Analytic solutions of a second-order functional differential equation*

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Abstract. In this paper, we study the existence of analytic solutions of a second-order differential equation

$$\alpha z + \beta x'(z) + \gamma x''(z) = x(az + bx''(z)),$$

in the complex field $\mathbb{C}$, where $\alpha, \beta, \gamma, a, b$ are complex numbers. We discuss not only that the constant λ at resonance, i.e. at a root of the unity, but also those λ near resonance (near a root of the unity) under the Brjuno condition.

AMS(2000) subject classifications. 34K05, 39B22, 34A25

Keywords and phrases: Iterative differential equation, analytic solution, resonance, Brjuno condition.

1 Introduction

Functional differential equation of the form

$$x'(z) = f(z, x(z - \tau(z)))$$

have been studied in [1] and [6]. However, such equations, when the delay function $\tau(z)$ depends not only on the argument of the unknown function, but also state or state derivative, $\tau(z) = \tau(z, x(z), x'(z))$, have been relatively little researched. In [5], [7], [11]-[13] and [16]-[22], analytic solutions of the state dependent functional

*This work was partially supported by the Natural Science Foundation of Chongqing Normal University (Grant No. 12XLZ04)

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differential equations are found. In particular, under the Bruno condition, the quasi-periodic solutions and stability of a reversible and Hamiltonian systems has been discussed by Hanssmann and Si in [23]. In [9], [14]-[15], the authors studied the existence of analytic solutions of the equations with state derivative dependent delay

$$\alpha z + \beta x'(z) = x(az + bx'(z)),$$

$$x''(z) = x(az + bx'(z)),$$

and

$$\alpha z + \beta x'(z) = x(az + bx''(z)),$$

respectively.

For the inner function with second order derivative ([9]), to the best of our knowledge, there are little results about it. If there is no derivative for the inner function, we can transform the original equation into an iterative equation, which can be solved. If it is the inner function with first derivative, we can transform it into a new iterative equation with an integral. However, when it comes to the inner function with second derivative, there will be a double integral after transformation, which is complicated and worth our effort to deal with.

In this paper, we will deal with a more general functional differential equation of the form

$$\alpha z + \beta x'(z) + \gamma x''(z) = x(az + bx''(z)), \quad z \in \mathbb{C} \tag{1.1}$$

where we assume that $\alpha, \beta, \gamma, a, b$ are complex numbers and $\gamma \neq 0$. We will establish existence therorem of analytic solutions of Eq. (1.1) in the complex field.

2 Auxiliary solutions of the auxiliary equation

In this section, we discuss local invertible analytic solutions of Eq. (1.1) with $b \neq 0$. In order to construct analytic solutions of Eq. (1.1), we first let

$$y(z) = az + bx''(z). \tag{2.1}$$

Then for any numbers z_0, we have

$$x'(z) = x'(z_0) + \frac{1}{b}\int_{z_0}^{z}(y(s) - as)ds, \tag{2.2}$$

and

$$x(z) = x(z_0) + x'(z_0)(z - z_0) + \frac{1}{b}\int_{z_0}^{z}\int_{z_0}^{s}(y(t) - at)dtds, \tag{2.3}$$

so $x(y(z)) = x(z_0) + x'(z_0)(y(z) - z_0) + \frac{1}{b}\int_{z_0}^{y(z)}\int_{z_0}^{s}(y(t) - at)dtds$. Therefore, in view of Eq. (1.1), we have

$$\begin{aligned} & b\alpha z + b\beta x'(z_0) + \beta\int_{z_0}^{z}(y(s) - as)ds + \gamma(y(z) - az) \\ = \;& bx(z_0) + bx'(z_0)(y(z) - z_0) + \int_{z_0}^{y(z)}\int_{z_0}^{s}(y(t) - at)dtds. \end{aligned} \tag{2.4}$$

If z_0 is a fixed point of $y(z)$, we see that

$$\begin{aligned} x(z_0) &= \frac{\beta\gamma}{b} + (\gamma + b\alpha - a\gamma)\frac{z_0}{b} + \frac{(b\alpha - a\gamma)\beta + (1-a)\beta^2 z_0}{by'(z_0)}, \\ x'(z_0) &= \frac{\gamma}{b} + \frac{(b\alpha - a\gamma) + (1-a)\beta z_0}{by'(z_0)}. \end{aligned} \tag{2.5}$$

Furthermore, differentiating both sides of (2.4) with respect to z, we obtain

$$\beta(y'(z) - a)y'(z) - (\beta y(z) - a\beta z + b\alpha - a\gamma)y''(z) = (y(y(z)) - ay(z))(y'(z))^3. \tag{2.6}$$

To find analytic solution of (2.6), as in [14]-[15], we reduce Eq. (2.6) with $y(z) = g(\lambda g^{-1}(z))$, called the Schröder transformation sometimes (see [8]), to the auxiliary equation

$$\begin{aligned} & \beta\lambda\frac{(g'(\lambda z))^2}{(g'(z))^2} - a\beta\frac{g'(\lambda z)}{g'(z)} - \Big(\beta g(\lambda z) - a\beta g(z) + b\alpha - a\gamma\Big) \times \\ & \times\frac{\lambda g''(\lambda z)g'(z) - g'(\lambda z)g''(z)}{(g'(z))^3} \\ = \;& \lambda^2(g(\lambda^2 z) - ag(\lambda z))\frac{(g'(\lambda z))^3}{(g'(z))^3}, \end{aligned} \tag{2.7}$$

with the initial value conditions

$$g(0) = 0, \quad g'(0) = \eta \neq 0. \tag{2.8}$$

We will assume that λ in (2.7) satisfies one of the following conditions:

(C1) $0 < |\lambda| < 1$;

(C2) $\lambda = e^{2\pi i\theta}, \theta \in \mathbb{R}\backslash\mathbb{Q}$ and θ is a Brjuno number ([2], [10]): $B(\theta) = \sum_{n=0}^{\infty}\frac{\log q_{n+1}}{q_n} < \infty$, where $\{p_n/q_n\}$ denotes the sequence of partial fraction of the continued fraction expansion of θ;

(C3) $\lambda = e^{2\pi i q/p}$ for some integer $p \in \mathbb{N}$ with $p \geq 2$ and $q \in \mathbb{Z}\backslash\{0\}$, and $\lambda \neq e^{2\pi i\xi/v}$ for all $1 \leq v \leq p-1$ and $\xi \in \mathbb{Z}\backslash\{0\}$.

We observe that λ is inside the unit circle S^1 in case **(C1)** but on S^1 in the rest cases. More difficulties are encountered for λ on S^1 since the small divisor $\lambda^n - 1$

is involved in the latter (2.13). Under Diophantine condition: "$\lambda = e^{2\pi i\theta}$, *where* $\theta \in \mathbb{R}\backslash\mathbb{Q}$ *and there exist constants* $\zeta > 0$ *and* $\delta > 0$ *such that* $|\lambda^n - 1| \geq \zeta^{-1}n^{-\delta}$ *for all* $n \geq 1$," the number $\lambda \in S^1$ is "far" from all roots of the unity. Since then, we have been striving to give a result of analytic solutions for those λ "near" a root of the unity, i.e., neither being roots of the unity nor satisfying the Diophantine condition. The Brjuno condition in **(C2)** provides such a chance for us. Moreover, we also discuss the so-called the resonance case, i.e. the case of **(C3)**.

Theorem 2.1 *Suppose* **(C1)** *holds and* $\mu = a\gamma - b\alpha \neq 0$, *then for any* $\eta \in \mathbb{C}\backslash\{0\}$, *Eq.* (2.7) *in a neighborhood of the origin has an analytic solution of the form*

$$g(z) = \eta z + \sum_{n=2}^{\infty} g_n z^n. \tag{2.9}$$

Proof. As in [9], we rewrite (2.7) in the form

$$\left(\frac{(\beta g(\lambda z) - a\beta g(z) + b\alpha - a\gamma)g'(z)}{g'(\lambda z)}\right)' = \lambda^2\Big(g(\lambda^2 z) - ag(\lambda z)\Big)g'(\lambda z). \tag{2.10}$$

When $g'(0) = \eta \neq 0$, Eq. (2.10) is reduced equivalent to the integro-differential equation

$$\begin{aligned} & \mu\Big(g'(\lambda z) - g'(z)\Big) \\ = \ & \lambda^2 g'(\lambda z)\int_0^z \Big(g(\lambda^2 s) - ag(\lambda s)\Big)g'(\lambda s)ds - \beta\Big(g(\lambda z) - ag(z)\Big)g'(z). \end{aligned} \tag{2.11}$$

We now seek a solution of (2.7) in the form a power series (2.9). Substituting (2.9) into (2.11), since $\mu \neq 0$ after comparing coefficients, we obtain

$$\mu(\lambda^0 - 1)g_1 = 0, \tag{2.12}$$

$$\begin{aligned} & \mu(n+1)(\lambda^n - 1)g_{n+1} \\ = \ & \sum_{j=0}^{n-2}\sum_{k=0}^{n-j-2}\frac{(j+1)(k+1)}{n-j}\lambda^{j+k+2}(\lambda^{2(n-j-k-1)} - a\lambda^{n-j-k-1})g_{j+1}g_{k+1}g_{n-j-k-1} \\ & -\sum_{k=0}^{n-1}(k+1)\beta(\lambda^{n-k} - a)g_{k+1}g_{n-k}, \qquad n \geq 1. \end{aligned} \tag{2.13}$$

Then for arbitrarily chosen $g_1 = \eta \neq 0$, the sequence $\{g_n\}_{n=2}^{\infty}$ is successively determined by (2.13) in a unique manner. Now we show that the power series (2.9) converges in a neighborhood of the origin. Since $0 < |\lambda| < 1$, we have

$$\lim_{n\to\infty} \frac{1}{|\lambda^n - 1|} = 1,$$

there exists $L > 0$ such that $\frac{1}{|\lambda^n-1|} \leq L, \quad \forall n \geq 1$. It follows from (2.13) that

$$|g_{n+1}| \leq \frac{L(1+|a|)}{|\mu|}\Big(\sum_{j=0}^{n-2}\sum_{k=0}^{n-j-2} |g_{j+1}||g_{k+1}||g_{n-j-k-1}| + |\beta|\sum_{k=0}^{n-1} |g_{k+1}||g_{n-k}|\Big) \tag{2.14}$$

for $n \geq 1$.

We consider the implicit function equation

$$G(z) = |\eta|z + \frac{L(1+|a|)}{|\mu|}[|\beta|G^2(z) + G^3(z)].$$

Define the function

$$\Theta(z, \omega; L, a, \mu, \eta, \beta) = |\eta|z - \omega + \frac{L(1+|a|)}{|\mu|}(|\beta|\omega^2 + \omega^3) \tag{2.15}$$

for (z, ω) from a neighborhood of $(0, 0)$, then the function $G(z)$ satisfies

$$\Theta(z, G(z); L, a, \mu, \eta, \beta) = 0. \tag{2.16}$$

In view of $\Theta(0, 0; L, a, \mu, \eta, \beta) = 0$,

$$\Theta'_\omega(0, 0; L, a, \mu, \eta, \beta) = -1 \neq 0,$$

and the implicit function theorem, there exists a unique function $\Phi(z)$, analytic in a neighborhood of zero, such that

$$\Phi(0) = 0, \quad \Phi'(0) = -\frac{\Theta'_z(0, 0; L, a, \mu, \eta, \beta)}{\Theta'_\omega(0, 0; L, a, \mu, \eta, \beta)} = |\eta|,$$

and $\Theta(z, \Phi(z); L, a, \mu, \eta, \beta) = 0$. According to (2.16), we have $G(z) = \Phi(z)$.

If we assume that the power series expansion of $G(z)$ is as follows

$$G(z) = \sum_{n=1}^{\infty} G_n z^n, \quad G_1 = |\eta|, \tag{2.17}$$

substituting the series in (2.16) we have

$$\sum_{n=0}^{\infty} G_{n+1} z^{n+1} = |\eta| z + \frac{L}{|\mu|}(1+|a|)\Big[\sum_{n=0}^{\infty}\Big(\sum_{j=0}^{n}\sum_{k=0}^{n-j} G_{j+1}G_{k+1}G_{n-j-k+1}\Big) z^{n+3} + |\beta| \sum_{n=0}^{\infty}\Big(\sum_{k=0}^{n} G_{k+1}G_{n-k+1}\Big) z^{n+2}\Big].$$

Comparing coefficients , we obtain $G_1 = |\eta|$ and

$$G_{n+1} = \frac{L}{|\mu|}(1+|a|)\Big(\sum_{j=0}^{n-2}\sum_{k=0}^{n-j-2} G_{j+1}G_{k+1}G_{n-j-k-1} + |\beta| \sum_{k=0}^{n-1} G_{k+1}G_{n-k}\Big), \qquad n \geq 1, \tag{2.18}$$

from (2.14) we obtain immediately that $|g_{n+1}| \leq G_{n+1}$ for all n by induction. This implies that (2.9) converges in a neighborhood of the origin. This completes the proof. □

Next we devote to the existence of analytic solutions of Eq. (2.7) under the Brjuno condition. First, we recall briefly the definition of Brjuno numbers and some basic facts. As stated in [3], for a real number θ, we let $[\theta]$ denote its integer part and $\{\theta\} = \theta - [\theta]$ its fractional part. Then every irrational number θ has a unique expression of the Gauss' continued fraction

$$\theta = d_0 + \theta_0 = d_0 + \frac{1}{d_1 + \theta_1} = \cdots,$$

denoted simply by $\theta = [d_0, d_1, \cdots, d_n, \cdots]$, where d_j's and θ_j's are calculated by the algorithm: (**a**) $d_0 = [\theta]$, $\theta_0 = \{\theta\}$, and (**b**) $d_n = \left[\frac{1}{\theta_{n-1}}\right]$, $\theta_n = \left\{\frac{1}{\theta_{n-1}}\right\}$ for all $n \geq 1$. Define the sequences $(p_n)_{n\in\mathbb{N}}$ and $(q_n)_{n\in\mathbb{N}}$ as follows:

$$q_{-2} = 1, q_{-1} = 0, q_n = d_n q_{n-1} + q_{n-2}$$

$$p_{-2} = 0, p_{-1} = 1, p_n = d_n p_{n-1} + p_{n-2}.$$

It is easy to show that $p_n/q_n = [d_0, d_1, \cdots, d_n]$. Thus, for every $\theta \in \mathbb{R}\backslash\mathbb{Q}$ we associate, using its convergence, an arithmetical function $B(\theta) = \sum_{n\geq 0} \frac{\log q_{n+1}}{q_n}$. We say that θ is a Brjuno number or that it satisfies Brjuno condition if $B(\theta) < +\infty$. The Brjuno condition is weaker than the Diophantine condition. For example, if $d_{n+1} \leq ce^{d_n}$ for all $n \geq 0$, where $c > 0$ is a constant, then $\theta = [d_0, d_1, \cdots, d_n, \cdots]$ is a Brjuno number but is not a Diophantine number. So, the case **(C2)** contains both Diophantine condition and a part of α "near" resonance. Let $\theta \in \mathbb{R}\backslash\mathbb{Q}$ and $(q_n)_{n\in\mathbb{N}}$ be the sequence of partial denominators of the Gauss's continued fraction for θ. As in [3], let

$$A_k = \{n \geq 0 | \|n\theta\| \leq \frac{1}{8q_k}\}, \qquad E_k = \max(q_k, \frac{q_{k+1}}{4}), \qquad \eta_k = \frac{q_k}{E_k}.$$

Let A_k^* be the set of integers $j \geq 0$ such that either $j \in A_k$ or for some j_1 and j_2 in A_k, with $j_2 - j_1 < E_k$, one has $j_1 < j < j_2$ and q_k divides $j - j_1$. For any integer $n \geq 0$, define

$$l_k(n) = \max\left((1+\eta_k)\frac{n}{q_k} - 2, \quad (m_n\eta_k + n)\frac{1}{q_k} - 1\right),$$

where $m_n = \max\{j | 0 \leq j \leq n, j \in A_k^*\}$. We then define function $h_k : \mathbb{N} \to \mathbb{R}_+$ as follows:

$$h_k(n) = \begin{cases} \frac{m_n + \eta_k n}{q_k} - 1, & \text{if} \quad m_n + q_k \in A_k^*, \\ l_k(n), & \text{if} \quad m_n + q_k \notin A_k^*. \end{cases}$$

Let $g_k(n) := \max\left(h_k(n), \left[\frac{n}{q_k}\right]\right)$, and define $k(n)$ by the condition $q_{k(n)} \leq n \leq q_{k(n)+1}$. Clearly, $k(n)$ is non-decreasing. Then we are able to state the following result.

Lemma 2.1 (Davie's lemma [4]) *Let* $K(n) = n \log 2 + \sum_{k=0}^{k(n)} g_k(n) \log(2q_{k+1})$. *Then*

(a) *there is a universal constant* $\varrho > 0$ *(independent of* n *and* θ*) such that*

$$K(n) \leq n\left(\sum_{k=0}^{k(n)} \frac{\log q_{k+1}}{q_k} + \varrho\right),$$

(b) $K(n_1) + K(n_2) \leq K(n_1 + n_2)$ *for all* n_1 *and* n_2*, and*

(c) $-\log|\lambda^n - 1| \leq K(n) - K(n-1)$.

Theorem 2.2 *Suppose* **(C2)** *holds and* $\mu \neq 0$*, then for any* $\eta \in \mathbb{C}\backslash\{0\}$*, Eq.* (2.7) *has an analytic solution of the form* (2.9) *in a neighborhood of the origin.*

Proof. As in the Theorem 2.1, we seek a power series solution of the form (2.9). Then (2.13) again holds. From (2.13), we have

$$\begin{aligned} &|g_{n+1}| \\ \leq\ &\frac{(1+|a|)}{|\lambda^n - 1||\mu|}\Big(\sum_{j=0}^{n-2}\sum_{k=0}^{n-j-2} |g_{j+1}||g_{k+1}||g_{n-j-k-1}| + |\beta|\sum_{k=0}^{n-1}|g_{k+1}||g_{n-k}|\Big), n \geq 1 \end{aligned} \tag{2.19}$$

To construct a majorant series of (2.9), we consider the implicit functional equation

$$\Theta(z, \psi; L, a, \mu, \eta, \beta) = 0, \tag{2.20}$$

where Θ is defined in (2.15) and $L = 1$. Similarly to the proof of Theorem 2.1, using the implicit function theorem we can prove that (2.20) has a unique analytic solution $\psi(z)$ in a neighborhood of the origin such that $\psi(0) = 0$ and $\psi'(0) = |\eta|$. Thus $\psi(z)$ in (2.20) can be expanded into a convergent series

$$\psi(z) = \sum_{n=1}^{\infty} B_n z^n, \tag{2.21}$$

in a neighborhood of the origin. Replacing (2.21) into (2.20) and comparing coefficients, we obtain that $B_1 = |\eta|$ and

$$\begin{aligned} B_{n+1} &= \frac{1}{|\mu|}(1+|a|)\Big(\sum_{j=0}^{n-2}\sum_{k=0}^{n-j-2} B_{j+1}B_{k+1}B_{n-j-k-1} \\ &\quad +|\beta|\sum_{k=0}^{n-1} B_{k+1}B_{n-k}\Big), \qquad n \geq 1. \end{aligned} \tag{2.22}$$

Note that the series (2.21) converges in a neighborhood of the origin. Now, we can deduce, by induction, that $|g_n| \leq B_n e^{K(n-1)}$ for $n \geq 1$, where $K : \mathbb{N} \to \mathbb{R}$ is defined in Lemma 2.1.

In fact, $|g_1| = |\eta| = B_1$. For inductive proof we assume that $|g_j| \leq B_j e^{K(j-1)}$, for $j = 1, 2, \ldots, n$. From (2.19) we know

$$\begin{aligned} |g_{n+1}| &\leq \frac{(1+|a|)}{|1-\lambda^n||\mu|}\Big(\sum_{j=0}^{n-2}\sum_{k=0}^{n-j-2} |g_{j+1}||g_{k+1}||g_{n-j-k-1}| + |\beta|\sum_{k=0}^{n-1}|g_{k+1}||g_{n-k}|\Big) \\ &\leq \frac{(1+|a|)}{|1-\lambda^n||\mu|}\Big(\sum_{j=0}^{n-2}\sum_{k=0}^{n-j-2} B_{j+1}B_{k+1}B_{n-j-k-1}e^{K(j)}e^{K(k)}e^{K(n-j-k-2)} \\ &\quad +|\beta|\sum_{k=0}^{n-1} B_{k+1}B_{n-k}e^{K(k)}e^{K(n-k-1)}\Big), \qquad n \geq 1. \end{aligned}$$

Note that

$$\begin{gathered} K(j) + K(k) + K(n-j-k-2) \leq K(n-1), \\ K(k) + K(n-k-1) \leq K(n-1). \end{gathered}$$

Then from Lemma 2.1 we have

$$\begin{aligned} |g_{n+1}| &\leq \frac{(1+|a|)e^{K(n-1)}}{|1-\lambda^n||\mu|}\Big(\sum_{j=0}^{n-2}\sum_{k=0}^{n-j-2} B_{j+1}B_{k+1}B_{n-j-k-1} + |\beta|\sum_{k=0}^{n-1} B_{k+1}B_{n-k}\Big) \\ &\leq B_{n+1}e^{K(n)}, \qquad n \geq 1. \end{aligned}$$

Since $\sum_{n=1}^{\infty} B_n z^n$ is convergent in a neighborhood of the origin, there exists a constant $\Lambda > 0$ such that

$$B_n < \Lambda^n, \quad n \geq 1.$$

Moreover, from Lemma 2.1, we know that $K(n) \leq n(B(\theta)+\varrho)$ for some universal constant $\varrho > 0$. Then

$$|g_n| \leq B_n e^{K(n-1)} \leq \Lambda^n e^{(n-1)(B(\theta)+\varrho)},$$

that is,

$$\lim_{n\to\infty} \sup(|g_n|)^{\frac{1}{n}} \leq \lim_{n\to\infty} \sup(\Lambda^n e^{(n-1)(B(\theta)+\varrho)})^{\frac{1}{n}} = \Lambda e^{B(\theta)+\varrho}.$$

This implies that the convergence radius of (2.9) is at least $(\Lambda e^{B(\theta)+\varrho})^{-1}$. This completes the proof. □

In the case **(C3)** both the Diophantine condition and Brjuno condition are not satisfied. We need define a sequence $\{C_n\}_{n=1}^{\infty}$ by $C_1 = |\eta|$ and

$$C_{n+1} = \frac{(1+|a|)\Gamma}{|\mu|}\Big(\sum_{j=0}^{n-2}\sum_{k=0}^{n-j-2} C_{j+1}C_{k+1}C_{n-j-k-1} + |\beta|\sum_{k=0}^{n-1} C_{k+1}C_{n-k}\Big), n \geq 1, \tag{2.23}$$

where $\Gamma := \max\left\{1, \frac{1}{|1-\lambda|}, \frac{1}{|1-\lambda^2|}, \cdots, \frac{1}{|1-\lambda^{(p-1)}|}\right\}$, and p is defined in **(C3)**.

Theorem 2.3 *Assume that* **(C3)** *holds and* $\mu \neq 0$. *Let* $\{g_n\}_{n=0}^{\infty}$ *be determined by* $g_1 = \eta$ *and*

$$\mu(n+1)(\lambda^n - 1)g_{n+1} = \Xi(n,\lambda), \qquad n \geq 1, \tag{2.24}$$

where

$$\Xi(n,\lambda) = \sum_{j=0}^{n-2}\sum_{k=0}^{n-j-2} \frac{(j+1)(k+1)}{n-j}\lambda^{j+k+2}(\lambda^{2(n-j-k-1)} - a\lambda^{n-j-k-1})g_{j+1}g_{k+1}g_{n-j-k-1} - \sum_{k=0}^{n-1}(k+1)\beta(\lambda^{n-k} - a)g_{k+1}g_{n-k}.$$

If $\Xi(lp,\lambda) = 0$ *for all* $l = 1, 2, \cdots$, *then Eq.* (2.7) *has an analytic solution of the form*

$$g(z) = \eta z + \sum_{n=lp+1, l\in\mathbb{N}} \mu_{lp+1} z^n + \sum_{n\neq lp+1, l\in\mathbb{N}} g_n z^n, \quad \mathbb{N} = \{1, 2, 3, \cdots\}$$

in a neighborhood of the origin, where all μ_{lp+1}*'s are arbitrary constants satisfying the inequality* $|\mu_{lp+1}| \leq C_{lp+1}$ *and the sequence* $\{C_n\}_{n=1}^{\infty}$ *is defined in* (2.23). *Otherwise, if* $\Xi(lp,\alpha) \neq 0$ *for some* $l = 1, 2, \cdots$, *then Eq.* (2.7) *has no analytic solutions in any neighborhood of the origin.*

Proof. Analogously to the proof of Theorem 2.1, let (2.9) be the expansion of a formal solution $g(z)$ of Eq. (2.7), we also have (2.13) or (2.24). If $\Xi(lp,\lambda) \neq 0$ for some natural number l, then the equality in (2.24) does not hold for $n = lp$ since $\lambda^{lp} - 1 = 0$. In such a circumstance Eq. (2.7) has no formal solutions.

If $\Xi(lp,\lambda) = 0$ for all natural numbers l, then there are infinitely many choices of corresponding g_{lp+1} in (2.24) and the formal solutions (2.9) form a family of functions of infinitely many parameters. We can arbitrarily choose $g_{lp+1} = \mu_{lp+1}$ such that $|\mu_{lp+1}| \leq C_{lp+1}, l = 1, 2, \cdots$. In what follows we prove that the formal solution (2.9) converges in a neighborhood of the origin. First of all, note that $|\lambda^n - 1|^{-1} \leq \Gamma$. for $n \neq lp$. It follows from (2.24) that

$$|g_{n+1}| \leq \frac{(1+|a|)\Gamma}{|\mu|}\Big(\sum_{j=0}^{n-2}\sum_{k=0}^{n-j-2}|g_{j+1}||g_{k+1}||g_{n-j-k-1}| + |\beta|\sum_{k=0}^{n-1}|g_{k+1}||g_{n-k}|\Big), \tag{2.25}$$

for all $n \neq lp, l = 1, 2, \cdots$. Further, we can prove that

$$|g_n| \leq C_n, \quad n = 1, 2, \cdots. \tag{2.26}$$

In fact, for inductive proof we assume that $|g_r| \leq C_r$ for all $1 \leq r \leq n$. When $n = lp$, we have $|g_{n+1}| = |\mu_{n+1}| \leq C_{n+1}$. On the other hand, when $n \neq lp$, from (2.26) we get

$$\begin{aligned} |g_{n+1}| &\leq \frac{(1+|a|)\Gamma}{|\mu|}\Big(\sum_{j=0}^{n-2}\sum_{k=0}^{n-j-2}C_{j+1}C_{k+1}C_{n-j-k-1} + |\beta|\sum_{k=0}^{n-1}C_{k+1}C_{n-k}\Big) \\ &= C_{n+1} \end{aligned}$$

as desired. Set

$$F(z) = \sum_{n=1}^{\infty} C_n z^n, \quad C_1 = |\eta|. \tag{2.27}$$

It is easy to check that (2.27) satisfies

$$\Theta(z, \omega; \Gamma, a, \mu, \eta, \beta) = 0, \tag{2.28}$$

where the function Θ is defined in (2.15). Moreover, similarly to the proof of Theorem 2.1, we can prove that (2.28) has a unique analytic solution $F(z)$ in a neighborhood of the origin such that $F(0) = 0$ and $F'(0) = |\eta| \neq 0$. Thus (2.27) converges in a neighborhood of the origin. By the convergence of (2.27) and inequality (2.26), the series (2.9) converges in a neighborhood of the origin. This completes the proof. □

3 Analytic solutions of equation (1.1)

Theorem 3.1 *Suppose that conditions of either Theorem* 2.1, 2.2 *or* 2.3 *are fulfilled. Then Eq.* (2.6) *has an invertible analytic solution of the form*

$$y(z) = g(\lambda g^{-1}(z))$$

in a neighborhood of the origin, where $g(z)$ *is an analytic solutions of* (2.7) *satisfying the initial conditions* (2.8)*.*

Proof. In a view of Theorems 2.1-2.3, we may find an analytic solution $g(z)$ of the auxiliary equation (2.7) in the form of (2.9) such that $g(0) = 0$ and $g'(0) = \eta \neq 0$. Clearly the inverse $g^{-1}(z)$ exists and is analytic in a neighborhood of the $g(0) = 0$. Define

$$y(z) := g(\lambda g^{-1}(z)). \tag{3.1}$$

Then $y(z)$ is invertible analytic in a neighborhood of $z = 0$. From (3.1) it is easy to see

$$\begin{aligned} y(0) &= g(\lambda g^{-1}(0)) = g(0) = 0, \\ y'(0) &= \lambda g'(\lambda g^{-1}(0))(g^{-1})'(0) = \frac{\lambda g'(\lambda g^{-1}(0))}{g'(g^{-1}(0))} = \frac{\lambda g'(0)}{g'(0)} = \lambda \neq 0, \end{aligned}$$

and

$$y'(z) = \frac{\lambda g'(\lambda g^{-1}(z))}{g'(g^{-1}(z))}, \tag{3.2}$$

$$y''(z) = \frac{\lambda^2 g''(\lambda g^{-1}(z))g'(g^{-1}(z)) - \lambda g'(\lambda g^{-1}(z))g''(g^{-1}(z))}{(g'(g^{-1}(z))^3}, \tag{3.3}$$

from (3.2), (3.3) and (2.7), we have

$$\begin{aligned} &\beta(y'(z) - a)y'(z) - (\beta y(z) - a\beta z + b\alpha - a\gamma)y''(z) \\ =\ & \lambda^2 \beta \frac{(g'(\lambda g^{-1}(z)))^2}{(g'(g^{-1}(z)))^2} - a\lambda\beta \frac{g'(\lambda g^{-1}(z))}{g'(g^{-1}(z))} \\ & -\lambda\Big(\beta g(\lambda g^{-1}(z)) - a\beta z + b\alpha - a\gamma\Big)\frac{\lambda g''(\lambda g^{-1}(z))g'(g^{-1}(z)) - g'(\lambda g^{-1}(z))g''(g^{-1}(z))}{(g'(g^{-1}(z)))^3} \\ =\ & \lambda^3\Big(g(\lambda^2 g^{-1}(z)) - ag(\lambda g^{-1}(z))\Big)\frac{(g'(\lambda g^{-1}(z)))^3}{(g'(g^{-1}(z)))^3} \\ =\ & (y(y(z)) - ay(z))(y'(z))^3 \end{aligned}$$

as required. This completes the proof. □

By the Theorems 3.1 , we have shown that under the conditions of Theorems 2.1, 2.2 or 2.3, Eq. (2.6) has an analytic solution $y(z) = g(\lambda g^{-1}(z))$ in a neighborhood of the number 0, where $g(z)$ is an analytic solution of (2.7). Since the function $g(z)$ in (2.9) can be determined by (2.13), it is possible to calculate, at least in theory, the explicit form of $y(z)$, an analytic solution of (1.1), in a neighborhood of the fixed point 0 of $y(z)$ by means of (2.2) and (2.5). However, knowing that an analytic solution of (1.1) exists, we can take an alternative route as follows.

Assume that $\alpha, \beta, \gamma, a, b$ are unequal to 0 and $x(z)$ is of the form

$$x(z) = x(0) + x'(0)z + \frac{x''(0)}{2!}z^2 + \cdots, \tag{3.4}$$

we need to determine the derivatives $x^{(n)}(0), n = 0, 1, 2, \cdots$. First of all, in view of (2.5), we have

$$x(0) = \frac{\beta}{b\eta}(\eta\gamma + b\alpha - a\gamma) = \frac{\beta}{b\eta}(\eta\gamma - \mu)$$

and

$$x'(0) = \frac{1}{b\eta}(\eta\gamma + b\alpha - a\gamma) = \frac{1}{b\eta}(\eta\gamma - \mu).$$

Moreover, from (2.1) we have

$$x''(0) = 0.$$

Next by calculating the derivatives of both sides of (1.1) respectively, we obtain

$$\alpha + \beta x''(z) + \gamma x'''(z) = x'(az + bx''(z))(a + bx'''(z)),$$

$$\beta x'''(z) + \gamma x^{(4)}(z) = x''(az + bx''(z))(a + bx'''(z))^2 + bx'(az + bx''(z))x^{(4)}(z),$$

$$\cdots \qquad \cdots.$$

Thus,

$$x'''(0) = \frac{ax'(0) - \alpha}{\gamma - bx'(0)} = \frac{\eta - a}{b},$$

$$x^{(4)}(0) = \frac{\beta x'''(0)}{bx'(0) - \gamma} = \frac{(a - \eta)\beta\eta}{b\mu},$$

$$\cdots \qquad \cdots.$$

Then, the explicit from of solution $x(z)$ is:

$$x(z) = \frac{\beta}{b\eta}(\eta\gamma - \mu) + \frac{1}{b\eta}(\eta\gamma - \mu)z + \frac{\eta - a}{6b}z^3 + \frac{(a - \eta)\beta\eta}{24b\mu}z^4 + \cdots.$$

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