [8] gave an answer to the problem of Ulam for additive functions defined on Banach spaces: Let X_1 and X_2 be real Banach spaces and $\varepsilon > 0$. Then for every function $f : X \to Y$ satisfying

$$\|f(x+y) - f(x) - f(y)\| \leq \varepsilon \quad (\text{for } x, y \in X_1),$$

there exists a unique additive function $A : X_1 \to X_2$ with the property

$$\|f(x) - A(x)\| \leq \varepsilon \quad (\text{for } x \in X_1).$$

After Hyers's result, many mathematicians have extended the Ulam's problem to other functional equations and generalized the Hyers's result in various directions

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(see [2, 6, 9, 14]). A generalization of the Ulam's problem was recently proposed by replacing functional equations with differential equations: The differential equation $\varphi\big(f(t), y(t), y'(t), \ldots, y^{(n)}(t)\big) = 0$ has the Hyers-Ulam stability if, for any given $\varepsilon > 0$ and any function y satisfying $\big|\varphi\big(f(t), y(t), y'(t), \ldots, y^{(n)}(t)\big)\big| \leq \varepsilon$, there exists a solution y_0 of the differential equation such that $|y(t) - y_0(t)| \leq K(\varepsilon)$ and $\lim_{\varepsilon \to 0} K(\varepsilon) = 0$.

Obłoza seems to be the first author who has investigated the Hyers-Ulam stability of linear differential equations (see [20, 21]). Thereafter, Alsina and Ger published their paper [1], which handles the Hyers-Ulam stability of the linear differential equation $y'(t) = y(t)$: If a differentiable function $y(t)$ is a solution of the inequality $|y'(t) - y(t)| \leq \varepsilon$ for any $t \in (a, \infty)$, then there exists a constant c such that $|y(t) - ce^t| \leq 3\varepsilon$ for all $t \in (a, \infty)$.

Those previous results were extended to the Hyers-Ulam stability of linear differential equations of first order and higher order with constant coefficients in [7, 16, 17, 18, 25, 26] and in [19], respectively. Furthermore, Jung [10, 11, 12] has also proved the Hyers-Ulam stability of linear differential equations. Rus investigated the Hyers-Ulam stability of differential and integral equations using Gronwall lemma and the technique of weakly Picard operators (see [23, 24]). Recently, the Hyers-Ulam stability problems of linear differential equations of first order and second order with constant coefficients were studied by using the method of integral factor (see [15, 28]). The results given in [11, 15, 18] have been generalized by Cimpean and Popa [5] for the linear differential equations of nth order with constant coefficients.

Definition 1.1 *Assume that X is a normed space over a scalar field $\mathbb{F}$ and I is an arbitrary interval. Let $p_0, p_1, \ldots, p_n : I \to \mathbb{F}$ and $q : I \to X$ be continuous functions. Let $y : I \to X$ be any n times continuously differentiable function satisfying the inequality*

$$\left\| \sum_{k=0}^{n} p_k(t) y^{(k)}(t) - q(t) \right\| \leq \varphi(t)$$

for all $t \in I$, where $\varphi : I \to [0, \infty)$ is **an (Lebesgue) integrable** *function. Then we say that the differential equation $\sum_{k=0}^{n} p_k(t) y^{(k)}(t) = q(t)$ has the Hyers-Ulam-Rassias stability provided there exists a solution $y_0 : I \to X$ of the differential equation*

$$\sum_{k=0}^{n} p_k(t) y^{(k)}(t) = q(t) \tag{1.1}$$

and

$$\|y(t) - y_0(t)\| \leq M \|\varphi\|_1$$

for any $t \in I$, where M is a positive number and $\|\varphi\|_1 = \int_I |\varphi(t)| dt$. When $\varphi(t) = \varepsilon > 0$ for all $t \in I$, the differential equation is said to have the Hyers-Ulam stability.

In this paper, we apply the fixed point method used in [3, 4, 13, 22] to investigate the Hyers-Ulam-Rassias stability of the nth order linear differential equations of the form (1.1).

2 Preliminaries

Throughout this section, let X be a Banach space over a scalar field $\mathbb{F}$, where $\mathbb{F}$ denotes either $\mathbb{R}$ or $\mathbb{C}$. For any interval I of real numbers and any $n \in \mathbb{N}$, $C^n(I, X)$ stands for the set of all n times continuously differentiable functions from I into X. We denote by $f^{(i)}$ the ith derivative of f with respect to t and we define

$$C_b^n(I, X) = \Big\{ f \in C^n(I, X) : f^{(i)} \text{ is bounded for } i = 0, 1, \ldots, n \Big\}.$$

It is easy to see that $C_b^n(I, X)$ equipped with the norm

$$\|f\| = \max \Big\{ \big\|f^{(i)}\big\|_\infty : i = 0, 1, \ldots, n \Big\}$$

is a Banach space. Note that $C^n(I, X) = C_b^n(I, X)$ provided I is a closed interval.

Following the ideas of Cădariu and Radu [3, 4, 22] and Jung [13], we prove the Hyers-Ulam-Rassias stability of the nth order linear differential equations of the form (1.1). Before starting with our main theorem, we need the following fixed point alternative theorem:

THEOREM 2.1 *Let (Ω, d) be a generalized complete metric space. Assume that $\Lambda : \Omega \to \Omega$ is a strictly contractive operator with the Lipschitz constant $L < 1$, i.e.,*

$$d(\Lambda f_1, \Lambda f_2) \leq L d(f_1, f_2)$$

for all $f_1, f_2 \in \Omega$. If there exists an integer $n_0 \geq 0$ such that $d(\Lambda^{n_0+1} y, \Lambda^{n_0} y) < \infty$ for some $y \in \Omega$, then the following statements are true:

(i) The sequence $\{\Lambda^n y\}$ converges to a fixed point f of Λ;

(ii) f is the unique fixed point of Λ in $\Omega^* = \{g \in \Omega : d(\Lambda^{n_0} y, g) < \infty\}$;

(iii) If $g \in \Omega^*$, then

$$d(g, f) \leq \frac{1}{1-L} d(\Lambda g, g).$$

For a $c \in [a, b]$ and for every integrable function $f : [a, b] \to X$, the Volterra type operator V_c can be defined by

$$V_c(f)(t) = \int_c^t f(\tau) d\tau \quad (\text{for } t \in [a, b]).$$

Note that $(V_c(f))'(t) = f(t)$ for any $c \in [a, b]$.

Let $V_c^0 = \text{id}$ be the identity operator and $V_c^m(f)(t) = V_c(V_c^{m-1}(f))(t)$ for all $m \in \mathbb{N}$ and $t \in [a, b]$. Then, **for all integrable and continuous functions** $f, g : [a, b] \to X$, scalars α, β, and for any $m \in \mathbb{N}$, it holds that

(a) $V_c(\alpha f + \beta g) = \alpha V_c(f) + \beta V_c(g)$;

(b) $\|\mathbf{V_c(f)}\| \le \mathbf{V_c}(\|\mathbf{f}\|)$;

(c) $(V_c^m(f))'(t) = V_c^{m-1}(f)(t)$;

(d) $(V_c^m(f))^{(i)}(t) = V_c^{m-i}(f)(t)$ for $i = 0, 1, \dots, m$;

(e) If I is a closed interval of $\mathbb{R}$ and f is integrable on I, then $V_c^m(f) < \infty$.

In what follows, the notation

$$\|f\|_I = \sup\left\{|f(t)| : t \in I\right\}$$

is used for every bounded function $f : I \to \mathbb{F}$.

Definition 2.2 *Given an integer $n \ge 1$ and an interval I, let $p_0, p_1, \dots, p_n : I \to \mathbb{F}$ be bounded continuous functions, where $p_n(t) \ne 0$ for $t \in I$. Assume that $q \in C_b(I, X)$ and $\varphi : I \to [0, \infty)$ is an integrable function. We define*

$$S_n(I, \varphi) = \left\{ y \in C_b^n(I, X) : \left\| \sum_{k=0}^{n} p_k(t) y^{(k)}(t) - q(t) \right\| \le \varphi(t) \text{ for all } t \in I \right\}.$$

Lemma 2.3 *Under the assumptions of Definition 2.2, let I be any interval whose interior contains the open interval (a, b) with $0 \le b - a < 1$. Let $\varphi : I \to [0, \infty)$ be a bounded integrable function and*

$$L_{a,b} = (b-a) \sum_{k=0}^{n-1} \left\| \frac{p_k}{p_n} \right\|_I < 1. \tag{2.1}$$

Then it holds:

(i) *If either $I = [a, b]$ or $I = [a, b)$ and $y \in S_n(I, \varphi)$, then there exists a unique solution $f \in C_b^n(I, X)$ of Eq. (1.1) such that*

$$f^{(i)}(a) = y^{(i)}(a) \quad (\text{for } 1 = 0, 1, \dots, n-1)$$

and

$$\|f(t) - y(t)\| \le \frac{\|1/p_n\|_I}{1 - L_{a,b}} \int_a^b |\varphi(\tau)| d\tau. \tag{2.2}$$

(ii) *If either $I = [a, b]$ or $I = (a, b]$ and $y \in S_n(I, \varphi)$, then there exists a unique solution $f \in C_b^n(I, X)$ of Eq. (1.1) satisfying (2.2) and*

$$f^{(i)}(b) = y^{(i)}(b) \quad (\text{for } 1 = 0, 1, \dots, n-1).$$

(iii) *If $I = (a, b)$ and $y \in S_n(I, \varphi)$, then there exists a unique solution $f \in C_b^n(I, X)$ of Eq. (1.1) satisfying (2.2).*

Proof. Let $\|\varphi\|_1 = \int_a^b \varphi(t)dt$.

(i) Consider the set

$$\Omega_a = \Big\{ f \in C_b^{n-1}(I,X) : f^{(i)}(a) = y^{(i)}(a) \text{ for } i = 0,1,\ldots,n-1 \Big\}$$

equipped with the metric d defined by

$$d(f_1,f_2) = \max\left\{ \left\| f_1^{(i)} - f_2^{(i)} \right\|_I : i = 0,1,\ldots,n-1 \right\}.$$

Then it is easy to show that (Ω_a, d) is a complete metric space and

$$\max\left\{ \left\| f_1^{(i)}(t) - f_2^{(i)}(t) \right\| : i = 0,1,\ldots,n-1 \right\} \leq d(f_1,f_2) \quad (\text{for } t \in [a,b]). \tag{2.3}$$

Now we define the mapping $\Lambda_a : \Omega_a \to \Omega_a$ by

$$(\Lambda_a f)(t) = \sum_{k=0}^{n-1} \frac{y^{(k)}(a)}{k!}(t-a)^k - V_a^n\left(\sum_{k=0}^{n-1} \frac{p_k}{p_n} f^{(k)} - \frac{q}{p_n} \right)(t)$$

for all $f \in \Omega_a$ and $t \in I$. Note that the ith derivative of $\Lambda_a f$ is given by

$$(\Lambda_a f)^{(i)}(t) = \sum_{k=i}^{n-1} \frac{y^{(k)}(a)}{(k-i)!}(t-a)^{k-i} - V_a^{n-i}\left(\sum_{k=0}^{n-1} \frac{p_k}{p_n} f^{(k)} - \frac{q}{p_n} \right)(t) \tag{2.4}$$

for $i = 0,1,\ldots,n-1$. If $f_1, f_2 \in \Omega_a$ are given, then it follows from (b), (2.1), (2.3), and (2.4) that

$$\begin{aligned}
\left\|(\Lambda_a f_1)^{(i)}(t) - (\Lambda_a f_2)^{(i)}(t)\right\| &= \left\| V_a^{n-i}\left(\sum_{k=0}^{n-1} \frac{p_k}{p_n}\left(f_1^{(k)} - f_2^{(k)} \right) \right)(t) \right\| \\
&\leq V_a^{n-i}\left(\sum_{k=0}^{n-1} \left|\frac{p_k}{p_n}\right| \left\| f_1^{(k)} - f_2^{(k)} \right\| \right)(t) \\
&\leq V_a^{n-i}\left(\sum_{k=0}^{n-1} \left|\frac{p_k}{p_n}\right| d(f_1,f_2) \right)(t) \\
&\leq (b-a)^{n-i} \sum_{k=0}^{n-1} \left\|\frac{p_k}{p_n}\right\|_I d(f_1,f_2) \\
&\leq L_{a,b} d(f_1,f_2)
\end{aligned}$$

for all $t \in I$ and $i = 0,1,\ldots,n-1$. Hence, it follows from the definition of d that

$$d(\Lambda_a f_1, \Lambda_a f_2) \leq L_{a,b} d(f_1,f_2), \tag{2.5}$$

which implies that Λ_a is a strict contraction mapping on Ω_a.

An easy computation (using induction on i) shows that

$$V_a^{n-i}\big(y^{(n)}\big)(t) = y^{(i)}(t) - \sum_{k=i}^{n-1} \frac{y^{(k)}(a)}{(k-i)!}(t-a)^{k-i} \quad (\text{for } i = 0,1,\ldots,n-1). \tag{2.6}$$

Thus, it follows from (a), (2.4), (2.6), and from the definition of $S_n(I,\varphi)$ that

$$
\begin{aligned}
&\big\|y^{(i)}(t) - (\Lambda_a y)^{(i)}(t)\big\| \\
&= \left\| y^{(i)}(t) - \sum_{k=i}^{n-1} \frac{y^{(k)}(a)}{(k-i)!}(t-a)^{k-i} + V_a^{n-i}\left(\sum_{k=0}^{n-1} \frac{p_k}{p_n} y^{(k)} - \frac{q}{p_n} \right)(t) \right\| \\
&= \left\| V_a^{n-i}\big(y^{(n)}\big)(t) + V_a^{n-i}\left(\sum_{k=0}^{n-1} \frac{p_k}{p_n} y^{(k)} - \frac{q}{p_n} \right)(t) \right\| \\
&= \left\| V_a^{n-i}\left(y^{(n)} + \sum_{k=0}^{n-1} \frac{p_k}{p_n} y^{(k)} - \frac{q}{p_n} \right)(t) \right\| \\
&= \left\| V_a^{n-i}\left(\frac{1}{p_n} \sum_{k=0}^{n} p_k y^{(k)} - \frac{q}{p_n} \right)(t) \right\| \\
&\le \left\| \frac{1}{p_n} \right\|_I V_a^{n-i}\left(\left\| \sum_{k=0}^{n} p_k y^{(k)} - q \right\| \right)(t) \\
&\le \left\| \frac{1}{p_n} \right\|_I V_a^{n-i}(\varphi)(t) \\
&\le \|\varphi\|_1 \left\| \frac{1}{p_n} \right\|_I
\end{aligned}
$$

for any $i = 0, 1, \ldots, n-1$ and $t \in I$, where we note that

$$V_a(\varphi)(t) = \int_a^t \varphi(\tau)d\tau \le \int_a^b \varphi(\tau)d\tau = \|\varphi\|_1$$

and

$$V_a^{n-i}(\varphi)(t) \le (b-a)^{n-i-1}\|\varphi\|_1 \le \|\varphi\|_1.$$

Hence, we get

$$d(\Lambda_a y, y) \le \|\varphi\|_1 \left\| \frac{1}{p_n} \right\|_I < \infty. \tag{2.7}$$

By Theorem 2.1 (i), there exists a mapping $f \in \Omega_a$ (and so $f \in C_b^{n-1}(I,X)$) which is a fixed point of Λ_a, i.e.,

$$f(t) = (\Lambda_a f)(t) = \sum_{k=0}^{n-1} \frac{y^{(k)}(a)}{k!}(t-a)^k - V_a^n\left(\sum_{k=0}^{n-1} \frac{p_k}{p_n} f^{(k)} - \frac{q}{p_n} \right)(t). \tag{2.8}$$

Since $V_a^n(F) \in C_b^n(I,X)$ for every function $F \in C_b(I,X)$, we conclude that $f \in C_b^n(I,X)$. Now by differentiating both sides of (2.8) n times, we obtain

$$f^{(n)}(t) = (\Lambda_a f)^{(n)}(t) = -\sum_{k=0}^{n-1} \frac{p_k(t)}{p_n(t)} f^{(k)}(t) + \frac{q(t)}{p_n(t)},$$

i.e.,

$$\sum_{k=0}^{n} p_k(t) f^{(k)}(t) = q(t) \quad (\text{for } t \in I).$$

Therefore, f is a solution of Eq. (1.1). Since $f \in \Omega_a$, by the definition of Ω_a, we have

$$f^{(i)}(a) = y^{(i)}(a) \quad (\text{for } i = 0, 1, \ldots, n-1).$$

Moreover, by Theorem 2.1 (ii), f is a unique fixed point of Λ_a in the set $\Omega_a^* = \{g \in \Omega_a : d(g, y) < \infty\}$. Hence, $d(y, f) < \infty$ and by Theorem 2.1 (iii) and considering (2.1) and (2.7), we conclude that

$$d(y, f) \leq \frac{1}{1 - L_{a,b}} d(\Lambda_a y, y) \leq \frac{\|1/p_n\|_I}{1 - L_{a,b}} \|\varphi\|_1.$$

On the other hand, by (2.3), we have

$$\|y(t) - f(t)\| \leq d(y, f) \leq \frac{\|1/p_n\|_I}{1 - L_{a,b}} \|\varphi\|_1 \quad (\text{for } t \in [a, b]),$$

which completes the proof of part (i).

(ii) Let us define

$$\Omega_b = \left\{ f \in C_b^{n-1}(I, X) : f^{(i)}(b) = y^{(i)}(b) \text{ for } i = 0, 1, \ldots, n-1 \right\}$$

and

$$(\Lambda_b f)(t) = \sum_{k=0}^{n-1} \frac{y^{(k)}(b)}{k!} (t - b)^k - V_b^n \left(\sum_{k=0}^{n-1} \frac{p_k}{p_n} f^{(k)} - \frac{q}{p_n} \right)(t).$$

Then, we get

$$(\Lambda_b f)^{(i)}(t) = \sum_{k=i}^{n-1} \frac{y^{(k)}(b)}{(k-i)!} (t - b)^{k-i} - V_b^{n-i} \left(\sum_{k=0}^{n-1} \frac{p_k}{p_n} f^{(k)} - \frac{q}{p_n} \right)(t)$$

and

$$V_b^{n-i}\big(y^{(n)}\big)(t) = y^{(i)}(t) - \sum_{k=i}^{n-1} \frac{y^{(k)}(b)}{(k-i)!} (t - b)^{k-i} \quad (\text{for } i = 0, 1, \ldots, n-1).$$

Applying Λ_b and Ω_b instead of Λ_a and Ω_a, we follow the steps in part (i) to show that there exists a unique solution $f \in C_b^n(I, X)$ of Eq. (1.1) satisfying (2.2) and $f^{(i)}(b) = y^{(i)}(b)$ for every $i = 0, 1, \ldots, n-1$.

(iii) Write $(a, b) = (a, c] \cup [c, b)$ for some $c \in (a, b)$. Since $y \in S_n(I, \varphi)$, it holds that $y \in S_n((a, c], \varphi)$ and $y \in S_n([c, b), \varphi)$. Since $\max\{L_{a,c}, L_{c,b}\} < L_{a,b} < 1$, by parts (i) and (ii), there exist solutions $f_1 \in C_b^n((a, c], X)$ and $f_2 \in C_b^n([c, b), X)$ of Eq. (1.1) such that

$$f_1^{(i)}(c) = y^{(i)}(c) = f_2^{(i)}(c) \quad (\text{for } i = 0, 1, \ldots, n-1),$$

where f_1 and f_2 are uniquely determined. We define $f : (a, b) \to X$ by $f(t) = f_1(t)$ for $t \in (a, c]$ and $f(t) = f_2(t)$ for $t \in [c, b)$. Then, f is a solution of (1.1) and by the above relation, $f \in C_b^n(I, X)$. Moreover, by (2.2), we have

$$\|y(t) - f(t)\| \leq \frac{\|1/p_n\|_I}{1 - L_{a,b}} \|\varphi\|_1 \quad (\text{for } t \in (a, b)),$$

which completes the proof. □

3 Hyers-Ulam-Rassis stability of Eq. (1.1)

We now prove the main theorem of this paper.

THEOREM 3.1 *Let I be any interval, $q \in C(I, X)$, and let $p_0, p_1, \ldots, p_n : I \to \mathbb{F}$ be continuous functions such that $p_n(t) \neq 0$ for each $t \in I$. Then the differential equation*

$$\sum_{k=0}^{n} p_k(t) y^{(k)}(t) = q(t)$$

has the Hyers-Ulam-Rassias stability.

PROOF. Without loss of generality, let $I = (a, b)$ and $p_n(t) \equiv 1$. Assume that $\varphi : I \to [0, \infty)$ is a bounded integrable function and $y \in C_b^n(I, X)$ such that

$$\left\| y^{(n)}(t) + \sum_{k=0}^{n-1} p_k(t) y^{(k)}(t) - q(t) \right\| \leq \varphi(t) \quad (\text{for } t \in I). \tag{3.1}$$

Let $\{a_0, a_1, \ldots, a_m\}$ be a partition of the interval $[a, b]$ with the properties:

(1) $a_0 = a$, $a_m = b$, and $0 < a_j - a_{j-1} < 1$ for $j = 1, 2, \ldots, m$;

(2) $I_1 = (a_0, a_1]$, $I_m = [a_{m-1}, a_m)$, and $I_j = [a_{j-1}, a_j]$ for $j = 2, 3, \ldots, m-1$;

(3) $L_{a_{j-1}, a_j} = (a_j - a_{j-1}) \sum_{k=0}^{n-1} \|p_k\|_{I_j} < 1$ for $j = 1, 2, \ldots, m$.

Restricting the inequality (3.1) to the interval $I_1 = (a_0, a_1]$, it follows from Lemma 2.3 (ii) that there exists a unique solution $y_1 \in C_b^n(I_1, X)$ of Eq. (1.1) such that

$$\|y(t) - y_1(t)\| \leq \frac{1}{1 - L_{a_0, a_1}} \int_{a_0}^{a_1} |\varphi(\tau)| d\tau \quad (\text{for } t \in I_1)$$

and

$$y_1^{(i)}(a_1) = y^{(i)}(a_1) \quad (\text{for } i = 0, 1, \ldots, n-1). \tag{3.2}$$

If the inequality (3.1) is restricted to $I_2 = [a_1, a_2]$, then Lemma 2.3 (i) and (3.1) imply that there exists a unique solution $y_2 \in C_b^n(I_2, X)$ of Eq. (1.1) such that

$$\|y(t) - y_2(t)\| \leq \frac{1}{1 - L_{a_1, a_2}} \int_{a_1}^{a_2} |\varphi(\tau)| d\tau \quad (\text{for } t \in I_2)$$

and

$$y_2^{(i)}(a_1) = y^{(i)}(a_1) \quad (\text{for } i = 0, 1, \ldots, n-1). \tag{3.3}$$

Comparing (3.2) and (3.3), we get

$$y_2^{(i)}(a_1) = y_1^{(i)}(a_1) \quad (\text{for } i = 0, 1, \ldots, n-1).$$

By a similar way, we obtain a solution $y_j \in C_b^n(I_j, X)$ of Eq. (1.1) on I_j such that

$$\|y(t) - y_j(t)\| \le \frac{1}{1 - L_{a_{j-1},a_j}} \int_{a_{j-1}}^{a_j} |\varphi(\tau)| d\tau \quad (\text{for } t \in I_j) \tag{3.4}$$

and

$$y_{j+1}^{(i)}(a_j) = y_j^{(i)}(a_j) \quad (\text{for } j = 1, 2, \ldots, m-1 \text{ and } i = 0, 1, \ldots, n-1).$$

Applying the last relation and using the fact that y_j $(j = 1, 2, \ldots, m)$ is a solution of Eq. (1.1) on I_j, it follows from Eq. (1.1) with $p_n(t) \equiv 1$ that

$$y_{j+1}^{(n)}(a_j) = -\sum_{k=0}^{n-1} p_k(a_j) y_{j+1}^{(k)}(a_j) + q(a_j) = -\sum_{k=0}^{n-1} p_k(a_j) y_j^{(k)}(a_j) + q(a_j) = y_j^{(n)}(a_j).$$

Hence

$$y_{j+1}^{(i)}(a_j) = y_j^{(i)}(a_j) \quad (\text{for } j = 1, 2, \ldots, m-1 \text{ and } i = 0, 1, \ldots, n). \tag{3.5}$$

Now, we define $y_s : I \to X$ by $y_s(t) = y_j(t)$ for $t \in I_j$. In view of (3.5), the function y_s is well defined and n times continuously differentiable. Let us define

$$M = \max\left\{ \frac{1}{1 - L_{a_{j-1},a_j}} : j = 1, 2, \ldots, m \right\}.$$

Then, (3.4) implies that

$$\|y(t) - y_s(t)\| \le M \|\varphi\|_1 \quad (\text{for } t \in I)$$

and this completes the proof. □

When $\varphi(t) \equiv \varepsilon > 0$, we obtain the following corollary.

COROLLARY 3.2 *Under the assumptions of Theorem 3.1, the differential equation*

$$\sum_{k=0}^{n} p_k(t) y^{(k)}(t) = q(t)$$

has the Hyers-Ulam stability.

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