SOLVABILITY OF A THIRD-ORDER THREE-POINT BOUNDARY VALUE PROBLEM ON A HALF-LINE*

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Abstract

In this paper, we consider the solvability of a third-order three-point boundary value problem on a half-line of the form:

$$\begin{cases} x'''(t) = f(t, x(t), x'(t), x''(t)), \quad 0 < t < +\infty, \\ x(0) = \alpha x(\eta), \quad \lim\limits_{t\to+\infty} x^{(i)}(t) = 0, \quad i = 1, 2, \end{cases}$$

where $\alpha \neq 1$ and $\eta \in (0, +\infty)$, while $f : [0, +\infty) \times \mathbb{R}^3 \to \mathbb{R}$ is S^2- Carathéodory function. The existence and uniqueness of solutions for the boundary value problems are obtained by the Leray-Schauder continuation theorem. As an application, an example is given to demonstrate our results.

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1. Introduction

In this paper we consider the existence and uniqueness of solutions for third-order three-point boundary value problems on a half-line

$$x'''(t) = f(t, x(t), x'(t), x''(t)), \quad 0 < t < +\infty, \tag{1.1}$$

$$x(0) = \alpha x(\eta), \quad \lim_{t\to+\infty} x^{(i)}(t) = 0, \quad i = 1, 2, \tag{1.2}$$

where $\alpha \neq 1$ and $\eta \in (0, +\infty)$.

The third-order differential equations arise in many areas, such as the deflection of a curved beam having a constant or a varying cross-section, three layer beam, electromagnetic waves or gravity-driven flows [10]. Meanwhile, the third order boundary value problems in an infinite interval has been widely used to describe

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the evolution of physical phenomena, for example some draining or coating fluid-flow problems, see [3, 24, 25]. We refer the reader to [2, 4–9, 11–16, 20, 21, 23, 26, 27] for the study of the finite interval problems of third-order differential equations, and to [1, 3, 17, 22, 24, 25] for the study of the infinite interval problems.

Motivated by the above works and [19], in this paper we discuss the solvability of third-order three-point boundary value problems (1.1),(1.2). Based upon the Leray-Schauder continuation theorem, the existence and uniqueness of solutions for BVP(1.1),(1.2) were obtained.

The rest of this paper is organized as follows. In section 2, as the preliminary we give some lemmas which help us to simplify the proofs of our main results. In section 3, firstly we discuss the existence of solutions for BVP(1.1),(1.2) by Leray-Schauder continuation theorem, and then investigate the uniqueness of solutions to BVP(1.1),(1.2). Finally, as an application, we give an example to demonstrate our results.

2. Preliminary

In this section, we present some definitions and lemmas which are useful in the proof of our main results.

Definition 2.1. *The function* $f : [0, +\infty) \times \mathbb{R}^3 \to \mathbb{R}$ *is called an* S^2-*Carathéodory function, if and only if*

(i) *for each* $(u, v, w) \in \mathbb{R}^3, t \mapsto f(t, u, v, w)$ *is measurable on* $[0, +\infty)$;
(ii) *for a.e.* $t \in [0, +\infty), (u, v, w) \mapsto f(t, u, v, w)$ *is continuous on* $\mathbb{R}^3$;
(iii) *for each* $r > 0$, *there exists* $\varphi_r(t) \in L^1[0, +\infty)$ *with* $\varphi_r(t) > 0$ *on* $(0, +\infty)$ *and* $t\varphi_r(t)$, $t^2\varphi_r(t) \in L^1[0, +\infty)$, *such that* $\forall u, v, w \in [-r, r]$,

$$|f(t, u, v, w)| \le \varphi_r(t), \quad a.e.\ t \in [0, +\infty).$$

Lemma 2.1. *For any* $h(t) \in L^1[0, +\infty)$ *with* $th(t), t^2h(t) \in L^1[0, +\infty)$, *the BVP*

$$\begin{cases} x'''(t) = h(t), \quad 0 < t < +\infty, \\ x(0) = \alpha x(\eta), \quad \lim\limits_{t \to +\infty} x^{(i)}(t) = 0, \ i = 1, 2 \end{cases} \tag{2.1}$$

has a unique solution

$$x(t) = \int_0^{+\infty} G(t,s)h(s)\mathrm{d}s,$$

where

$$G(t,s) = \begin{cases} \frac{1}{2(1-\alpha)}s^2, & s \le \min\{\eta, t\}; \\ \frac{\alpha}{2(1-\alpha)}s^2 - \frac{1}{2}t^2 + ts, & t \le s \le \eta; \\ \frac{\alpha}{\alpha-1}(\frac{1}{2}\eta^2 - \eta s) + \frac{1}{2}s^2, & \eta \le s \le t; \\ \frac{\alpha}{\alpha-1}(\frac{1}{2}\eta^2 - \eta s) - \frac{1}{2}t^2 + ts, & \max\{\eta, t\} \le s. \end{cases}$$

Proof. Noticing that $h(t), th(t), t^2h(t) \in L^1[0,+\infty)$, integrate the differential equation in BVP(2.1) from t to $+\infty$, we have

$$-x''(t) = \int_t^{+\infty} h(s)\mathrm{d}s.$$

Integrating this differential equation on $[t,+\infty)$, and applying the Fubini theorem we obtain that

$$x'(t) = \int_t^{+\infty} (s-t)h(s)\mathrm{d}s.$$

Also integrating the above differential on $[0,t]$ one has

$$x(t) = x(0) + \frac{1}{2}\int_0^t s^2h(s)\mathrm{d}s + \int_t^{+\infty}(ts - \frac{1}{2}t^2)h(s)\mathrm{d}s. \tag{2.2}$$

Since $x(0) = \alpha x(\eta)$, we have

$$x(0) = \alpha[x(0) + \frac{1}{2}\int_0^\eta s^2h(s)\mathrm{d}s + \int_\eta^{+\infty}(\eta s - \frac{1}{2}\eta^2)h(s)\mathrm{d}s],$$

and thus

$$x(0) = \frac{\alpha}{1-\alpha}[\frac{1}{2}\int_0^\eta s^2h(s)\mathrm{d}s + \int_\eta^{+\infty}(\eta s - \frac{1}{2}\eta^2)h(s)\mathrm{d}s].$$

Hence from (2.2) it follows that

$$\begin{aligned} x(t) &= \tfrac{\alpha}{1-\alpha}[\tfrac{1}{2}\textstyle\int_0^\eta s^2h(s)\mathrm{d}s + \int_\eta^{+\infty}(\eta s - \frac{1}{2}\eta^2)h(s)\mathrm{d}s] \\ &\quad + \tfrac{1}{2}\textstyle\int_0^t s^2h(s)\mathrm{d}s + \int_t^{+\infty}(ts - \frac{1}{2}t^2)h(s)\mathrm{d}s. \end{aligned}$$

Therefore, when $0 \le t \le \eta$,

$$\begin{aligned} x(t) &= \tfrac{\alpha}{1-\alpha}[\tfrac{1}{2}(\textstyle\int_0^t + \int_t^\eta)s^2h(s)\mathrm{d}s + \int_\eta^{+\infty}(\eta s - \frac{1}{2}\eta^2)h(s)\mathrm{d}s] \\ &\quad + \tfrac{1}{2}\textstyle\int_0^t s^2h(s)\mathrm{d}s + (\int_t^\eta + \int_\eta^{+\infty})(ts - \frac{1}{2}t^2)h(s)\mathrm{d}s, \end{aligned}$$

when $\eta \le t \le +\infty$,

$$\begin{aligned} x(t) &= \tfrac{\alpha}{1-\alpha}[\tfrac{1}{2}\textstyle\int_0^\eta s^2h(s)\mathrm{d}s + (\int_\eta^t + \int_t^{+\infty})(\eta s - \frac{1}{2}\eta^2)h(s)\mathrm{d}s] \\ &\quad + \tfrac{1}{2}(\textstyle\int_0^\eta + \int_\eta^t)s^2h(s)\mathrm{d}s + \int_t^{+\infty}(ts - \frac{1}{2}t^2)h(s)\mathrm{d}s, \end{aligned}$$

that is,

$$x(t) = \int_0^{+\infty} G(t,s)h(s)\mathrm{d}s, \quad \forall t \in [0,+\infty).$$

This completes the proof of the lemma. □

Lemma 2.2. *Let the Green function $G(t,s)$ be as in Lemma 2.1. Then $G(t,s)$ has two properties*:

(1) *For all $t, s \in [0,+\infty)$,*

$$|G(t,s)| \le \begin{cases} \frac{1}{2}s^2, & \alpha < 0; \\ \frac{1}{2(1-\alpha)}s^2, & 0 \le \alpha < 1; \\ \frac{\alpha}{2(\alpha-1)}s^2, & \alpha > 1. \end{cases}$$

(2)

$$\lim_{t\to+\infty} G(t,s) = \overline{G}(s) := \begin{cases} \frac{1}{2(1-\alpha)}s^2, & s \le \eta; \\ \frac{\alpha}{\alpha-1}(\frac{1}{2}\eta^2 - \eta s) + \frac{1}{2}s^2, & \eta \le s. \end{cases}$$

Proof. (1) It is easy to see that for each fixed $s \in [0,+\infty)$,

$$\frac{\partial}{\partial t}G(t,s) \ge 0,\ \forall t \in [0,s] \quad \text{and} \quad \frac{\partial}{\partial t}G(t,s) \equiv 0,\ \forall t \in [s,+\infty),$$

it follows that

$$G(0,s) \le G(t,s) \le G(s,s), \quad \forall t, s \in [0,+\infty).$$

Since

$$G(0,s) = \begin{cases} \frac{\alpha}{2(1-\alpha)}s^2, & s \le \eta; \\ \frac{\alpha}{\alpha-1}(\frac{1}{2}\eta^2 - \eta s), & \eta \le s, \end{cases}$$

$$G(s,s) = \begin{cases} \frac{1}{2(1-\alpha)}s^2, & s \le \eta; \\ \frac{\alpha}{\alpha-1}(\frac{1}{2}\eta^2 - \eta s) + \frac{1}{2}s^2, & \eta \le s, \end{cases}$$

we have $\forall (t,s) \in [0,+\infty) \times [0,\eta]$,

$$\frac{\alpha}{2(1-\alpha)}s^2 \le G(t,s) \le \frac{1}{2(1-\alpha)}s^2,$$

and $\forall (t,s) \in [0,+\infty) \times [\eta,+\infty)$,

$$\frac{\alpha}{\alpha-1}(\frac{1}{2}\eta^2 - \eta s) \le G(t,s) \le \frac{\alpha}{\alpha-1}(\frac{1}{2}\eta^2 - \eta s) + \frac{1}{2}s^2.$$

Now we have three cases to consider:

Case 1. $\alpha < 0$. In this case, since $\alpha/(\alpha-1) > 0$ and $\eta^2/2 - \eta s < 0$ for $s \ge \eta$, we have

$$\frac{\alpha}{\alpha-1}(\frac{1}{2}\eta^2 - \eta s) + \frac{1}{2}s^2 \le \frac{1}{2}s^2 \quad \text{for} \quad s \ge \eta.$$

But from $1/(1-\alpha) < 1$, it follows that

$$\frac{1}{2(1-\alpha)}s^2 \leq \frac{1}{2}s^2 \quad \text{for} \quad s \in [0,\eta].$$

Therefore

$$G(t,s) \leq \frac{1}{2}s^2, \quad \forall t,s \in [0,+\infty). \tag{2.3}$$

Also $\forall s \in [0,+\infty)$,

$$\frac{\alpha}{\alpha-1}(\frac{1}{2}\eta^2 - \eta s) - \frac{\alpha}{2(1-\alpha)}s^2 = \frac{\alpha}{2(\alpha-1)}(\eta - s)^2 \geq 0,$$

then

$$G(t,s) \geq \frac{\alpha}{2(1-\alpha)}s^2, \quad \forall t,s \in [0,+\infty).$$

This together with (2.3) implies that

$$\frac{\alpha}{2(1-\alpha)}s^2 \leq G(t,s) \leq \frac{1}{2}s^2, \quad \forall t,s \in [0,+\infty).$$

Hence from the fact $0 < \alpha/(\alpha-1) < 1$, it follows that

$$|G(t,s)| \leq \frac{1}{2}s^2, \quad \forall t,s \in [0,+\infty).$$

Case 2. $0 \leq \alpha < 1$. In this case, obviously $G(t,s) \geq 0$ on $[0,+\infty) \times [0,+\infty)$. Since $\forall s \in [0,+\infty)$,

$$\frac{\alpha}{\alpha-1}(\frac{1}{2}\eta^2 - \eta s) + \frac{1}{2}s^2 - \frac{1}{2(1-\alpha)}s^2 = \frac{\alpha}{2(\alpha-1)}(\eta - s)^2 \leq 0,$$

it follows that

$$G(t,s) \leq \frac{1}{2(1-\alpha)}s^2, \quad \forall t,s \in [0,+\infty).$$

Hence

$$0 \leq G(t,s) \leq \frac{1}{2(1-\alpha)}s^2, \quad \forall t,s \in [0,+\infty),$$

which implies that

$$|G(t,s)| \leq \frac{1}{2(1-\alpha)}s^2, \quad \forall t,s \in [0,+\infty).$$

Case 3. $\alpha > 1$. In this case, obviously $s^2/(2(1-\alpha)) \leq 0$ for $s \in [0,+\infty)$. On the other hand,

$$\frac{\alpha}{\alpha-1}(\frac{1}{2}\eta^2 - \eta s) < 0 \quad \text{for} \quad s \geq \eta,$$

and thus

$$\frac{\alpha}{\alpha-1}(\frac{1}{2}\eta^2 - \eta s) + \frac{1}{2}s^2 < \frac{1}{2}s^2 \quad \text{for} \quad s \geq \eta.$$

Therefore

$$G(t,s)\le \frac{1}{2}s^2,\quad \forall t,s\in[0,+\infty). \tag{2.4}$$

Also since

$$\frac{\alpha}{2(1-\alpha)}s^2-\frac{\alpha}{\alpha-1}(\frac{1}{2}\eta^2-\eta s)=\frac{\alpha}{2(1-\alpha)}(s-\eta)^2\le 0,\quad \forall s\in[0,+\infty),$$

it follows from (2.4) that

$$\frac{\alpha}{2(1-\alpha)}s^2\le G(t,s)\le \frac{1}{2}s^2,\quad \forall t,s\in[0,+\infty).$$

This together with the fact $\alpha/(1-\alpha)<-1$ implies that

$$|G(t,s)|\le \frac{\alpha}{2(\alpha-1)}s^2,\quad \forall t,s\in[0,+\infty).$$

(2) From the definition of $G(t,s)$, it follows that

$$G(t,s)=\begin{cases}\frac{1}{2(1-\alpha)}s^2, & s\le\eta\le t;\\ \frac{\alpha}{\alpha-1}(\frac{1}{2}\eta^2-\eta s)+\frac{1}{2}s^2, & \eta\le s\le t,\end{cases}$$

and thus

$$\lim_{t\to+\infty}G(t,s)=\begin{cases}\frac{1}{2(1-\alpha)}s^2, & s\le\eta;\\ \frac{\alpha}{\alpha-1}(\frac{1}{2}\eta^2-\eta s)+\frac{1}{2}s^2, & \eta\le s.\end{cases}$$

This completes the proof of the lemma. □

Consider the space

$$C^2_\infty[0,+\infty)=\{x\in C^2[0,+\infty):\lim_{t\to+\infty}x^{(i)}(t)\text{ exists}, i=0,1,2\}$$

with the norm $\|x\|:=\|x\|_\infty+\|x'\|_\infty+\|x''\|_\infty$, $\forall x\in C^2_\infty[0,+\infty)$. Then by the standard arguments, we can prove that $(C^2_\infty[0,+\infty),\|\cdot\|)$ is a Banach space. Now, we introduce the subspace X of $C^2_\infty[0,+\infty)$ as follows:

$$X=\{x\in C^2_\infty[0,+\infty):x(0)=\alpha x(\eta),\lim_{t\to+\infty}x^{(i)}(t)=0, i=1,2\}.$$

Then it is clear that X is closed in $C^2_\infty[0,+\infty)$, and hence is itself a Banach space.

Lemma 2.3. [1] *Let $M\subset C_\infty=\{x\in C[0,+\infty):\lim_{t\to+\infty}x(t)$ exists$\}$. Then M is relatively compact if the following conditions hold:*

(i) *all functions from M are uniformly bounded;*

(ii) *all functions from M are equicontinuous on any compact interval of* $[0,+\infty)$;

(iii) *all functions from M are equiconvergent at infinity, that is, for any given $\varepsilon > 0$, there exists a $T(\varepsilon) > 0$, such that $|f(t) - f(+\infty)| < \varepsilon$ for all $t > T$ and $f \in M$.*

Let us denote the operator T as

$$(Tx)(t) = \int_0^{+\infty} G(t,s) f(s, x(s), x'(s), x''(s))\mathrm{d}s, \quad 0 < t < +\infty.$$

It is easy to check from Lemma 2.1, 2.2 and Lebesgue's dominated convergence theorem that if f satisfies the S^2- Carathéodory condition, then $T : X \to X$ is well defined.

Lemma 2.4. *Let $f : [0, +\infty) \times \mathbb{R}^3 \to \mathbb{R}$ be an S^2- Carathéodory function, then $T : X \to X$ is compact.*

Proof. First we show that T is continuous. To do this, let $x_n, x_0 \in X (n = 1, 2, \cdots)$ and $x_n \to x_0 (n \to +\infty)$. Then there exists $r_0 > 0$, such that

$$\|x_n\| \le r_0, \quad n = 0, 1, 2, \cdots .$$

Since f is an S^2- Carathéodory function, then for the above $r_0 > 0$, there exists a positive function $\varphi_{r_0} \in L^1[0, +\infty)$ with $t\varphi_{r_0}(t), t^2\varphi_{r_0}(t) \in L^1[0, +\infty)$ such that for each $n = 0, 1, 2, \cdots$,

$$|f(t, x_n(t), x_n'(t), x_n''(t))| \le \varphi_{r_0}(t), \quad \text{a.e. } t \in [0, +\infty).$$

Thus

$$\begin{aligned} & \int_0^{+\infty} |\overline{G}(s)(f(s, x_n(s), x_n'(s), x_n''(s)) - f(s, x_0(s), x_0'(s), x_0''(s)))|\mathrm{d}s \\ \le\ & 2\int_0^{+\infty} |\overline{G}(s)|\varphi_{r_0}(s)\mathrm{d}s < +\infty, \quad n = 1, 2, \cdots . \end{aligned}$$

Consequently from the Lebesgue's dominated convergence theorem, it follows that

$$\begin{aligned} & |(Tx_n)(t) - (Tx_0)(t)| \\ \le\ & \int_0^{+\infty} |G(t,s)||f(s, x_n(s), x_n'(s), x_n''(s)) - f(s, x_0(s), x_0'(s), x_0''(s))|\mathrm{d}s \\ \le\ & \tfrac{1}{2}(1 + \tfrac{1+|\alpha|}{|1-\alpha|}) \int_0^{+\infty} s^2|f(s, x_n(s), x_n'(s), x_n''(s)) - f(s, x_0(s), x_0'(s), x_0''(s))|\mathrm{d}s \\ \rightrightarrows\ & 0 \quad \text{on } [0, +\infty)\ (n \to +\infty), \end{aligned}$$

$$\begin{aligned}
&|(Tx_n)'(t)-(Tx_0)'(t)|\\
\leq\ & \int_t^{+\infty}(s-t)|f(s,x_n(s),x_n'(s),x_n''(s))-f(s,x_0(s),x_0'(s),x_0''(s))|\mathrm{d}s\\
\leq\ & \int_0^{+\infty}s|f(s,x_n(s),x_n'(s),x_n''(s))-f(s,x_0(s),x_0'(s),x_0''(s))|\mathrm{d}s\\
\rightrightarrows\ & 0 \quad \text{on } [0,+\infty)\ (n\to+\infty)
\end{aligned}$$

and

$$\begin{aligned}
&|(Tx_n)''(t)-(Tx_0)''(t)|\\
=\ & |-\int_t^{+\infty}(f(s,x_n(s),x_n'(s),x_n''(s))-f(s,x_0(s),x_0'(s),x_0''(s)))\mathrm{d}s|\\
\leq\ & \int_0^{+\infty}|f(s,x_n(s),x_n'(s),x_n''(s))-f(s,x_0(s),x_0'(s),x_0''(s))|\mathrm{d}s\\
\rightrightarrows\ & 0 \quad \text{on } [0,+\infty)\ (n\to+\infty).
\end{aligned}$$

Therefore, $T:X\to X$ is continuous.

Next we show that T maps bounded sets into relatively compact set. Let $B\subset X$ be a bounded set. Then there exists $r_1>0$, such that $\forall x\in B$,

$$\|x\|=\|x\|_\infty+\|x'\|_\infty+\|x''\|_\infty\leq r_1.$$

Since f is an S^2- Carathéodory function, there exists a positive function $\varphi_{r_1}\in L^1[0,+\infty)$ with $t\varphi_{r_1}(t),t^2\varphi_{r_1}(t)\in L^1[0,+\infty)$, such that $\forall x\in B$,

$$|f(t,x(t),x'(t),x''(t))|\leq\varphi_{r_1}(t),\quad \text{a.e. } t\in[0,+\infty).$$

Therefore $\forall x\in B$, we have

$$\begin{aligned}
|(Tx)(t)| &\leq \int_0^{+\infty}|G(t,s)||f(s,x(s),x'(s),x''(s))|\mathrm{d}s\\
&\leq \tfrac{1}{2}\left(1+\tfrac{1+|\alpha|}{|1-\alpha|}\right)\int_0^{+\infty}s^2\varphi_{r_1}(s)\mathrm{d}s<+\infty,\quad t\in[0,+\infty),
\end{aligned}$$

$$\begin{aligned}
|(Tx)'(t)| &\leq \int_t^{+\infty}s|f(s,x(s),x'(s),x''(s))|\mathrm{d}s\\
&\leq \int_0^{+\infty}s\varphi_{r_1}(s)\mathrm{d}s<+\infty,\quad t\in[0,+\infty)
\end{aligned}\tag{2.5}$$

and

$$\begin{aligned}
|(Tx)''(t)| &\leq \int_t^{+\infty}|f(s,x(s),x'(s),x''(s))|\mathrm{d}s\\
&\leq \int_0^{+\infty}\varphi_{r_1}(s)\mathrm{d}s<+\infty,\quad t\in[0,+\infty).
\end{aligned}\tag{2.6}$$

Thus $\{(Tx)(t):x\in B\}$, $\{(Tx)'(t):x\in B\}$ and $\{(Tx)''(t):x\in B\}$ are uniformly bounded. Also, from (2.5) and (2.6) it follows that, $\{(Tx)(t):x\in B\}$ and

$\{(Tx)'(t) : x \in B\}$ are equicontinuous on any compact interval of $[0, +\infty)$. Meanwhile, $\forall t_1, t_2 \in [0, +\infty)$ and $\forall x \in B$, we have

$$\begin{aligned} |(Tx)''(t_2) - (Tx)''(t_1)| &= |\int_{t_1}^{t_2} f(s, x(s), x'(s), x''(s))\mathrm{d}s| \\ &\leq |\int_{t_1}^{t_2} \varphi_{r_1}(s)\mathrm{d}s|, \end{aligned}$$

and so by the absolute continuity of Lebesgue integral, $\{(Tx)''(t) : x \in B\}$ is equicontinuous on any compact interval of $[0, +\infty)$.

On the other hand, from Lebesgue's dominated convergence theorem, it follows that

$$\begin{aligned} |(Tx)(t) - (Tx)(+\infty)| &\leq \int_0^{+\infty} |G(t, s) - \overline{G}(s)||f(s, x(s), x'(s), x''(s))|\mathrm{d}s \\ &\leq \int_0^{+\infty} |G(t, s) - \overline{G}(s)|\varphi_{r_1}(s)\mathrm{d}s \\ &\to 0 \quad (t \to +\infty), \quad \forall x \in B, \end{aligned}$$

$$\begin{aligned} |(Tx)'(t) - (Tx)'(+\infty)| &\leq \int_t^{+\infty} (s - t)|f(s, x(s), x'(s), x''(s))|\mathrm{d}s \\ &\leq \int_t^{+\infty} s\varphi_{r_1}(s)\mathrm{d}s \\ &\to 0 \quad (t \to +\infty), \quad \forall x \in B \end{aligned}$$

and

$$\begin{aligned} |(Tx)''(t) - (Tx)''(+\infty)| &\leq \int_t^{+\infty} |f(s, x(s), x'(s), x''(s))|\mathrm{d}s \\ &\leq \int_t^{+\infty} \varphi_{r_1}(s)\mathrm{d}s \\ &\to 0 \quad (t \to +\infty), \quad \forall x \in B, \end{aligned}$$

we have that $\{(Tx)(t) : x \in B\}, \{(Tx)'(t) : x \in B\}$ and $\{(Tx)''(t) : x \in B\}$ are equiconvergent at infinity. Hence from Lemma 2.4, TB is relatively compact in X.

In summary, $T : X \to X$ is compact. This completes the proof of the lemma. □

Lemma 2.5. [18](Leray-Schauder continuation theorem) *Let X be a real Banach space and let Ω be a bounded open neighborhood of* 0 *in X. Let $T : \bar{\Omega} \to X$ be a completely continuous operator such that for all $\lambda \in (0, 1)$ and $x \in \partial\Omega$, $x \neq \lambda Tx$. Then the operator equation*

$$x = Tx$$

has a solution $x \in \bar{\Omega}$.

3. Main Results

Now we apply the Leray-Schauder continuation theorem to establish the existence theorems for BVP(1.1),(1.2).

Theorem 3.1. *Assume that* $f : [0, +\infty) \times \mathbb{R}^3 \to \mathbb{R}$ *is* $S^2 -$ *Carathéodory function. Suppose also that there exist nonnegative functions* $p(t), q(t), r(t), e(t) \in L^1[0, +\infty)$ *with* $t^i p(t), t^i q(t), t^i r(t), t^i e(t) \in L^1[0, +\infty)(i = 1, 2)$*, such that for all* $(u, v, w) \in \mathbb{R}^3$,

$$|f(t, u, v, w)| \leq p(t)|u| + q(t)|v| + r(t)|w| + e(t), \quad a.e.\ t \in [0, +\infty).$$

Then $BVP(1.1), (1.2)$ *has at least one solution, provided*

$$\max\{P, Q, R\} < 1,$$

where

$$P = \begin{cases} P_0 + P_1 + \frac{1}{2}P_2, & \alpha < 0; \\ P_0 + P_1 + \frac{1}{2(1-\alpha)}P_2, & 0 \leq \alpha < 1; \\ P_0 + P_1 + \frac{\alpha}{2(\alpha-1)}P_2, & \alpha > 1, \end{cases} \quad P_i = \int_0^{+\infty} s^i p(s)\mathrm{d}s,\ i = 0, 1, 2,$$

$$Q = \begin{cases} Q_0 + Q_1 + \frac{1}{2}Q_2, & \alpha < 0; \\ Q_0 + Q_1 + \frac{1}{2(1-\alpha)}Q_2, & 0 \leq \alpha < 1; \\ Q_0 + Q_1 + \frac{\alpha}{2(\alpha-1)}Q_2, & \alpha > 1, \end{cases} \quad Q_i = \int_0^{+\infty} s^i q(s)\mathrm{d}s,\ i = 0, 1, 2,$$

$$R = \begin{cases} R_0 + R_1 + \frac{1}{2}R_2, & \alpha < 0; \\ R_0 + R_1 + \frac{1}{2(1-\alpha)}R_2, & 0 \leq \alpha < 1; \\ R_0 + R_1 + \frac{\alpha}{2(\alpha-1)}R_2, & \alpha > 1, \end{cases} \quad R_i = \int_0^{+\infty} s^i r(s)\mathrm{d}s,\ i = 0, 1, 2.$$

Proof. By Lemma 2.1, it is easy to see that $x \in X$ is a solution of BVP$(1.1), (1.2)$ if and only if x is a fixed point of T. Now, we apply Leray-Schauder continuation theorem, to prove that T has a fixed point in X. To do this, it is sufficient to show that operator equations

$$x = \lambda T x, \quad \lambda \in (0, 1) \tag{3.1}$$

has a priori bound M independently of λ, that is we need only to show that boundary value problems

$$\begin{cases} x'''(t) = \lambda f(t, x(t), x'(t), x''(t)), \quad 0 < t < +\infty,\ \lambda \in (0,1), \\ x(0) = \alpha x(\eta), \quad \lim_{t\to+\infty} x'(t) = 0, \quad \lim_{t\to+\infty} x''(t) = 0 \end{cases} \tag{3.2}$$

has a priori bound M independently of $\lambda \in (0,1)$.

Suppose that $x = x(t)$ is a possible solution of (3.2), and let

$$E_0 = \int_0^{+\infty} e(s)\mathrm{d}s, \quad E_1 = \int_0^{+\infty} se(s)\mathrm{d}s, \quad E_2 = \int_0^{+\infty} s^2 e(s)\mathrm{d}s.$$

We will now divide the proof into three cases.

Case 1. $\alpha < 0$. In this case, by Lemma 2.2 we have

$$|G(t,s)| \le \frac{1}{2}s^2, \quad \forall t, s \in [0, +\infty).$$

From (3.2) and Lemma 2.1 it follows that

$$\begin{aligned} |x(t)| &= \left|\lambda \int_0^{+\infty} G(t,s) f(s, x(s), x'(s), x''(s))\mathrm{d}s\right| \\ &\le \int_0^{+\infty} |G(t,s)||f(s, x(s), x'(s), x''(s))|\mathrm{d}s \\ &\le \int_0^{+\infty} \tfrac{1}{2}s^2(p(s)|x(s)| + q(s)|x'(s)| + r(s)|x''(s)| + e(s))\mathrm{d}s \\ &\le \tfrac{1}{2}(P_2\|x\|_\infty + Q_2\|x'\|_\infty + R_2\|x''\|_\infty + E_2), \quad \forall t \in [0, +\infty), \end{aligned}$$

$$\begin{aligned} |x'(t)| &= \left|\lambda \int_t^{+\infty} (s-t) f(s, x(s), x'(s), x''(s))\mathrm{d}s\right| \\ &\le \int_0^{+\infty} s|f(s, x(s), x'(s), x''(s))|\mathrm{d}s \\ &\le \int_0^{+\infty} s(p(s)|x(s)| + q(s)|x'(s)| + r(s)|x''(s)| + e(s))\mathrm{d}s \\ &\le P_1\|x\|_\infty + Q_1\|x'\|_\infty + R_1\|x''\|_\infty + E_1, \quad \forall t \in [0, +\infty) \end{aligned}$$

and

$$\begin{aligned} |x''(t)| &= \left|-\lambda \int_t^{+\infty} f(s, x(s), x'(s), x''(s))\mathrm{d}s\right| \\ &\le \int_0^{+\infty} |f(s, x(s), x'(s), x''(s))|\mathrm{d}s \\ &\le \int_0^{+\infty} (p(s)|x(s)| + q(s)|x'(s)| + r(s)|x''(s)| + e(s))\mathrm{d}s \\ &\le P_0\|x\|_\infty + Q_0\|x'\|_\infty + R_0\|x''\|_\infty + E_0, \quad \forall t \in [0, +\infty). \end{aligned}$$

Thus

$$\begin{aligned} &\|x\|_\infty \le \tfrac{1}{2}(P_2\|x\|_\infty + Q_2\|x'\|_\infty + R_2\|x''\|_\infty + E_2), \\ &\|x'\|_\infty \le P_1\|x\|_\infty + Q_1\|x'\|_\infty + R_1\|x''\|_\infty + E_1, \\ &\|x''\|_\infty \le P_0\|x\|_\infty + Q_0\|x'\|_\infty + R_0\|x''\|_\infty + E_0. \end{aligned}$$

Consequently

$$\begin{aligned}\|x\| &\le P\|x\|_\infty + Q\|x'\|_\infty + R\|x''\|_\infty + E_0 + E_1 + \tfrac{1}{2}E_2 \\ &\le \max\{P,Q,R\}\,\|x\| + E_0 + E_1 + \tfrac{1}{2}E_2.\end{aligned}$$

Therefore

$$\|x\| \le \frac{E_0 + E_1 + \frac{1}{2}E_2}{1 - \max\{P,Q,R\}} =: M_1.$$

Case 2. $0 \le \alpha < 1$. In this case, by Lemma 2.2 we have

$$|G(t,s)| \le \frac{1}{2(1-\alpha)}s^2, \quad \forall t,s \in [0,+\infty).$$

From (3.2) and Lemma 2.1 it follows that

$$\begin{aligned}|x(t)| &\le \int_0^{+\infty} |G(t,s)||f(s,x(s),x'(s),x''(s))|\mathrm{d}s \\ &\le \int_0^{+\infty} \tfrac{1}{2(1-\alpha)}s^2(p(s)|x(s)| + q(s)|x'(s)| + r(s)|x''(s)| + e(s))\mathrm{d}s \\ &\le \tfrac{1}{2(1-\alpha)}(P_2\|x\|_\infty + Q_2\|x'\|_\infty + R_2\|x''\|_\infty + E_2), \quad \forall t \in [0,+\infty), \\ |x'(t)| &= \left|\lambda \int_t^{+\infty}(s-t)f(s,x(s),x'(s),x''(s))\mathrm{d}s\right| \\ &\le P_1\|x\|_\infty + Q_1\|x'\|_\infty + R_1\|x''\|_\infty + E_1, \quad \forall t \in [0,+\infty)\end{aligned}$$

and

$$\begin{aligned}|x''(t)| &= \left|-\lambda \int_t^{+\infty} f(s,x(s),x'(s),x''(s))\mathrm{d}s\right| \\ &\le P_0\|x\|_\infty + Q_0\|x'\|_\infty + R_0\|x''\|_\infty + E_0, \quad \forall t \in [0,+\infty).\end{aligned}$$

Thus

$$\|x\| \le \max\{P,Q,R\}\,\|x\| + E_0 + E_1 + \frac{1}{2(1-\alpha)}E_2,$$

and hence

$$\|x\| \le \frac{E_0 + E_1 + \frac{1}{2(1-\alpha)}E_2}{1 - \max\{P,Q,R\}} =: M_2.$$

Case 3. $\alpha > 1$. In this case, by Lemma 2.2 we have

$$|G(t,s)| \le \frac{\alpha}{2(\alpha-1)}s^2, \quad \forall t,s \in [0,+\infty).$$

From (3.2) and Lemma 2.1 it follows that

$$\begin{aligned}|x(t)| &\le \int_0^{+\infty} |G(t,s)||f(s,x(s),x'(s),x''(s))|\mathrm{d}s \\ &\le \int_0^{+\infty} \tfrac{\alpha}{2(\alpha-1)}s^2(p(s)|x(s)| + q(s)|x'(s)| + r(s)|x''(s)| + e(s))\mathrm{d}s \\ &\le \tfrac{\alpha}{2(\alpha-1)}(P_2\|x\|_\infty + Q_2\|x'\|_\infty + R_2\|x''\|_\infty + E_2), \quad \forall t \in [0,+\infty), \\ |x'(t)| &= \left|\lambda \int_t^{+\infty}(s-t)f(s,x(s),x'(s),x''(s))\mathrm{d}s\right| \\ &\le P_1\|x\|_\infty + Q_1\|x'\|_\infty + R_1\|x''\|_\infty + E_1, \quad \forall t \in [0,+\infty)\end{aligned}$$

and

$$\begin{aligned} |x''(t)| &= \left|-\lambda \int_t^{+\infty} f(s,x(s),x'(s),x''(s))\mathrm{d}s\right| \\ &\le P_0\|x\|_\infty + Q_0\|x'\|_\infty + R_0\|x''\|_\infty + E_0, \quad \forall t\in[0,+\infty). \end{aligned}$$

Thus

$$\|x\| \le \max\{P,Q,R\}\|x\| + E_0 + E_1 + \frac{\alpha}{2(\alpha-1)}E_2,$$

and hence

$$\|x\| \le \frac{E_0+E_1+\frac{\alpha}{2(\alpha-1)}E_2}{1-\max\{P,Q,R\}} =: M_3.$$

In summary, the operator equations (3.1) has a priori bound $M := M_1 + M_2 + M_3$ which is independent of $\lambda \in (0,1)$. Hence by Lemma 2.5(Leray-Schauder continuation theorem), BVP(1.1), (1.2) has at least one solution. This completes the proof of the theorem. □

Next, we give a result on the uniqueness of solutions for BVP(1.1),(1.2).

Theorem 3.2. *Assume that $f : [0,+\infty)\times\mathbb{R}^3 \to \mathbb{R}$ is S^2-Carathéodory function. Suppose also that there exist nonnegative functions $p(t), q(t), r(t) \in L^1[0,+\infty)$ with $t^ip(t), t^iq(t), t^ir(t) \in L^1[0,+\infty)(i = 1,2)$, such that*

(3.3) $|f(t,u_1,v_1,w_1)-f(t,u_2,v_2,w_2)| \le p(t)|u_1-u_2|+q(t)|v_1-v_2|+r(t)|w_1-w_2|$

for a.e. $t\in[0,+\infty)$ and all $(u_i,v_i,w_i)\in\mathbb{R}^3, i = 1,2$.

Then BVP(1.1), (1.2) has a unique solution, provided

$$\max\{P,Q,R\} < 1,$$

where P, Q, R as in Theorem 3.1.

Proof. We note that assumption (3.3) implies

$$|f(t,u,v,w)| \le p(t)|u| + q(t)|v| + r(t)|w| + |f(t,0,0,0)|$$

for a.e. $x\in[0,+\infty)$ and all $(u,v,w)\in\mathbb{R}^3$. Accordingly from Theorem 3.1, BVP (1.1),(1.2) has at least one solution.

Now, suppose that $x_1(t)$, $x_2(t)$ are two solutions of BVP(1.1),(1.2). Let $y(t) = x_1(t) - x_2(t)$. Then $y(t)$ satisfies the boundary condition (1.2) and

$$y'''(t) = f(t,x_1(t),x_1'(t),x_1''(t)) - f(t,x_2(t),x_2'(t),x_2''(t)), \quad 0 < t < +\infty.$$

Hence from Lemma 2.1 we have

$$y(t) = \int_0^{+\infty} G(t,s)[f(s, x_1(s), x_1'(s), x_1''(s)) - f(t, x_2(s), x_2'(s), x_2''(s))]\mathrm{d}s.$$

Similar to the proof of Theorem 3.1, we can show easily that

$$\|y\| \leq \max\{P, Q, R\} \|y\|,$$

that is

$$(1 - \max\{P, Q, R\})\|y\| \leq 0.$$

Since $\max\{P, Q, R\} < 1$, it follows that $\|y\| = 0$, and hence $y(t) \equiv 0$ on $[0, +\infty)$, i.e., $x_1(t) \equiv x_2(t)$ on $[0, +\infty)$. This completes the proof of the theorem. □

Finally, as an application, we give an example to demonstrate our results.

Example 3.1. *Consider the third-order boundary value problem*

$$x''' = \frac{1}{4}\min\{1, t^{-4}\}\frac{1}{1+x^2} + \frac{1}{3}\min\{t, t^{-5}\}\sqrt{(x')^2+1} + \frac{1}{2}e^{-2t}|x''|, \tag{3.4}$$

$$x(0) = \frac{1}{2}x(\eta), \quad \lim_{t\to+\infty} x'(t) = 0, \quad \lim_{t\to+\infty} x''(t) = 0, \tag{3.5}$$

where $0 < \eta < +\infty$.

Let

$$f(t,u,v,w) = \tfrac{1}{4}\min\{1, t^{-4}\}\tfrac{1}{1+u^2} + \tfrac{1}{3}\min\{t, t^{-5}\}\sqrt{v^2+1} + \tfrac{1}{2}e^{-2t}|w|,$$
$$p(t) = \tfrac{1}{4}\min\{1, t^{-4}\}, \quad q(t) = \tfrac{1}{3}\min\{t, t^{-5}\}, \quad r(t) = \tfrac{1}{2}e^{-2t}.$$

Then it is easy to check that $f : [0, +\infty) \times \mathbb{R}^3 \to \mathbb{R}$ is an $S^2 -$ Carathéodory function, and

$$|f(t, u_1, v_1, w_1) - f(t, u_2, v_2, w_2)| \leq p(t)|u_1 - u_2| + q(t)|v_1 - v_2| + r(t)|w_1 - w_2|$$

for all $t \in [0, +\infty)$ and all $(u_i, v_i, w_i) \in \mathbb{R}^3, i = 1, 2$. Meanwhile, obviously $t^i p(t)$, $t^i q(t)$, $t^i r(t) \in L^1[0, +\infty)$, $i = 0, 1, 2$.

It is easy to compute that

$$P_0 = \tfrac{1}{3}, \quad P_1 = \tfrac{1}{4}, \quad P_2 = \tfrac{1}{3},$$
$$Q_0 = \tfrac{1}{4}, \quad Q_1 = \tfrac{2}{9}, \quad Q_2 = \tfrac{1}{4},$$
$$R_0 = \tfrac{1}{4}, \quad R_1 = \tfrac{1}{8}, \quad R_2 = \tfrac{1}{8}.$$

It follows that

$$\begin{aligned}\max\{P,Q,R\} &= \max\{P_0+P_1+P_2, Q_0+Q_1+Q_2, R_0+R_1+R_2\}\\ &= \max\left\{\tfrac{11}{12},\tfrac{13}{18},\tfrac{1}{2}\right\} = \tfrac{11}{12} < 1.\end{aligned}$$

In summary, all conditions of Theorem 3.2 are satisfied for BVP(3.4),(3.5) and hence BVP(3.4),(3.5) has a unique solution $x = x(t)$.

Noticing that the Green's function corresponding to BVP(3.4),(3.5) satisfy

$$G(t,s) \geq G(0,s) = \begin{cases} \frac{1}{2}s^2 \geq 0, & s \leq \eta; \\ \eta s - \frac{1}{2}\eta^2 > 0, & s \geq \eta, \end{cases}$$

and

$$f(t,u,v,w) > 0, \quad \forall (t,u,v,w) \in [0,+\infty) \times \mathbb{R}^3.$$

It follows that

$$\begin{aligned} x(t) &= \int_0^{+\infty} G(t,s) f(s,x(s),x'(s),x''(s))\mathrm{d}s \\ &\geq \int_0^{+\infty} G(0,s) f(s,x(s),x'(s),x''(s))\mathrm{d}s > 0, \quad t \in [0,+\infty). \end{aligned}$$

Also since

$$x'(t) = \int_t^{+\infty} (s-t) f(s,x(s),x'(s),x''(s))\mathrm{d}s > 0, \quad t \in [0,+\infty),$$
$$x''(t) = -\int_t^{+\infty} f(s,x(s),x'(s),x''(s))\mathrm{d}s < 0, \quad t \in [0,+\infty),$$

we have that the unique solution $x = x(t)$ is strictly monotone increasing convex positive on $[0,+\infty)$, that is, BVP(3.4),(3.5) has a unique solution which is strictly monotone increasing, convex and positive on $[0,+\infty)$.

Reference

[1] R.P. Agarwal, D. O'Regan, *Infinite Interval Problems for Differential, Difference and Integral Equations*, Kluwer Academic, 2001.

[2] D.R. Anderson, J.M. Davis, *Multiple solutions and eigenvalues for third-order right focal boundary value problems*, J. Math. Anal. Appl., **267**(2002), 135–157.

[3] F. Bernis, L.A. Peletier, *Two problems from draining flows involving third-order ordinary differential equations*, SIAM J. Math. Anal., **27**(1996), 515–527.

[4] A. Cabada, M.R. Grossinho, F. Minhós, *Extremal solutions for third-order nonliner problems with upper and lower solutions in reversed order*, Nonlinear Anal., **62**(2005), 1109–1121.

[5] J. Chu, Z. Zhou, *Positive solutions for singular non-linear third-order periodic boundary value problems*, Nonlinear Anal., **86**(2006), 1528–1542.

[6] Z. Du, X. Lin, W. Ge, *On a third-order multi-point boundary value problem at resonance*, J. Math. Anal. Appl., **302**(2005), 217–229.

[7] W. Feng, J.R.L. Webb, *Solvability of three point boundary value problems at resonace*, Nonlinear Anal., **30**(1997), 3227–3238.

[8] Y. Feng, G. Qu, *A new comparison theorem and the solvability of a third-order two-point boundary value problem*, Bull. Malays. Math. Sci. Soc. (2), **34**(2011), 435–444.
[9] J.R. Graef, J.R.L. Webb, *Third order boundary value problems with nonlocal boundary conditions*, Nonlinear Anal., **71**(2009), 1542-1551.
[10] M. Greguš, *Third Order Linear Differential Equations, Mathematics and its Applications*, Reidel Publishing Co., Dordrecht, 1987.
[11] M.D.R. Grossinho, F.M. Minós. *Existence result for some third-order separated boundary value problems*, Nonlinear Anal., **47**(2001), 2407–2418.
[12] J. Henderson, *Best interval lengths for third order Lipschitz equations*, SIAM J. Math. Anal., **18**(1987), 293–305.
[13] J. Henderson, C.C. Tisdale, *Five-point boundary value problems for third-order differential equations by solution matching*, Math. Comput. Modelling, **42**(2005), 133–137.
[14] B. Hopkins, N. Kosmatov, *Third-order boundary value problems with sign-changing solutions*, Nonlinear Anal., **67**(2007), 126–137.
[15] Z. Hu, W. Liu, T. Chen, *Two-point boundary value problems for fractional differential equations at resonance*, Bull. Malays. Math. Sci. Soc. (2), accepted.
[16] G. Infante, P. Pietramala, *A third order BVP subject to nonlinear boundary conditions*, Math. Bohem., **135**(2010), 113–121.
[17] D. Jiang, R.P. Agarwal, *A uniqueness and existence theorem for a singular third-order boundary value problem on* $[0, +\infty)$, Appl. Math. Lett., **15**(2002), 445–451.
[18] G.S. Ladde, V. Lakshmikantham, A.S. Vatsala, Monotone Iterative Techniques for Nonlinear Differential Equations, Boston: Pitman Advanced Publishing, 1985.
[19] H. Lian, W. Ge, *Solvability for second-order three-point boundary value problems on a half-line*, Appl. Math. Lett., **19**(2006), 1000–1006.
[20] R. Ma, *Multiplicity results for a third-order boundary value problem at resonance*, Nonlinear Anal., **32**(1998), 493–499.
[21] F. M. Minhós, *Periodic solutions for a third order differential equation under conditions on the potential*, Port. Math.(N.S.), **55**(1998), 475–484.
[22] P.K. Palamides, R.P. Agarwal, *An existence theorem for a singular third-order boundary value problem on* $[0, +\infty)$, Appl. Math. Lett., **21**(2008), 1254–1259.
[23] M. Pei, S.K. Chang, *Existence and uniqueness of solutions for third-order nonlinear boundary value problems*, J. Math. Anal. Appl., **327**(2007), 23–35.
[24] William C. Troy, *Solutions of third-order differential equations relevant to draining and coating flows*, SIAM J. Math. Anal., **24**(1993), 155–171.
[25] E.O. Tuck, L.W. Schwartz, *A boundary value problem from draining and coating flows involving a third-order differential equation relevant to draining and coating flows*, SIAM Rev., **32**(1990), 453–469.
[26] Q. Yao, *Positive solutions of singular third-order three-point boundary value problems*, J. Math. Anal. Appl., **354**(2009), 207–212.
[27] P.J.Y. Wong, *Constant-sign solutions for a system of third-order generalized right focal problems*, Nonlinear Anal., **63**(2005), 2153–2163.