Multipliers of Commutative $F-$Algebras of Continuous Vector-valued Functions

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Abstract. Characterizations of multipliers on algebras of continuous functions with values in a commutative Banach or C^*-algebra A have been obtained by several authors. In this paper, we investigate the extent to which these characterizations can be made beyond Banach algebras. We shall focus mainly on the algebras of continuous functions with values in an $F-$algebra A (not necessarily locally convex), in particular in a complete $p-$normed algebra, $0 < p \leq 1$, having a minimal approximate identity. We include a few examples related to our results. Most of our initial results remain valid without the commutativity of A.

Keywords. multiplier, $F-$algebra, minimal approximate identity, algebra of continuous vector-valued functions.

AMS Subject Classification (2000): Primary 43A22; Secondary 46A16, 46E40, 47B38.

1 Introduction

The general notion of multipliers on a commutative Banach algebra A (into itself) was first systematically studied by Wang [24] in 1961. In the non-commutative setting, Johnson [11] in 1964 introduced the notions of left multipliers, right multipliers, multipliers, and double multipliers on semigroups, rings, algebras, Banach algebras and topological algebras. Since then the theory of multipliers has been extensively studied in various settings (see, e.g., [3, 7, 10, 16, 20].) We mention that the essential ingredients of arguments used by Wang [24] and later authors is the assumption of an approximate identity in the algebra A or a kind of Cohn factorization in A.

Wang [24] showed that if A is taken as the commutative C^*-algebra $C_o(X)$ with X a locally compact Hausdorff space, then $M(C_o(X)) \cong C_b(X)$. A similar result for a topological algebra A has been obtained by Husain [10] in the following form: "Let A be a faithful semisimple commutative topological algebra and $\Delta(A)$ the set of non-zero continuous multiplicative linear functionals on A endowed with the w^*-topology. If $\Delta(A)$ is non-empty, then each $T \in M(A)$ can be represented by a continuous map $h_T \in C(\Delta(A))$." The above result of Wang has also been generalized to vector-valued functions by several authors (see, e.g. [3, 8, 18, 19]). For instance, Lai [19] showed that if X is a locally compact abelian group and A is a commutative Banach algebra with a bounded approximate identity, then $M(C_o(X,A)) \cong C_b(X, M(A)_u)$.

In this paper, we investigate the extent to which the above study of multipliers for vector-valued functions can be made beyond Banach algebras. We shall focus mainly on the class of $F-$algebras, in particular on complete $p-$normed algebras, $0 < p \leq 1$, having a minimal approximate identity. We mention that the arguments of above authors relied heavily on the fact that, in the case of A a Banach algebra, $C_o(X,A)$ is isometrically isomorphic to the completed tensor product $C_o(X) \otimes_\lambda A$ with respect to the smallest cross norm λ (see [3, 18, 19]). We shall avoid the use of this technique as it need not work in our case. In fact, when A is not locally convex, $\otimes_\lambda$ is no longer appropriate; even for A a complete $p-$normed space, many complications arise (see [21], Section 10.4; [23], p. 100).

For a background, the reader is referred to [20] for multipliers on Banach algebras and to [10, 11, 16] for multipliers on topological algebras

2 Preliminaries

In this section, we give basic definitions and study various classes of topological algebras considered in this paper.

Definition 1 *[17, 22] Let E be a vector space over the field $\mathbb{K}$ $(= \mathbb{R}$ or $\mathbb{C})$.*

(1) A function $q : E \to \mathbb{R}$ is called an $F-$seminorm on E if it satisfies

(F_1) $q(x) \geq 0$ for all $x \in E$;

(F_2) $q(x) = 0$ if $x = 0$;

(F_3) $q(\alpha x) \leq q(x)$ for all $x \in E$ and $\alpha \in \mathbb{K}$ with $|\alpha| \leq 1$;

(F_4) $q(x+y) \leq q(x) + q(y)$ for all $x, y \in E$;

(F_5) if $\alpha_n \to 0$ in $\mathbb{K}$, then $q(\alpha_n x) \to 0$ for all $x \in E$.

(2) An $F-$seminorm q on E is called an $F-$norm if, for any $x \in E$, $q(x) = 0$ implies $x = 0$.

(3) An $F-$seminorm (or $F-$norm) q on E is called $p-$homogeneous, where $0 < p \leq 1$, if it also satisfies

(F_3') $q(\alpha x) = |\alpha|^p \, q(x)$ for all $x \in E$ and $\alpha \in \mathbb{K}$.

(4) A $p-$homogeneous $F-$seminorm (resp. $F-$norm) on E is called, in short, a $p-$seminorm (resp. $p-$norm).

Definition 2 *(1) A vector space with an $F-$norm q is called an $F-$normed space and is denoted by (E,q); if it is also complete, it is called an $F-$space. Clearly, any $F-$normed space (E,q) is a metrizable TVS with metric given by*

$$d(x,y) = q(x-y), \quad x, y \in E.$$

(2) An $F-$seminorm (or $F-$norm) q on an algebra A is called submultiplicative if

$$q(xy) \leq q(x)q(y) \text{ for all } x, y \in A.$$

An algebra with a submultiplicative $F-$norm q is called an $F-$normed algebra; if it is also complete it is called an $F-$algebra. An algebra with a submultiplicative $p-$norm q is called a $p-$normed algebra.

For the general theory and undefined terms, the reader is referred to [17, 22] for $F-$normed and $p-$normed spaces, and to ([21]; [26], p. 32–35) for various classes of topological algebras.

If E and F are topological vector spaces over the field $\mathbb{K}$ ($= \mathbb{R}$ or $\mathbb{C}$), then the set of all continuous linear mappings $T : E \to F$ is denoted by $CL(E, F)$. Clearly, $CL(E, F)$ is a vector space over $\mathbb{K}$ with the usual pointwise operations. Further, if $F = E$, $CL(E) = CL(E, E)$ is an algebra under composition (i.e. $(ST)(x) = S(T(x)), x \in E$) and has the identity $I : E \to E$ given by $I(x) = x$ ($x \in E$).

Definition 3 *(1) A net $\{e_\lambda : \lambda \in I\}$ in a topological algebra A is called an approximate identity, if*

$$\lim_{\lambda} e_\lambda a = \lim_{\lambda} a e_\lambda \quad \textit{for all } a \in A.$$

(2) An approximate identity $\{e_\lambda : \lambda \in I\}$ in an $F-$normed algebra (A, q) is said to be minimal if $q(e_\lambda) \leq 1$ for all $\lambda \in I$.

(3) An algebra A is said to be left (resp. right) faithful if, for any $a \in A$, $aA = \{0\}$ (resp. $Aa = \{0\}$) implies that $a = 0$; A is called faithful if it is both left and right faithful.

We mention that A is faithful in each of the following cases:

(i) A is a topological algebra with an approximate identity (e.g., A is a locally C^*-algebra).

(ii) A is a topological algebra with an orthogonal basis [10].

Definition 4 *A topological algebra A is called:*

(1) factorable if, for each $a \in A$, there exist $b, c \in A$ such that $a = bc$;

(2) strongly factorable if, for any sequence $\{a_n\}$ in A with $a_n \to 0$, there exist $a \in A$ and a sequence $\{b_n\}$ (resp. $\{c_n\}$) in A with $b_n \to 0$ (resp. $c_n \to 0$) such that $a_n = ab_n$ (resp. $a_n = c_n a$) for all $n \geq 1$.

Clearly, every strongly factorable algebra is factorable. The two notions coincide on a Banach algebra A having a bounded approximate identity ([5], Corollary 12, p. 61). Following Ansari-Piri [4], a TVS X is called *fundamental,* if there exists a constant $M > 1$ such that, for every sequence $\{x_n\}$ in X, the convergence of $M^n(x_{n+1} - x_n)$ to 0 in X implies that $\{x_n\}$ is a Cauchy sequence in X. Every locally convex and every locally bounded TVS is fundamental.

Theorem 5 *[4] Let A be a fundamental $F-$algebra with a uniformly bounded left approximate identity. Then A is strongly factorable.*

Definition 6 *Let (A, q) be an $F-$normed algebra. For any $T \in CL(A)$, let*

$$||T||_{A_q} := \sup_{x \in A, q(x) \leq 1} q(T(x)). \qquad (*)$$

In general, $\|T\|_q$ need not exist since the set $\{x \in A : q(x) \leq 1\}$ may not be bounded in an $F-$algebra (A, q) (see ([2], Remark 2.12; [22]. p. 8) for counter-examples). However, it exists in the case of q a $p-$norm $(0 < p \leq 1)$ or a seminorm. In the former case, the existence and other useful properties of $||.||_q$ are summarized in the following theorem (cf. [2, 22]).

Theorem 7 *Let (A, q) be a $p-$normed algebra, where $0 < p \leq 1$. Then:*
(a) A linear mapping $T : A \to A$ is continuous $\Leftrightarrow ||T||_{A_q} < \infty$.
(b) $\|.\|_q$ is a $p-$norm on $CL(A)$.
(c) For any $T \in CL(A)$, $q(T(x)) \leq ||T||_{A_q}.q(x)$ for all $x \in A$.
(d) For any $S, T \in CL(A)$, $||ST||_{A_q} \leq ||S||_{A_q}||T||_{A_q}$; hence $(CL(A), \|.\|_{A_q})$ is a $p-$normed algebra.
(e) If A is complete, then $(CL(A), \|.\|_{A_q})$ is also complete.
(f) If A has a minimal approximate identity, then, for any $a \in A$,

$$||L_a||_{A_q} = ||R_a||_{A_q} = q(a),$$

where $L_a, R_a : A \to A$ are the maps given by $L_a(x) = ax$ and $R_a(x) = xa$, $x \in A$.

3 Multipliers on $F-$ normed algebras

In this section, we recall basic definitions and results regarding the study of multipliers on various classes of topological algebras, as given in [7, 10, 11, 12, 16].

Definition 8 *[11] Let A be an algebra over the field $\mathbb{K}$ ($\mathbb{R}$ or $\mathbb{C}$). A mapping $T : A \to A$ is called a*
(i) multiplier on A if $aT(b) = T(a)b$ for all $a, b \in A$;
(ii) left multplier on A if $T(ab) = T(a)b$ for all $a, b \in A$;
(iii) right multiplier on A if $T(ab) = aT(b)$ for all $a, b \in A$.

Some authors use the term *centralizer* instead of *multiplier* (see, e.g. [7, 11]).

Let $M(A)$ (resp. $M_\ell(A), M_r(A)$) denote the set of all multipliers (resp. left multipliers, right multipliers) on A. Clearly, $M_\ell(A) \cap M_r(A) \subseteq M(A)$; if A is faithful, then $M(A) = M_\ell(A) \cap M_r(A)$. Both $M_\ell(A)$ and $M_r(A)$ are algebras with composition as multiplication (i.e. $(T_1T_2)(x) = T_1(T_2(x))$) and have the identity $I : A \to A$, $I(x) = x$ $(x \in A)$. If A is faithful, then $M(A)$ is a commutative algebra (without A being commutative) with identity I.

As an example, for any $a \in A$, consider the maps $L_a, R_a : A \to A$ given by $L_a(x) = ax$ and $R_a(x) = xa$, $x \in A$. Then $L_a \in M_\ell(A)$ and $R_a \in M_r(A)$.

Regarding the continuity of multipliers, we state:

Theorem 9 *(a) [12, 16] Suppose A is a strongly factorable $F-$ normed algebra. Then each $T \in M_\ell(A)$ (resp. $M_r(A)$) is linear and continuous.*

(b) [11, 16] Suppose A is a faithful $F-$ algebra. Then each $T \in M(A)$ is linear and continuous.

Convention. In the sequel, we shall always assume, unless stated otherwise, that A is an $F-$algebra having a bounded approximate identity $\{e_\lambda : \lambda \in I\}$. In view of the above remarks and Theorem 9, $M(A) = M_\ell(A) \cap M_r(A)$ and $M_\ell(A)$, $M_r(A)$ and $M(A))$ are algebras consisting of all continuous linear left multipliers, right multipliers and multipliers, respectively, on A.

Definition 10 *Let (A, q) be a $p-$normed algebra.*

(i) The uniform topology u on each of $M_\ell(A)$, $M_r(A)$, $M(A)$ is defined as the linear topology given by the $p-$norm

$$||T||_{A_q} = \sup \ \{q(T(x)) : x \in A, q(x) \leq 1\},\ T \in M_\ell(A),\ M_r(A) \text{ or } M(A).$$

(ii) The strong topology (or pointwise topology) s on each of $M_\ell(A)$, $M_r(A)$, $M(A)$ is defined as the linear topology given by the family $\{P_a : a \in A\}$ of $F-$seminorms, where

$$P_a(T) = q(T(a)),\ T \in M_\ell(A), M_r(A) \text{ or } M(A).$$

Clearly, $s \leq u$. Further properties are summarized, as follows.

Theorem 11 *(cf. [11, 16]) Let A be a faithful $p-$normed algebra, and let $M_t(A)$ denote any one of the algebras $M_\ell(A)$, $M_r(A)$ and $M(A)$. Then*

(a) If A is complete, $(M_t(A), u)$ and $(M_t(A), s)$ are complete.

(b) s and u have the same bounded sets.

(c) If $(M_t(A), s)$ is metrizable, then $s = u$ on $M_t(A)$.

(d) If A has a two-sided approximate identity, then A can be embedded as an s-dense set in $M_t(A)$.

4 Multipliers of $F-$Algebras of Vector-Valued Functions

We begin by recalling some terminology on vector-valued function spaces.

Let X be a Hausdorff topological space and E a topological vector space (TVS) over the field $\mathbb{K}$ ($= \mathbb{R}$ or $\mathbb{C}$) with a base $\mathcal{W}$ of neighbourhoods of 0 in E. A function $f : X \to E$ is said to *vanishe at infinity* if, for each $W \in \mathcal{W}$, there exists a compact set $K = K_W \subseteq X$ such that

$$f(x) \in W \text{ for all } x \in X \backslash K.$$

We shall denote by $C_b(X, E)$ the vector space of all continuous bounded $E-$valued functions on X and by $C_o(X, E)$ the subspace of $C_b(X, E)$ consisting of those functions which vanish at infinity. When $E = \mathbb{K}$ ($= \mathbb{R}$ or $\mathbb{C}$), these spaces will be denoted by $C_b(X)$ and $C_o(X)$. Let $C_b(X) \otimes E$ denote the vector subspace of $C_b(X, E)$ spanned by the set of all functions of the form $\varphi \otimes u$, where $\varphi \in C_b(X)$, $u \in E$, and

$$(\varphi \otimes u)(x) = \varphi(x)u, \quad x \in X.$$

If E is an algebra, then $C_b(X,E)$ is also an algebra with respect to the pointwise multiplication defined by

$$(fg)(x) = f(x)g(x), \quad x \in X.$$

If E is commutative, then $C_b(X,E)$ is also commutative. In particular, $C_b(X)$ is a commutative algebra. Note that if E is only a vector space, then $C_b(X,E)$ is a $C_b(X)$-bimodule with respect to the module multiplications $(\varphi, f) \to \varphi.f$ and $(f,\varphi) \to f.\varphi$ defined by

$$(\varphi.f)(x) = \varphi(x)f(x) = (f.\varphi)(x), \quad x \in X.$$

We mention that, if X is not locally compact, then $C_o(X,E)$ may be the trivial vector space $\{0\}$. For example, if $X = \mathbb{Q}$, the space of rationals, and $E = \mathbb{R}$, then $C_o(\mathbb{Q},\mathbb{R}) = \{0\}$.

Definition 12 *Let X be a Hausdorff space and E a Hausdorff topological vector space (TVS) over $\mathbb{K}$ $(= \mathbb{R}$ or $\mathbb{C})$. The uniform topology u on $C_b(X,E)$ is the linear topology which has a base of neighbourhoods of 0 consisting of all sets of the form*

$$N(X,G) = \{f \in C_b(X,E) : f(X) \subseteq G\},$$

where G vareies over $\mathcal{W}$.

In particular, if E is an $F-$normed space, the $u-$topology on $C_b(X,E)$ is given by the $F-$norm.

$$\|f\|_{q,\infty} = \sup_{x\in X} q(f(x)), \quad f \in C_b(X,E).$$

It is easy to see that, if (E,q) is an $F-$space, then so is $(C_b(X,E), \|\cdot\|_{q,\infty})$; further, if (E,q) is an $F-$normed algebra, then $(C_b(X,E), \|.\|_{q,\infty})$ is also an $F-$normed algebra.

We now state a useful consequence of the vector-valued version of Stone-Weierstrass theorem [1, 13, 14, 23] for reference purpose.

Theorem 13 *Let X be a locally compact Hausdorff space and E a TVS. Then $C_o(X) \otimes E$ is u-dense in $C_o(X,E)$ in each of the following cases:*

(a) E is locally convex.

(b) Every compact subset of X has a finite covering dimension and E is any TVS.

(c) E is an $F-$space with a basis (e.g. $E = \ell^p$ for $p > 0$)

(d) E has the approximation property.

Remark 14 *In view of the above result, we shall always assume in the sequel that $C_o(X) \otimes E$ is $u-$dense in $C_o(X,E)$. It is well-known that $C_o(X)$ has a minimal approximate identity ([9], p. 75–76). A useful consequence of this assumption is the following result.*

Lemma 15 *Let (A,q) be an $F-$normed algebra having a minimal approximate identity $\{e_\lambda : \lambda \in I\}$. Then $C_o(X,A)$ also has a minimal approximate identity.*

Proof. Let $\{h_\alpha : \alpha \in J\}$ be a minimal approximate identity in $C_o(X)$. We claim that $\{h_\alpha \otimes e_\lambda : (\alpha,\lambda) \in J \times I)$ is a minimal approximate identity for $C_o(X,A)$. First we see that, for any $\alpha \in J$ and $x \in X$, $|h_\alpha(x)| \leq 1$ and $q(e_\lambda) \leq 1$, and so by property (F_3) of an $F-$norm

$$||h_\alpha \otimes e_\lambda||_{q,\infty} = \sup_{x \in X} q[h_\alpha(x)e_\lambda] \leq q(e_\lambda) \leq 1; \tag{1}$$

hence $\{h_\alpha \otimes e_\lambda : (\alpha,\lambda) \in J \times I\}$ is minimal. Next, let $f \in C_o(X,A)$, and let $\varepsilon > 0$. Since $C_o(X) \otimes A$ is assumed to be dense in $C_o(X,A)$, there exists $g = \Sigma_{i=1}^n g_i \otimes a_i$ in $C_o(X) \otimes A$ such that

$$||g - f||_{q,\infty} < \frac{\varepsilon}{3}. \tag{2}$$

Now

$$\begin{aligned} \lim_{(\alpha,\lambda)} (h_\alpha \otimes e_\lambda)g &= \lim_{(\alpha,\lambda)} \Sigma_{i=1}^n (h_\alpha g_i \otimes e_\lambda a_i) \\ &= \Sigma_{i=1}^n (\lim_\alpha h_\alpha g_i \otimes \lim_\lambda e_\lambda a_i) = \Sigma_{i=1}^n g_i \otimes a_i = g. \end{aligned}$$

So there exists $(\alpha_o, \lambda_o) \in J \times I$ such that

$$||(h_\alpha \otimes e_\lambda)g - g||_{q,\infty} < \frac{\varepsilon}{3} \text{ whenever } (\alpha,\lambda) > (\alpha_o,\lambda_o). \tag{3}$$

Hence for $(\alpha,\lambda) > (\alpha_o,\lambda_o)$, using (1), (2), (3) and the fact that $|| \cdot ||_{q,\infty}$ is submultiplicative,

$$\begin{aligned} ||(h_\alpha \otimes e_\lambda)f - f||_{q,\infty} &\leq ||(h_\alpha \otimes e_\lambda)f - (h_\alpha \otimes e_\lambda)g||_{q,\infty} \\ &\quad + ||(h_\alpha \otimes e_\lambda)g - g||_{q,\infty} + ||g - f||_{q,\infty} \\ &\leq ||h_\alpha \otimes e_\lambda||_{q,\infty}.||f - g||_{q,\infty} \\ &\quad + ||(h_\alpha \otimes e_\lambda)g - g||_{q,\infty} + ||g - f||_{q,\infty} \\ &< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon. \end{aligned}$$

Thus $\lim_{(\alpha,\lambda)} (h_\alpha \otimes e_\lambda)f = f$. ■

Recall that, if A is a topological algebra, then $C_b(X,A)$ is a left A-module with respect to the module multiplication $(a,f) \to a \cdot f$ as pointwise action:

$$(a \cdot f)(x) = af(x), a \in A, f \in C_b(X,A), x \in X.$$

In particular, $C_o(X,A)$ is a left A-module.

Lemma 16 *Let X be a locally compact Hausdorff space and A a topological algebra. If $T \in M(C_o(X,A))$, then $T(a \cdot f) = a \cdot T(f)$ for $f \in C_o(X,A)$ and $a \in A$.*

Proof. We note that $C_o(X)$ is a Banach algebra with a bounded approximate identity, $\{h_\alpha\}$ (say). Let $g = \Sigma_{i=1}^n \varphi_i \otimes b_i \in C_o(X) \otimes A$. Then, for any $a \in A$ and $1 \leq i \leq n$,

$$\lim_\alpha \ [(h_\alpha \otimes a).(\varphi_i \otimes b_i)] = \lim_\alpha \ (h_\alpha \varphi_i \otimes ab_i) = \varphi_i \otimes ab_i = a(\varphi_i \otimes b_i).$$

So

$$\begin{aligned} T(a(\varphi_i \otimes b_i)) &= \lim_\alpha \ T[(h_\alpha \otimes a).(\varphi_i \otimes b_i)] \\ &= \lim_\alpha \ (h_\alpha \otimes a) \cdot T(\varphi_i \otimes b_i) = a \cdot T(\varphi_i \otimes b_i). \end{aligned}$$

By linearity of T, we have $T(a \cdot g) = a \cdot T(g)$. Since $C_o(X) \otimes A$ is assumed to be u-dense in $C_o(X, A)$, $T(a \cdot f) = a \cdot T(f)$ holds for all $f \in C_o(X, A)$ and $a \in A$. ■

Before stating the next result, we need to mention that, if (A, q) is an $F-$normed algebra having a minimal approximate identity, then, by Lemma 15, $C_o(X, A)$ has an approximate identity and hence it is a faithful topological algebra. Consequently, the results of Section 3 apply also to the multiplier algebras $M(C_o(X, A))$, $M_\ell(C_o(X, A))$ and $M_r(C_o(X, A))$. In particular, $M(C_o(X, A)) = M_\ell(C_o(X, A)) \cap M_r(C_o(X, A))$ and so for any $T \in M(C_o(X, A))$, $T(fg) = fT(g) = T(f)g$ for all $f, g \in C_o(X, A)$; we shall write

$$||T||_{C_q} := ||T||_{C_o(X,A)_q} = \sup \ \{q(T(f)) : f \in C_o(X, A), ||f||_q \leq 1\}.$$

Lemma 17 *Let (A, q) be a complete $p-$normed algebra with a bounded approximate identity $\{e_\lambda : \lambda \in I\}$. Then $C_b(X, M(A)_u)$ can be isometrically embedded in $M(C_o(X, A))$.*

Proof. Let $F \in C_b(X, M(A)_u)$ and $f \in C_o(X, A)$. Define $F * f : X \to A$ by

$$(F * f)(x) = F(x)[f(x)], x \in X.$$

Then $F * f$ is a continuous function on X vanishing at infinity, that is, $F * f \in C_o(X, A)$. Therefore the mapping $T_F \in C_o(X, A) \to C_o(X, A)$ defined by

$$T_F(f) = F * f, \ f \in C_o(X, A)$$

is well-defined. Clearly, T_F is linear. Further, for any $f, g \in C_o(X, A))$ and $x \in X$, since $F(x) \in M(A)$,

$$\begin{aligned} [T_F(fg)](x) &= F(x)[f(x)g(x)] = f(x)F(x)[g(x)] \\ &= f(x)(T_F g)(x) = [fT_F(g))](x). \end{aligned}$$

Hence $T_F \in M(C_o(X, A))$. Next, using the argument as in ([18], p. 449),

$$\begin{aligned} ||T_F||_{C_q} &= \sup\{||T_F(f)||_{q,\infty} : f \in C_o(X, A), ||f||_{q,\infty} \leq 1\} \\ &= \sup\{\sup_{x \in X} q[T_F(f)(x)] : f \in C_o(X, A), ||f||_{q,\infty} \leq 1\} \\ &= \sup\{\sup_{x \in X} q[F(x)(f(x))] : f \in C_o(X, A), ||f||_{q,\infty} \leq 1\} \\ &= \sup_{x \in X} ||F(x)||_{A_q} = ||F||_{q,\infty}. \end{aligned}$$

■

Theorem 18 *Let $A = (A, q)$ be a commutative complete $p-$normed algebra with a minimal approximate identity $\{e_\lambda : \lambda \in I\}$. Then*

$$M(C_o(X, A)) \simeq C_b(X, M(A)_u).$$

Proof. In view of Lemma 17, we only need to show that

$$M(C_o(X, A)) \subseteq C_b(X, M(A)_u).$$

Since $C_o(X, A)$ is faithful, we have

$$M(C_o(X, A)) = M_\ell(C_o(X, A)) \cap M_r(C_o(X, A)).$$

Let $T \in M(C_o(X, A))_u$. For any $a \in A$ and $\varphi \in C_o(X)$, $\varphi \otimes a \in C_o(X, A)$ and so $T(\varphi \otimes a) \in C_o(X, A)$. For any $\varphi \in C_o(X)$, the mapping $h_T(x) : A \to L(A)$

$$h_T(x)(a) = \frac{T(\varphi \otimes a)(x)}{\varphi(x)}, \text{ for } a \in A, \qquad (**)$$

defines an A-valued function whenever $\varphi(x) \neq 0$.

We claim that $h_T \in C_o(X, M(A)_u)$. First, the function $h_T(x)$ defined in this way is independent of the choice of $\varphi \in C_o(X)$. [Indeed, for any fixed $x \in X$, let $\varphi, \psi \in C_o(X)$ such that $\varphi(x) \neq 0$, $\psi(x) \neq 0$. We have by commutativity

$$\begin{aligned} T[(\varphi \otimes a).(\psi \otimes e_\lambda)](x) &= [(\psi \otimes e_\lambda).T(\varphi \otimes a)](x) = \psi(x)e_\lambda.T(\varphi \otimes a)(x), \\ T[(\varphi \otimes a).(\psi \otimes e_\lambda)](x) &= \varphi(x)e_\lambda.T(\psi \otimes a)(x); \end{aligned}$$

or,

$$e_\lambda.\frac{T(\varphi \otimes a)(x)}{\varphi(x)} = e_\lambda.\frac{T(\psi \otimes a)(x)}{\psi(x)}.$$

Taking $\lim_\lambda$

$$\frac{T(\varphi \otimes a)(x)}{\varphi(x)} = \frac{T(\psi \otimes a)(x)}{\psi(x)}.]$$

Therefore $h_T(x)$ is a linear operator on A and, by $(**)$, we can write

$$\begin{aligned} [T(\varphi \otimes a)](x) &= \varphi(x)h_T(x)(a) = h_T(x)(\varphi(x)a) \\ &= h_T(x)(\varphi \otimes a)(x) \text{ for all } a \in A, \varphi \in C_o(X); \end{aligned}$$

that is, $h_T \cdot (\varphi \otimes a) = T(\varphi \otimes a)$. Moreover, h_T is bounded and

$$\begin{aligned} ||h_T \cdot (\varphi \otimes a)||_{q,\infty} &= ||T(\varphi \otimes a)||_{q,\infty} \leq ||T||_{C_q}.||\varphi \otimes a||_{q,\infty} \\ &= ||T||_{C_q} \sup_{x \in X} q[\varphi(x)a] \leq ||T||_{C_q}. \sup_{x \in X} |\varphi(x)|^p q(a) \\ &= ||T||_{C_q}.||\varphi||^p_{q,\infty}.q(a). \end{aligned}$$

This shows that $h_T : X \to M(A)_s$ is bounded.

Now our main task is to show that $h_T : X \to M(A)_u$ is continuous. Let $x_o \in X$. Then there exists $\varphi \in C_o(X)$ such that $\varphi(x_o) \neq 0$ and

$$N = N(x_o) = \{x \in X : \varphi(x) \neq 0\} = X \backslash \varphi^{-1}(0)$$

is an open neighborhood of x_o. Thus

$$h_T(x)(a) = \frac{T(\varphi \otimes a)(x)}{\varphi(x)}, \text{ for } x \in N,$$

defines an s continuous function $h_T(x) : A \to A$. Let $\{x_\alpha : \alpha \in J\} \subseteq N$ with $x_\alpha \to x_o$ in X. We claim that

$$||h_T(x_\alpha) - h_T(x_o)||_{A_q} = \sup_{a \in A, q(a) \leq 1} q[h_T(x_\alpha)(a) - h_T(x_o)(a)] \to 0 \text{ as } x_\alpha \to x_o. \tag{4}$$

For any $a \in A$,

$$\begin{aligned} & q[h_T(x_\alpha)(a) - h_T(x_o)(a)] = q\left[\frac{T(\varphi \otimes a)(x_\alpha)}{\varphi(x_\alpha)} - \frac{T(\varphi \otimes a)(x_o)}{\varphi(x_o)}\right] \\ = & \frac{1}{|\varphi(x_\alpha)\varphi(x_o)|^p} q[\varphi(x_o)T(\varphi \otimes a)(x_\alpha) - \varphi(x_\alpha)T(\varphi \otimes a)(x_o)] \\ = & \frac{1}{|\varphi(x_\alpha)\varphi(x_o)|^p}\{q[\varphi(x_o)T(\varphi \otimes a)(x_\alpha)] - q[\varphi(x_o)T(\varphi \otimes a)(x_o)] \\ & +q[\varphi(x_o)T(\varphi \otimes a)(x_o) - \varphi(x_\alpha)T(\varphi \otimes a)(x_o)]\} \\ \leq & \frac{1}{|\varphi(x_\alpha)\varphi(x_o)|^p}\{|\varphi(x_o)|^p q[T(\varphi \otimes a)(x_\alpha) - T(\varphi \otimes a)(x_o)] \\ & +|\varphi(x_o) - \varphi(x_\alpha)|^p . q[T(\varphi \otimes a)(x_o)]\}. \end{aligned} \tag{5}$$

Since $\varphi \in C_o(X)$, $\varphi(x_\alpha) \to \varphi(x_o)$ as $x_\alpha \to x_o$ in X, it follows that the second term of (5) in the last inequality tends to 0 whenever $x_\alpha \to x_o$. Therefore, we need to show that the first term of (5) tends to 0 uniformly on $\{a \in A : q(a) \leq 1\}$.

Since $\{e_\lambda : \lambda \in I\}$ is a bounded approximate identity of A, for any $\varepsilon > 0$, there exists $\lambda_o = \lambda_o(\varepsilon) \in I$ such that

$$||e_{\lambda_o} T(\varphi \otimes a) - T(\varphi \otimes a)||_{q,\infty} < \frac{\varepsilon}{4}. \tag{6}$$

Since $x_\alpha \to x_o$, there exists $\alpha_o \in J$ such that

$$q[T(\varphi \otimes e_{\lambda_o})(x_\alpha) - T(\varphi \otimes e_{\lambda_o})(x_o)] < \frac{\varepsilon}{2} \text{ for all } \alpha \geq \alpha_o.$$

Then, for any $a \in A$ with $q(a) \leq 1$, using Lemma 16 and the fact that $T \in M_\ell(C_o(X, A)) \cap M_r(C_o(X, A))$,

$$\begin{aligned} & q[e_{\lambda_o} T(\varphi \otimes a)(x_\alpha) - e_{\lambda_o} T(\varphi \otimes a)(x_o)] \\ = & q[T(\varphi \otimes e_{\lambda_o} a)(x_\alpha) - T(\varphi \otimes e_{\lambda_o} a)(x_o)] \\ = & q[T(\varphi \otimes e_{\lambda_o})(x_\alpha)a - T(\varphi \otimes e_{\lambda_o})(x_o)a] \\ \leq & q[T(\varphi \otimes e_{\lambda_o})(x_\alpha) - T(\varphi \otimes e_{\lambda_o})(x_o)].q(a) < \frac{\varepsilon}{2}, \end{aligned} \tag{7}$$

for all $\alpha \geq \alpha_o$. Hence, by (6) and (7),

$$\begin{aligned}
& q[T(\varphi \otimes a)(x_\alpha) - T(\varphi \otimes a)(x_o)] \\
\leq\ & q[T(\varphi \otimes a)(x_\alpha) - e_{\lambda_o}T(\varphi \otimes a)(x_\alpha)] \\
& +q[e_{\lambda_o}T(\varphi \otimes a)(x_\alpha) - e_{\lambda_o}T(\varphi \otimes a)(x_o)] \\
& +q[e_{\lambda_o}T(\varphi \otimes a)(x_o) - T(\varphi \otimes a)(x_o)] \\
\leq\ & 2||T(\varphi \otimes a) - e_{\lambda_o}T(\varphi \otimes a)||_{q,\infty} \\
& +q[e_{\lambda_o}T(\varphi \otimes a)(x_\alpha) - e_{\lambda_o}T(\varphi \otimes a)(x_o)] \\
<\ & \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon,
\end{aligned}$$

for all $\alpha \geq \alpha_o$. Therefore

$$\lim_{x_\alpha \to x_o} \quad q[T(\varphi \otimes a)(x_\alpha) - T(\varphi \otimes a)(x_o)] = 0$$

uniformly on $\{a \in A : q(a) \leq 1\}$. Hence

$$lim_{x_\alpha \to x_o}||h_T(x_\alpha) - h_T(x_o)||_{A_q} = 0,$$

which proves (4).

Finally, we show that $||T||_{C_q} = ||h_T||_{q,\infty}$. First, for any $x \in X$, we have

$$\begin{aligned}
||h_T(x)||_{A_q} &= \sup\{q[h_T(x).(\varphi \otimes a)(x))] : ||\varphi \otimes a(x)||_A \leq 1\} \\
&= \sup\{q[T(\varphi \otimes a)(x))] : q[\varphi \otimes a(x)] \leq 1\} \leq ||T||_{C_q},
\end{aligned}$$

and so

$$||h_T||_{q,\infty} = \sup_{x\in X} ||h_T(x)||_q \leq ||T||_{C_q}.$$

On the other hand,

$$\begin{aligned}
||T(\varphi \otimes a)||_{A_q,\infty} &= \sup_{x\in X} q[_T(x)(\varphi \otimes a(x))] \\
&\leq \sup_x ||h_T(x)||_q.||\varphi \otimes a||_{q,\infty} = ||h_T||_{q,\infty}.||\varphi||_\infty q(a).
\end{aligned}$$

Consequently, $||T||_{C_q} \leq ||h_T||_{q,\infty}$; hence $||T||_{C_q} = ||h_T||_{q,\infty}$. ∎

Remark 19 *If A has an identity, then A is identical with $M(A)$ and $s = u$.*

Example 20 *Let $\ell_p^{\pm}$, $0 < p \leq 1$, denote the algebra of all two-sided complex sequences $x = \{x_n\}_{n=-\infty}^{\infty}$ for which :*

$$\|x\|_p = \sum_{n=-\infty}^{\infty} |x_n|^p < \infty.$$

This is a commutative complete $p-$normed algebra with the multiplication defined as convolution:

$$\{x_n\}_{n=-\infty}^{\infty}.\{y_n\}_{n=-\infty}^{\infty} = \left\{ \sum_{k=-\infty}^{\infty} x_{n-k}y_k \right\}_{n=-\infty}^{\infty},$$

since the $p-$homogeneous norm $\|.\|_p$ defined above is also submultiplicative ([26], p. 33). The algebra $\ell_p^{\pm}$ possesses an identity element, namely the sequence $\{\delta_{n,0}\}_{n=-\infty}^{\infty}$, where $\delta_{n,m}$ is the kronecker symbol defined by $\delta_{n,m} = 0$ for $n \neq m$ and $\delta_{n,m} = 1$ for $n = m$. In this case $M(\ell_p^{\pm}) \simeq \ell_p^{\pm}$; hence $M(C_o(X, \ell_p^{\pm})) \simeq C_b(X, M(\ell_p^{\pm})) \simeq C_b(X, \ell_p^{\pm})$.

The above example can be generalized by taking $\ell_p(G)$ on any discrete group, with convolution multiplication ([26], p. 33).

Note that the algebra $\ell_p(\mathbb{N})$, $1 < p < \infty$), with the norm:

$$\|\{x_n\}\|_p = [\sum_{n=1}^{\infty} |x_n|^p]^{\frac{1}{p}},$$

is a Banach algebra without identity; however, it possesses an "unbounded" approximate identity ([26], p. 26). Also, for G a locally compact abelian group, $L_1(G)$ is a Banach algebra under the convolution multiplication and has a "bounded" approximate identity ([21], p. 232).

Example 21 *Let X be a topological Hausdorff space and k the compact-open topology on $C(X)$ given by the family $\{|| \cdot ||_K : K \subseteq X,\ K$ is compact$\}$ of submultiplicative seminorms, where*

$$||f||_K = \sup\{|f(x)| : x \in K\},\ \ f \in C(X).$$

$(C(X), k)$ is a commutative locally C^*- algebra with involution $f \to f^*$ given by $f^*(x) = \overline{f}(x)$; hence, by a famous result of Inoue, it has a (minimal) approximate identity $\{e_\lambda : \lambda \in I\}$ satisfying $||e_\lambda||_K \leq 1$ for all $\lambda \in I$ and compact $K \subseteq X$ (see [21], p. 490). In general, $(C(X), k)$ is not metrizable. In fact, if X is a locally compact Hausdorff space, then $(C(X), k)$ is metrizable iff X is hemicompact (i.e. X can be expressed as a countable union of compact sets K_n such that each compact subset of X is contained in some K_n). Therefore, in particular, $(C(\mathbb{R}), k)$ is a $F-$algebra. But $(C(\mathbb{R}), k)$ is not necessarily a $p-$normed algebra; so in this case, any multiplier $T \in M((C(\mathbb{R}), k))$ need not be continuous.

Example 22 *Let A_p, $0 < p \leq 1$, denote the algebra of all holomorphic functions in the unit disc*

$$\varphi(z) = \sum_{n=0}^{\infty} a_n z^n,$$

for which

$$\|\varphi\|_p = \sum_{n=0}^{\infty} |a_n|^p < \infty.$$

This is a commutative complete $p-$normed algebra with the pointwise multiplication and has an identity ([25], p. 80; [21], p. 135). In this case,

$$M(C_o(X, A_p)) \simeq C_b(X, M(A_p)_\beta) \simeq C_b(X, A_p).$$

Acknowledgement. This work has been done under the Project No. 3-059/429. The authors are grateful to the Deanship of Scientific Research of the King Abdulaziz University, Jeddah, for their financial support.

References

[1] M. Abel, Topological bimodule-algebras, Proc. 3rd Internat. Conf. Topological Algebras Appl. (Oulu, Finland, 2004), 25–42.

[2] M. Adib, A.H. Riazi and L.A. Khan, Quasi-multipliers on $F-$algebras, Abstract and Applied Analysis, Volume 2011, Article ID 235273, 30 pages.

[3] C. A. Akemann, G. K. Pedersen and J. Tomiyama, Multipliers of C^*-algebras, J. Func. Anal. 13(1973), 277–301.

[4] E. Ansari-Piri, A class of factorizable topological algebras, Proc. Edinbrugh Math. Soc. 33(1990), 53–59.

[5] F. F. Bonsall and J. Duncan, *Complete Normed Algebras* (Springer, 1973).

[6] R. C. Buck, Bounded continuous functions on a locally compact space, Michigan Math. J. 5(1958), 95–04.

[7] R. C. Busby, Double centralizers and extension of C^*-algebras, Trans. Amer. Math. Soc. 132(1968), 79–99.

[8] J. C. C. Haro and H. C. Lai, Multipliers in continuous vector-valued function spaces, Bull. Austral. Math. Soc. 46(1992), 199–204.

[9] A. Ya. Helemskii, *Banach and Locally Convex Algebras* (Clarendon Press, Oxford, 1993).

[10] T. Husain: Multipliers of Topological Algebras, Dissertations Math. 285(1989), 1–36.

[11] B. E. Johnson, An introduction to the theory of centralizers, Proc. London Math. Soc. 14(1964), 299–320.

[12] B. E. Johnson, Continuity of centralisers on Banach algebras, J. London Math. Soc. 41(1966), 639–640.

[13] A.K. Katsaras, On the strict topology in the non-locally convex setting, Math. Nachr. 102(1981), 321–329.

[14] L. A. Khan, Some approximation results for the compact-open topology, Periodica Math. Hungarica 30(1995), 81–86.

[15] L. A. Khan, Topological modules of continuous homomorphisms, J. Math. Anal. Appl. 343(2008), 141–150.

[16] L. A. Khan, N. Mohammad, and A.B. Thaheem, Double multipliers on topological algebras, Internat. J. Math. & Math. Sci. 22(1999), 629–636.

[17] G. Köthe, *Topological Vector Spaces I* (Springer-Verlag, 1969).

[18] H. C. Lai, Multipliers of a Banach algebra in the second conjugate algebra as an idealizer, Tohoku Math. J. 26(1974), 431–452.

[19] H. C. Lai, Multipliers for some spaces of Banach algebra valued functions, Rocky Mountain J. Math. 15(1985), 157–166.

[20] R. Larson, *An Introduction to the Theory of Multipliers* (Springer-Verlag, 1971).

[21] A. Mallios, *Topological Algebras - Selected Topics* (North-Holland, 1986).

[22] S. Rolewicz, *Metric Linear Spaces* (D. Reidel Publishing Company, 1985).

[23] A. H. Shuchat, Approximation of vector-valued continuous functions, Proc. Amer. Math. Soc., 31(1972), 97–103.

[24] J. K. Wang, Multipliers of commutative Banach algebras, Pacific. J. Math. 11(1961), 1131–1149.

[25] W. Żelazko, Metric Generalizations of Banach Algebras, Dissertations Math. 47(196589), 1–70.

[26] W. Żelazko, *Banach Algebras* (Elsevier, Amsterdam, 1973).