Critical exponents in a doubly degenerate nonlinear parabolic system with inner absorptions

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Abstract

This paper deals with critical exponents for a doubly degenerate nonlinear parabolic system coupled via local sources and with inner absorptions under null Dirichlet boundary conditions in a smooth bounded domain. The author first establishes the comparison principle and local existence theorem for the above problem. Then under appropriate hypotheses, the author proves that the solution either exists globally or blows up in finite time depends on the initial data and the relations of the parameters in the system. The critical exponent of the system is simply described via a characteristic matrix equation introduced.

Keyword: Critical exponents; Doubly degenerate nonlinear parabolic system; Local sources; Inner absorptions; Global existence; Finite time blow-up.
AMS(MOS). Mathematics Subject Classification: 35B35; 35K57; 35K60; 35K65.

1 Introduction and main results

In this paper, we consider the following nonlocal doubly degenerate nonlinear parabolic system with inner absorptions

$$\begin{aligned}
&u_t - \Delta_{m,p} u = u^{\alpha_1} v^{\beta_1} - a u^r, && (x,t) \in \Omega_T,\\
&v_t - \Delta_{n,q} v = u^{\alpha_2} v^{\beta_2} - b v^s, && (x,t) \in \Omega_T,\\
&u(x,t) = v(x,t) = 0, && (x,t) \in \partial\Omega \times (0,T],\\
&u(x,0) = u_0(x), \quad v(x,0) = v_0(x), && x \in \Omega,
\end{aligned} \tag{1.1}$$

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where for $k > 0, \gamma > 2$ and $N \geq 1$, $\Delta_{k,\gamma}\Theta = \nabla \cdot (|\nabla \Theta^k|^{\gamma-2} \cdot \nabla \Theta^k)$, $\nabla \Theta^k = k\Theta^{k-1}(\Theta_{x_1}, \cdots, \Theta_{x_N})$, $\Omega \subset \mathbb{R}^N (N \geq 1)$ is a bounded domain with appropriately smooth boundary $\partial\Omega$; $m, n, r, s \geq 1$, $p, q > 2$, $\alpha_i, \beta_i \geq 0$, $i = 1, 2$, $\Omega_T = \Omega \times (0, T]$ and a, b are positive constants and u_0, v_0 satisfies compatibility and the following conditions:

$$(H)\ u_0^m \in C(\overline{\Omega}) \cap W_0^{1,p}(\Omega), \quad v_0^n \in C(\overline{\Omega}) \cap W_0^{1,q}(\Omega) \text{ and } \nabla u_0^m \cdot \nu < 0,$$
$$\nabla v_0^n \cdot \nu < 0 \text{ on } \partial\Omega, \text{ where } \nu \text{ is unit outer normal vector on } \partial\Omega.$$

Parabolic systems like (1.1) arise in many applications in the fields of mechanics, physics and biology like, for instance, the description of turbulent filtration in porous media, the theory of non-Newtonian fluids perturbed by nonlinear terms and forced by rather irregular period in time excitations, the flow of a gas through a porous medium in a turbulent regime or the spread of biological (see [8, 15, 1, 6] and references therein); In the non-Newtonian fluids theory, the pair (p, q) is a characteristic quantity of medium. When $(m, n) \geq (1, 1)$ and $(p, q) > (2, 2)$, the system models the non-stationary, polytropic flow of a fluid in a porous medium; it has been intensively studied (see [18, 13, 16, 10, 2] and references therein).

The problems with nonlinear reaction term, absorption term and nonlinear diffusion include blow-up and global existence conditions of solutions, blow-up rates and blow-up sets, etc. This degenerate system exhibiting a doubly nonlinearity generalizes the porous medium system ($p = q = 2$) and the parabolic p-Laplace system ($m = n = 1$), which has been studied by many authors. For $p = q = 2$, $m = n = 1$, it is a classical reaction-diffusion system of Fujita type. Bedjaoui and Souplet [3] considered the critical blow-up exponents for the following system

$$u_t = \Delta u + v^p - b_1 u^r, \quad v_t = \Delta v + u^q - b_2 v^s, \qquad x \in \Omega, t > 0. \tag{1.2}$$

By constructing self-similar weak sub-solutions with compact supports, they obtained the critical exponent: $pq = \max(r, 1)\max(s, 1)$. Moreover scalar absorption-diffusion equations of the style $u_t - \Delta u = -u^r$ have also been widely studied (see [7, 9, 11] and references therein).

Zheng and Su [22] considered the quasilinear reaction-diffusion system with nonlocal sources and inner absorptions of the form

$$u_t = \Delta u^m + \int_\Omega v^p dx - au^r, \quad v_t = \Delta v^n + \int_\Omega u^q dx - bv^s, \quad x \in \Omega, t > 0. \tag{1.3}$$

They established the critical exponent and the blow-up rate for the system subject to homogeneous Dirichlet conditions and nonnegative initial data. It was found that the critical exponent is determined by the interaction among all the six nonlinear exponents from all the three kinds of the nonlinearities.

For p-Laplacian systems, Yang and Lu [19] studied the following equations

$$u_t - \operatorname{div}(|\nabla u|^{p-2}\nabla u) = u^{\alpha_1} v^{\beta_1}, \qquad (x, t) \in \Omega \times (0, T],$$

$$v_t - \mathrm{div}(|\nabla v|^{q-2}\nabla v) = u^{\alpha_2} v^{\beta_2}, \qquad (x,t) \in \Omega \times (0,T], \qquad (1.4)$$

with the homogeneous Dirichlet boundary value conditions, they derived some estimates near the blow-up point for positive solutions and non-existence of positive solutions of the relate elliptic systems.

Very recently, Zhang et al. [21] further studied the blow-up properties of positive solutions for system (1.1) with nonlocal sources

$$\begin{aligned} u_t - \mathrm{div}(|\nabla u|^{p-2}\nabla u) &= \int_\Omega v^m dx - u^r, \qquad (x,t) \in \Omega \times (0,T], \\ v_t - \mathrm{div}(|\nabla v|^{q-2}\nabla v) &= \int_\Omega u^n dx - v^s, \qquad (x,t) \in \Omega \times (0,T] \qquad (1.5) \end{aligned}$$

in a smooth bounded domain $\Omega \subset \mathbb{R}^N$. Under appropriate hypotheses, they discussed the global existence and blow-up of positive weak solutions by using a comparison principle. For $r = s = 0$, the system (1.1) is reduced to a local non-Newton polytropic filtration system without inner absorptions. And the author [16], [17] dealt with it under local and nonlocal sources. Under appropriate hypotheses, they all establish local theory of the solutions and prove that the solution either exists globally or blows up in finite time. More results for the non-Newton polytropic filtration system with sources can be found in [12], [23], [20] and the references therein.

However, as far as we know, there is little literature on the blow-up properties for problems (1.1) with the concentrated source and inner absorptions. Motivated by the above works, in this paper, we investigate the blow-up properties of solutions of the problem (1.1) and extend the results of [16, 3, 21, 23, 20] to more generalized cases. Due to the nonlinear diffusion terms and doubly degeneration for $u = v = 0$ and $|\nabla u| = |\nabla v| = 0$, we have some new difficulties to be overcome. Noticing that the system (1.1) includes the Newtonian filtration system ($p = q = 2$) and the non-Newtonian filtration system ($m = n = 1$) formally, so the method for it should be synthetic. In fact, we can use the methods for the above two systems to deal with it. In order to apply monotonicity, we establish the comparison principle for system (1.1) by choosing suitable test function and Gronwall's inequality. Then by the first eigenvalue and its corresponding eigenfunctions to the eigenvalue problem for the non-Newtonian filtration system, we construct a pair of well-ordered positive supersolution and subsolution. Using comparison principle, we achieve our purpose and obtain the global existence and blow-up of solutions to the problem. We will show that the critical exponent is determined by the interaction among all the nonlinear exponents from all the three nonlinearities. Correspondingly, two kinds of characteristic algebraic systems are introduced to get simple descriptions for the critical exponent and the blow-up considered.

In order to state our results, we introduce some useful symbols. Throughout this paper, we let $\zeta(x)$ and $\vartheta(x)$ be the unique solution of the following elliptic equation (see [23, 4]),

$$\begin{cases} -\Delta_{m,p}\zeta = 1, & x \in \Omega, \\ \zeta = 0, & x \in \partial\Omega, \end{cases} \qquad \begin{cases} -\Delta_{n,q}\vartheta = 1, & x \in \Omega, \\ \vartheta = 0, & x \in \partial\Omega. \end{cases} \tag{1.6}$$

Before starting the main results, we introduce a pair of parameters (μ_1, μ_2) solving the following characteristic algebraic system

$$\begin{pmatrix} -\mu_1 & \beta_1 \\ \alpha_2 & -\mu_2 \end{pmatrix} \begin{pmatrix} \tau \\ \theta \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix},$$

namely,

$$\tau = \frac{\beta_1 + \mu_2}{\beta_1\alpha_2 - \mu_1\mu_2}, \quad \theta = \frac{\alpha_2 + \mu_1}{\beta_1\alpha_2 - \mu_1\mu_2}$$

with

$$\mu_1 = \max\{m(p-1) - \alpha_1, r - \alpha_1\}, \ \mu_2 = \max\{n(q-1) - \beta_2, s - \beta_2\}.$$

It is obvious that $1/\tau$ and $1/\theta$ share the same signs. We claim that the critical exponent of problem (1.1) should be $(1/\tau, 1/\theta) = (0, 0)$, described by the following theorems.

Theorem 1.1 *Assume that* $(1/\tau, 1/\theta) < (0, 0)$, *then there exist solutions of (1.1) being globally bounded.*

Theorem 1.2 *Assume that* $(1/\tau, 1/\theta) > (0, 0)$, *then the nonnegative solution of (1.1) blows up in finite time for sufficiently large initial values and exists globally for sufficiently small initial values.*

Theorem 1.3 *Assume that* $(1/\tau, 1/\theta) = (0, 0)$, $\zeta(x)$ *and* $\vartheta(x)$ *are defined in (1.6), respectively.*

(i) Suppose that $r > m(p-1)$ *and* $s > n(q-1)$*. If*

$$a^{\alpha_2} b^{r-\alpha_1} \geq 1,$$

then the solutions are globally bounded for small initial data; if

$$\vartheta^{\beta_1} > a\zeta^{r-\alpha_1}, \zeta^{\alpha_2} > b\vartheta^{s-\beta_2},$$

then the solutions blow up in finite time for large data.

(ii) Suppose that $r < m(p-1)$ *and* $s < n(q-1)$*. If*

$$\zeta^{\frac{\alpha_2}{n(q-1)-\beta_2} + \frac{\alpha_1}{\beta_1}} \vartheta^{\frac{n(q-1)}{n(q-1)-\beta_2}} \leq 1,$$

then the solutions are globally bounded for small initial data; if

$$\zeta^{\alpha_1}\vartheta^{\beta_1} > 1, \zeta^{\alpha_2}\vartheta^{\beta_2} > 1,$$

then the solutions blow up in finite time for large data.

(iii) Suppose that $r < m(p-1)$ *and* $s > n(q-1)$. *If*

$$\zeta^{\alpha_2 + \frac{\alpha_1(s-\beta_2)}{\beta_1}} \leq b,$$

then the solutions are globally bounded for small initial data; if

$$\zeta^{\alpha_1}\vartheta^{\beta_1} > 1, \zeta^{\alpha_2} > b\vartheta^{s-\beta_2},$$

then the solutions blow up in finite time for large data.

(iv) Suppose that $r > m(p-1)$ *and* $s < n(q-1)$. *If*

$$\vartheta^{\alpha_1 + \frac{\alpha_2(r-\beta_1)}{\beta_2}} \leq a,$$

then the solutions are globally bounded for small initial data; if

$$\vartheta^{\beta_1} > a\zeta^{r-\alpha_1}, \zeta^{\alpha_2}\vartheta^{\beta_2} > 1,$$

then the solutions blow up in finite time for sufficiently large data.

The rest of this paper is organized as follows. In Section 2, we shall establish the comparison principle and local existence theorem for problem (1.1). Theorems 1.1 and 1.2 will be proved in Section 3 and Section 4, respectively. Finally, we will give the proof of Theorem 1.3 in Section 5.

2 Preliminaries

In order to study the globally existing and blowing-up solutions to problem (1.1), we need to firstly prove the comparison principle for the weak solution of the system (1.1). It worth to mention, this statement plays a crucial role in the investigation. Additions, the existence of local-in-time weak solutions of (1.1) under appropriate hypotheses is also studied in this section. From a physical point of view, we need only to consider the non-negative solutions. Moreover, if we assume that $u_0(x), v_0(x) \geq 0$ in Ω, by Lemma 2.1 (see it below), we can obtain that $(u(x,t), v(x,t)) \geq (0,0)$ a.e. in $(\Omega \times (0,T)) \times (\Omega \times (0,T))$. Thus we only consider the non-negative solutions in later sections.

As it is well known that doubly degenerate equations need not have classical solutions, we give a precise definition of a weak solution for problem (1.1). Let $\Omega_T = \Omega \times (0,T]$, $S_T = \partial\Omega \times [0,T]$, $T > 0$.

Definition 2.1 *A pair of functions (u,v) is called a solution of the problem (1.1) on $\overline{\Omega}_T \times \overline{\Omega}_T$ if and only if $u^m(x,t) \in C(0,T;L^\infty(\Omega)) \cap L^p(0,T;W_0^{1,p}(\Omega))$, $v^n(x,t) \in C(0,T;L^\infty(\Omega)) \cap L^q(0,T;W_0^{1,q}(\Omega))$, $(u^m)_t \in L^2(0,T;L^2(\Omega))$, $(v^n)_t \in L^2(0,T;L^2(\Omega))$, $u(x,0)=u_0(x), v(x,0)=v_0(x)$ and the equalities*

$$\begin{aligned}
&\int_\Omega u(x,t_2)\psi_1(x,t_2)dx - \int_\Omega u(x,t_1)\psi_1(x,t_1)dx = \int_{t_1}^{t_2}\int_\Omega u\psi_{1t}dxdt \\
&- \int_{t_1}^{t_2}\int_\Omega |\nabla u^m|^{p-2}\nabla u^m \cdot \nabla\psi_1 dxdt + a\int_{t_1}^{t_2}\int_\Omega \psi_1(x,t)(u^{\alpha_1}v^{\beta_1} - au^r)dxdt, \quad (2.1)\\
&\int_\Omega v(x,t_2)\psi_2(x,t_2)dx - \int_\Omega v(x,t_1)\psi_2(x,t_1)dx = \int_{t_1}^{t_2}\int_\Omega v\psi_{2t}dxdt \\
&- \int_{t_1}^{t_2}\int_\Omega |\nabla v^n|^{q-2}\nabla v^n \cdot \nabla\psi_2 dxdt + b\int_{t_1}^{t_2}\int_\Omega \psi_2(x,t)(u^{\alpha_2}v^{\beta_2} - bv^s)dxdt \quad (2.2)
\end{aligned}$$

hold for all $0 < t_1 < t_2 < T$, where $\psi_1(x,t), \psi_2(x,t) \in C^{1,1}(\overline{Q}_T)$ such that $\psi_1(x,T) = \psi_2(x,T) = 0$ and $\psi_1(x,t) = \psi_2(x,t) = 0$ on S_T.

Similarly, to define a subsolution $(\underline{u}(x,t), \underline{v}(x,t))$ we need only to require that $\psi_1(x,t) \geq 0, \psi_2(x,t) \geq 0$, $(\underline{u}(x,0), \underline{v}(x,0)) \leq (u_0(x), v_0(x))$ on $\Omega\times\Omega$, $(\underline{u}(x,t), \underline{v}(x,t)) \leq (0,0)$ on $S_T \times S_T$ and the equalities in (2.1) and (2.2) are replaced by $\leq$. A supersolution can be defined similarly.

Definition 2.2 *We say the solution (u,v) of the problem (1.1) blows up in finite time if there exists a positive constant $T^\star < \infty$, such that*

$$\lim_{t \to T^{\star -}} (|u(\cdot,t)|_{L^\infty(\Omega)} + |v(\cdot,t)|_{L^\infty(\Omega)}) = +\infty.$$

We say the solution (u,v) exists globally if

$$\sup_{t \in (0,+\infty)} (|u(\cdot,t)|_{L^\infty(\Omega)} + |v(\cdot,t)|_{L^\infty(\Omega)}) < +\infty.$$

By a modification of the method given in [18, 16, 17], we obtain the following results.

Theorem 2.1 *Suppose that $(u_0, v_0) \geq (0,0)$ and satisfies the conditions (H), then there exists a constant $T_0 > 0$ such that the problem (1.1) admits a unique solution $(u,v) \in Q_{T_0} \times Q_{T_0}$, $u^m \in C(0,T;L^\infty(\Omega)) \cap L^p(0,T;W_0^{1,p}(\Omega))$, $v^n \in C(0,T;L^\infty(\Omega)) \cap L^q(0,T;W_0^{1,q}(\Omega))$.*

Proof of Theorem 2.1. Consider the following approximate problems for the problem (1.1):

$$u_{it} - \operatorname{div}((|\nabla u_i^m|^2 + \varepsilon_i)^{\frac{p-2}{2}}\nabla u_i^m) = u_i^{\alpha_1}v_i^{\beta_1} - au_i^r, (x,t) \in \Omega_T,$$

$$
\begin{aligned}
&v_{it} - \operatorname{div}((|\nabla v_i^n|^2 + \sigma_i)^{\frac{q-2}{2}} \nabla v_i^n) = u_i^{\alpha_2} v_i^{\beta_2} - b v_i^s, (x,t) \in \Omega_T,\\
&u_i(x,t) = \varepsilon_i, v_i(x,t) = \sigma_i, (x,t) \in S_T,\\
&u_i(x,0) = u_{0\varepsilon_i}(x) + \varepsilon_i, v_i(x,0) = v_{0\sigma_i}(x) + \sigma_i, x \in \Omega. \qquad (2.3)
\end{aligned}
$$

Here ε_i, σ_i are strictly decreasing sequences, $0 < \varepsilon_i, \sigma_i < 1$, and $\varepsilon_i \to 0^+, \sigma_i \to 0^+$ as $i \to +\infty$. $u_{0\varepsilon_i}, v_{0\sigma_i} \in C_0^\infty(\Omega)$ are approximation functions for the initial data $u_0(x)$ and $v_0(x)$, respectively. $|u_{0\varepsilon_i} + \varepsilon_i|_{L^\infty(\Omega)} \le |u_0 + 1|_{L^\infty(\Omega)}$, $|\nabla u_{0\varepsilon_i}^m|_{L^\infty(\Omega)} \le |\nabla u_0^m|_{L^\infty(\Omega)}$, for all ε_i, and $(u_{0\varepsilon_i} + \varepsilon_i)^m \to u_0^m$ strongly in $W_0^{1,p}(\Omega)$; $|v_{0\sigma_i} + \sigma_i|_{L^\infty(\Omega)} \le |v_0 + 1|_{L^\infty(\Omega)}$, $|\nabla v_{0\sigma_i}^n|_{L^\infty(\Omega)} \le |\nabla v_0^n|_{L^\infty(\Omega)}$, for all σ_i, and $(v_{0\sigma_i} + \sigma_i)^n \to v_0^n$ strongly in $W_0^{1,q}(\Omega)$.

(2.3) is a non-degenerate problem for each fixed ε_i and σ_i; it is easy to prove that it admits a unique classic solution (u_i, v_i) by using the Schauder's fixed point theorem and $(u_i, v_i) \ge (\varepsilon_i, \sigma_i) > (0,0)$ by the classical theory for parabolic equations(see [10]). To find limit function $u(x,t)$ and $v(x,t)$ of the sequence $\{(u_i, v_i)\}$, we need some priori estimates for the nonnegative approximate solutions by carefully choosing special test functions and a scaling argument. The left arguments are as same as those of Theorem 1 in [16], so we omit them. We complete the existence part by a standard limiting process.

The uniqueness of the solution is obvious. In fact, assume that $(u_1, v_1), (u_2, v_2)$ are two non-negative solutions of (1.1), using Lemma 2.1 repeatedly, we can get $u_1 = u_2, v_1 = v_2$ a.e. in $\overline{\Omega} \times [0, T_0]$.

We first give a comparison lemma for the non-degenerate parabolic system, which plays a crucial role in the proof of our results.

Proposition 2.1 *(Comparison Principle.) Suppose that* $(\underline{u}(x,t), \underline{v}(x,t))$ *and* $(\overline{u}(x,t), \overline{v}(x,t))$ *are the lower and upper solution of problem (1.1) on* $\overline{\Omega}_T \times \overline{\Omega}_T$*, respectively. Then* $(\underline{u}(x,t), \underline{v}(x,t)) \le (\overline{u}(x,t), \overline{v}(x,t))$ *a.e. on* $\overline{\Omega}_T \times \overline{\Omega}_T$*.*

Proof of Proposition 2.1. For small $\sigma > 0$, set $\psi_\sigma(\xi) = \min\{1, \max\{\xi/\sigma, 0\}\}$, $\xi \in \mathbb{R}$. Then $\psi_\sigma(\xi)$ is a piecewise differentiable function. Let $\psi_1 = \psi_\sigma(\underline{u}^m - u^m)$, $\psi_2 = \psi_\sigma(\underline{v}^n - v^n)$, it is easy to verify that ψ_1 and ψ_2 are admissible test functions in (2.1) and (2.2).

Since $(\underline{u}, \underline{v})$ and $(\overline{u}, \overline{v})$ are subsolution and supersolution of (1.1), let $t_1 = \tau$, $t_2 = \tau + h$, $\tau, h > 0$, $\tau + h < T$ and $w = \underline{u} - \overline{u}$, $z = \underline{v} - \overline{v}$, $w_1 = \underline{u}^m - \overline{u}^m$, $z_1 = \underline{v}^n - \overline{v}^n$, then we obtain

$$
\begin{aligned}
&\int_\Omega w(x, \tau+h)\psi_1(x, \tau+h)dx - \int_\Omega w(x,\tau)\psi_1(x,\tau)dx\\
&= \int_\tau^{\tau+h}\int_\Omega w\psi_{1t}dxds - \int_\tau^{\tau+h}\int_\Omega (|\nabla \underline{u}^m|^{p-2}\nabla \underline{u}^m - |\nabla \overline{u}^m|^{p-2}\nabla \overline{u}^m) \cdot \nabla\psi_1 dxds\\
&+ \int_\tau^{\tau+h}\int_\Omega \psi_1(x,t)[(\underline{u}^{\alpha_1}\underline{v}^{\beta_1} - \overline{u}^{\alpha_1}\overline{v}^{\beta_1}) - a(\underline{u}^r - \overline{u}^r)]dxds, \qquad (2.4)
\end{aligned}
$$

$$
\begin{aligned}
&\int_\Omega z(x,\tau+h)\psi_2(x,\tau+h)dx-\int_\Omega z(x,\tau)\psi_2(x,\tau)dx\\
&=\int_\tau^{\tau+h}\int_\Omega z\psi_{2t}dxds-\int_\tau^{\tau+h}\int_\Omega(|\nabla\underline{v}^n|^{q-2}\nabla\underline{v}^n-|\nabla\overline{v}^m|^{q-2}\nabla\overline{v}^n)\cdot\nabla\psi_2dxds\\
&+\int_\tau^{\tau+h}\int_\Omega\psi_2(x,t)[(\underline{u}^{\alpha_2}\underline{v}^{\beta_2}-\overline{u}^{\alpha_2}\overline{v}^{\beta_2})-b(\underline{v}^s-\overline{v}^s)]dxds, \qquad (2.5)
\end{aligned}
$$

Dividing (2.4) and (2.5) by h and integrating τ over $(0,t)$ gives

$$
\begin{aligned}
&\int_0^t\frac{1}{h}\int_\Omega(w(x,\tau+h)\psi_1(x,\tau+h)-w(x,\tau)\psi_1(x,\tau))dxd\tau\\
&=\int_0^t\frac{1}{h}\int_\tau^{\tau+h}\int_\Omega w\psi_{1t}dxdsd\tau\\
&-\int_0^t\frac{1}{h}\int_\tau^{\tau+h}\int_\Omega(|\nabla\underline{u}^m|^{p-2}\nabla\underline{u}^m-|\nabla\overline{u}^m|^{p-2}\nabla\overline{u}^m)\cdot\nabla\psi_1dxdsd\tau\\
&+\int_0^t\frac{1}{h}\int_\tau^{\tau+h}\int_\Omega\psi_1(x,t)[(\underline{u}^{\alpha_1}\underline{v}^{\beta_1}-\overline{u}^{\alpha_1}\overline{v}^{\beta_1})-a(\underline{u}^r-\overline{u}^r)]dxdsd\tau, \qquad (2.6)\\
&\int_0^t\frac{1}{h}\int_\Omega z(x,\tau+h)\psi_2(x,\tau+h)dx-\int_\Omega z(x,\tau)\psi_2(x,\tau)dxd\tau\\
&=\int_0^t\frac{1}{h}\int_\tau^{\tau+h}\int_\Omega z\psi_{2t}dxdsd\tau\\
&-\int_0^t\frac{1}{h}\int_\tau^{\tau+h}\int_\Omega(|\nabla\underline{v}^n|^{q-2}\nabla\underline{v}^n-|\nabla\overline{v}^m|^{q-2}\nabla\overline{v}^n)\cdot\nabla\psi_2dxdsd\tau\\
&+\int_0^t\frac{1}{h}\int_\tau^{\tau+h}\int_\Omega\psi_2(x,t)[(\underline{u}^{\alpha_2}\underline{v}^{\beta_2}-\overline{u}^{\alpha_2}\overline{v}^{\beta_2})-b(\underline{v}^s-\overline{v}^s)]dxdsd\tau. \qquad (2.7)
\end{aligned}
$$

By the properties of Steklov's averages ([5], Lemma 1.3.2), we get

$$
\begin{aligned}
&\int_0^t\frac{1}{h}\int_\tau^{\tau+h}\int_\Omega w\psi_{1t}dxdsd\tau\to\int_0^t\int_\Omega w\psi_{1t}dxds \text{ as } h\to0^+, \qquad (2.8)\\
&\int_0^t\frac{1}{h}\int_\tau^{\tau+h}\int_\Omega z\psi_{2t}dxdsd\tau\to\int_0^t\int_\Omega z\psi_{2t}dxds \text{ as } h\to0^+, \qquad (2.9)\\
&\int_0^t\frac{1}{h}\int_\tau^{\tau+h}\int_\Omega(|\nabla\underline{u}^m|^{p-2}\nabla\underline{u}^m-|\nabla\overline{u}^m|^{p-2}\nabla\overline{u}^m)\cdot\nabla\psi_1dxdsd\tau\\
&\to\int_0^t\int_\Omega(|\nabla\underline{u}^m|^{p-2}\nabla\underline{u}^m-|\nabla\overline{u}^m|^{p-2}\nabla\overline{u}^m)\cdot\nabla\psi_1dxds \text{ as } h\to0^+, \qquad (2.10)\\
&\int_0^t\frac{1}{h}\int_\tau^{\tau+h}\int_\Omega(|\nabla\underline{v}^n|^{q-2}\nabla\underline{v}^n-|\nabla\overline{v}^m|^{q-2}\nabla\overline{v}^n)\cdot\nabla\psi_2dxdsd\tau\\
&\to\int_0^t\int_\Omega(|\nabla\underline{v}^n|^{q-2}\nabla\underline{v}^n-|\nabla\overline{v}^m|^{q-2}\nabla\overline{v}^n)\cdot\nabla\psi_2dxds \text{ as } h\to0^+, \qquad (2.11)\\
&\int_0^t\frac{1}{h}\int_\tau^{\tau+h}\int_\Omega\psi_1(x,t)[(\underline{u}^{\alpha_1}\underline{v}^{\beta_1}-\overline{u}^{\alpha_1}\overline{v}^{\beta_1})-a(\underline{u}^r-\overline{u}^r)]dxdsd\tau
\end{aligned}
$$

$$\to \int_0^t \int_\Omega \psi_1(x,t)[(\underline{u}^{\alpha_1}\underline{v}^{\beta_1} - \overline{u}^{\alpha_1}\overline{v}^{\beta_1}) - a(\underline{u}^r - \overline{u}^r)]dxds \text{ as } h \to 0^+,$$
$$\int_0^t \frac{1}{h}\int_\tau^{\tau+h} \int_\Omega \psi_2(x,t)[(\underline{u}^{\alpha_2}\underline{v}^{\beta_2} - \overline{u}^{\alpha_2}\overline{v}^{\beta_2}) - b(\underline{v}^s - \overline{v}^s)]dxdsd\tau,$$
$$\to \int_0^t \int_\Omega \psi_2(x,t)[(\underline{u}^{\alpha_2}\underline{v}^{\beta_2} - \overline{u}^{\alpha_2}\overline{v}^{\beta_2}) - b(\underline{v}^s - \overline{v}^s)]dxds \text{ as } h \to 0^+. \tag{2.12}$$

Now we claim that

$$\int_0^t \frac{1}{h}\int_\Omega (w(x,\tau+h)\psi_1(x,\tau+h) - w(x,\tau)\psi_1(x,\tau))dxd\tau$$
$$\to \int_\Omega (w(x,t)\psi_1(x,t) - w(x,0)\psi_1(x,0))dx, \tag{2.13}$$
$$\int_0^t \frac{1}{h}\int_\Omega z(x,\tau+h)\psi_2(x,\tau+h)dx - \int_\Omega z(x,\tau)\psi_2(x,\tau)dxd\tau$$
$$\to \int_\Omega z(x,t)\psi_2(x,t)dx - \int_\Omega z(x,0)\psi_2(x,0)dx. \tag{2.14}$$

By (2.4)-(2.14), we obtain

$$\int_\Omega w(x,t)\psi_\sigma(w_1(x,t))dx \le \int_\Omega w(x,0)\psi_\sigma(w_1(x,0))dx + \int_0^t\int_\Omega w\psi'_\sigma(w_1)w_{1s}dxds$$
$$- \int_0^t\int_\Omega (|\nabla \underline{u}^m|^{p-2}\nabla\underline{u}^m - |\nabla\overline{u}^m|^{p-2}\nabla\overline{u}^m)\cdot\nabla\psi_\sigma(\underline{u}^m - \overline{u}^m)dxds$$
$$+ \int_0^t\int_\Omega \psi_\sigma(w_1(x,t))[(\underline{u}^{\alpha_1}\underline{v}^{\beta_1} - \overline{u}^{\alpha_1}\overline{v}^{\beta_1}) - a(\underline{u}^r - \overline{u}^r)]dxds, \tag{2.15}$$
$$\int_\Omega z(x,t)\psi_\sigma(z_1(x,t))dx \le \int_\Omega z(x,0)\psi_\sigma(z_1(x,0))dx + \int_0^t\int_\Omega z\psi'_\sigma(z_1)z_{1s}dxds$$
$$- \int_0^t\int_\Omega (|\nabla \underline{v}^n|^{q-2}\nabla\underline{v}^n - |\nabla\overline{v}^n|^{q-2}\nabla\overline{v}^n)\cdot\nabla\psi_\sigma(\underline{v}^n - \overline{v}^n)dxds$$
$$+ \int_0^t\int_\Omega \psi_\sigma(z_1(x,t))[(\underline{u}^{\alpha_2}\underline{v}^{\beta_2} - \overline{u}^{\alpha_2}\overline{v}^{\beta_2}) - b(\underline{v}^s - \overline{v}^s)]dxds. \tag{2.16}$$

Now we deal with the terms in (2.15) and (2.16). First, we have

$$\int_0^t\int_\Omega \psi_\sigma(w_1(x,t))[(\underline{u}^{\alpha_1}\underline{v}^{\beta_1} - \overline{u}^{\alpha_1}\overline{v}^{\beta_1}) - a(\underline{u}^r - \overline{u}^r)]dxds$$
$$\le \beta_1 M_1^{\alpha_1}M_2^{\beta_1-1}\int_0^t\int_\Omega (\underline{v}-\overline{v})_+dx + \alpha_1 M_1^{\alpha_1-1}M_2^{\beta_1}\int_0^t\int_\Omega(\underline{u}-\overline{u})_+dxds$$
$$+ arM_1^{r-1}\int_0^t\int_\Omega(\underline{u}-\overline{u})_+dxds,$$
$$\int_0^t\int_\Omega \psi_\sigma(z_1(x,t))[(\underline{u}^{\alpha_2}\underline{v}^{\beta_2} - \overline{u}^{\alpha_2}\overline{v}^{\beta_2}) - b(\underline{v}^s - \overline{v}^s)]dxds$$

$$
\begin{aligned}
&\leq \alpha_2 M_1^{\alpha_2-1}M_2^{\beta_2}\int_0^t\int_\Omega(\underline{u}-\overline{u})_+dx+\beta_2 M_1^{\alpha_2}M_2^{\beta_2-1}\int_0^t\int_\Omega(\underline{v}-\overline{v})_+dxds\\
&+bsM_2^{s-1}\int_0^t\int_\Omega(\underline{v}-\overline{v})_+dxds
\end{aligned}
$$

for some positive constants M_1, M_2, and as $\sigma\to 0^+$,

$$
\begin{aligned}
&\left|\int_0^t\int_\Omega w\psi'_\sigma(w_1)w_{1s}dxds\right|\leq\int_0^t\int_\Omega w_+|\psi'_\sigma(w_1)||w_{1s}|dxds\\
&=\frac{1}{\sigma}\int_0^\sigma\int_\Omega w_+|w_{1s}|dxds\to 0,\\
&\left|\int_0^t\int_\Omega z\psi'_\sigma(z_1)z_{1s}dxds\right|\leq\int_0^t\int_\Omega z_+|\psi'_\sigma(z_1)||z_{1s}|dxds\\
&=\frac{1}{\sigma}\int_0^\sigma\int_\Omega z_+|z_{1s}|dxds\to 0.
\end{aligned}
$$

Second, by Lemma 1.4.4 in [5], we get

$$
\begin{gathered}
(|\nabla\underline{u}^m|^{p-2}\nabla\underline{u}^m-|\nabla\overline{u}^m|^{p-2}\nabla\overline{u}^m)\cdot\nabla\psi_\sigma(\underline{u}^m-\overline{u}^m)\geq\min\{0,\gamma_1|\nabla(\underline{u}^m-\overline{u}^m)_+|^p\},\\
(|\nabla\underline{v}^n|^{q-2}\nabla\underline{v}^n-|\nabla\overline{v}^n|^{q-2}\nabla\overline{v}^n)\cdot\nabla\psi_\sigma(\underline{v}^n-\overline{v}^n)\geq\min\{0,\gamma_2|\nabla(\underline{v}^n-\overline{v}^n)_+|^q\}
\end{gathered}
$$

for some $\gamma_1,\gamma_2>0$.

Finally, we have $\int_\Omega w(x,0)\psi_\sigma(w_1(x,0))dx\equiv 0$, $\int_\Omega z(x,0)\psi_\sigma(z_1(x,0))dx\equiv 0$ and $\psi'_\sigma\geq 0$ a.e. in R, $w\psi'_\sigma(w_1)w_{1s}$, $z\psi'_\sigma(z_1)z_{1s}$ increase and tend to w_+, z_+ as $\sigma\to 0^+$, respectively. Hence we may let $\sigma\to 0^+$ in (2.15) and (2.16) to yield

$$
\begin{aligned}
&\int_\Omega w_+(x,t)dx\leq C_1\int_0^t\int_\Omega w_+(x,s)dxds+C_2\int_0^t\int_\Omega z_+(x,s)dxds,\\
&\int_\Omega z_+(x,t)dx\leq C_3\int_0^t\int_\Omega w_+(x,s)dxds+C_4\int_0^t\int_\Omega z_+(x,s)dxds.
\end{aligned}
$$

Hence,

$$
\int_\Omega(w_+(x,t)+z_+(x,t))dx\leq C\int_0^t\int_\Omega(w_+(x,s)+z_+(x,s))dxds.
$$

By the Gronwall's inequality we obtain $\int_\Omega(w_+(x,t)+z_+(x,t))dx=0$, i.e. $\underline{u}\leq\overline{u}$, $\underline{v}\leq\overline{v}$, a.e. on $\overline{\Omega}_T$. This completes the proof.

3 Proof of Theorem 1.1

In this section, we investigate the global existence property of the solutions to problem (1.1) and prove Theorem 1.1. The main method is constructing a globally upper solution and using comparison principle to achieve our purpose.

In order to study the globally existing solutions to problem (1.1), we need to study the following elliptic system

$$-\Delta_{k,\gamma}\Theta = 1,\ x \in \Omega, \quad \Theta = 1,\ x \in \partial\Omega, \tag{3.1}$$

where $\Delta_{k,\gamma}\Theta$ is defined in (1.1), and we obtain the following lemma.

Lemma 3.1 *problem (3.1) has a unique solution $\Theta(x)$, and satisfies the following relations,*

$$\Theta(x) > 1 \ \text{in}\ \Omega,\ \nabla\Theta\cdot\nu < 0 \ \text{on}\ \partial\Omega,\ \sup_{x\in\Omega}\Theta = M < +\infty,$$

where M is a positive constant.

Proof of this lemma is similar to that given in [23], we omit it here.

Proof of Theorem 1.1. Let $\varphi(x)$ and $\psi(x)$ be the unique solution of the following elliptic problem

$$\begin{cases} -\Delta_{m,p}\varphi = 1, & x \in \Omega, \\ \varphi = 1, & x \in \partial\Omega, \end{cases} \qquad \begin{cases} -\Delta_{n,q}\psi = 1, & x \in \Omega, \\ \psi = 1, & x \in \partial\Omega. \end{cases} \tag{3.2}$$

Then from Lemma 3.1, we obtain the following relations

$$\varphi(x), \psi(x) > 1 \text{ in } \Omega, \quad \nabla\varphi\cdot\nu, \nabla\psi\cdot\nu < 0 \text{ on } \partial\Omega, \tag{3.3}$$

$$M_1 = \min\{\inf_{x\in\Omega}\varphi,\ \inf_{x\in\Omega}\psi\} < +\infty, M_2 = \max\{\sup_{x\in\Omega}\varphi,\ \sup_{x\in\Omega}\psi\} < +\infty, \tag{3.4}$$

where $M_1, M_2 > 0$ is a positive constant.

Notice that $(1/\tau, 1/\theta) < (0,0)$ implies

$$\beta_1\alpha_2 < \mu_1\mu_2 = \max\{m(p-1)-\alpha_1, r-\alpha_1\}\max\{n(q-1)-\beta_2, s-\beta_2\}.$$

We will prove Theorem 1.1 in four subcases.

(a) For $\mu_1 = r-\alpha_1$, $\mu_2 = s-\beta_2$, we then have $\beta_1\alpha_2 < (r-\alpha_1)(s-\beta_2)$. Let $(\overline{u}, \overline{v}) = (\Lambda_1, \Lambda_2)$, where $\Lambda_1 \geq \max_{x\in\overline{\Omega}} u_0(x)$, $\Lambda_2 \geq \max_{x\in\overline{\Omega}} v_0(x)$ will be determined later. After a simple computation, we have

$$\overline{u}_t - \Delta_{m,p}\overline{u} - \overline{u}^{\alpha_1}\overline{v}^{\beta_1} + a\overline{u}^r = a\Lambda_1^r - \Lambda_1^{\alpha_1}\Lambda_2^{\beta_1},$$
$$\overline{v}_t - \Delta_{n,q}\overline{v} - \overline{u}^{\alpha_2}\overline{v}^{\beta_2} + b\overline{v}^s = b\Lambda_2^s - \Lambda_1^{\alpha_2}\Lambda_2^{\beta_2}.$$

So, $(\overline{u}, \overline{v}) = (\Lambda_1, \Lambda_2)$ is a time-independent supersolution of problem (1.1) if

$$a\Lambda_1^{r-\alpha_1} \geq \Lambda_2^{\beta_1} \text{ and } b\Lambda_2^{s-\beta_2} \geq \Lambda_1^{\alpha_2},$$

i.e.

$$\Lambda_2^{\frac{\beta_1}{r-\alpha_1}}(\frac{1}{a})^{\frac{1}{r-\alpha_1}} \le \Lambda_1 \le \Lambda_2^{\frac{s-\beta_2}{\alpha_2}}(b)^{\frac{1}{\alpha_2}}.$$

(b) For $\mu_1 = m(p-1) - \alpha_1$, $\mu_2 = n(q-1) - \beta_2$, we then have $\beta_1\alpha_2 < mn(p-1)(q-1)$. Let $(\overline{u}, \overline{v}) = (\Lambda_1\varphi(x), \Lambda_2\psi(x))$, where $\Lambda_1, \Lambda_2 > 0$ will be determined later. Then with a direct computation we obtain

$$\begin{aligned}
\overline{u}_t - \Delta_{m,p}\overline{u} - \overline{u}^{\alpha_1}\overline{v}^{\beta_1} + a\overline{u}^r &\ge \Lambda_1^{m(p-1)} - \Lambda_1^{\alpha_1}\Lambda_2^{\beta_1}M_2^{\alpha_1+\beta_1},\\
\overline{v}_t - \Delta_{n,q}\overline{v} - \overline{u}^{\alpha_2}\overline{v}^{\beta_2} + b\overline{v}^s &\ge \Lambda_2^{n(q-1)} - \Lambda_1^{\alpha_2}\Lambda_2^{\beta_2}M_2^{\alpha_2+\beta_2},
\end{aligned}$$

So, $(\overline{u}(x,t), \overline{v}(x,t))$ is an upper solution of problem (1.1), if

$$\begin{aligned}
&\Lambda_1^{m(p-1)} \ge \Lambda_1^{\alpha_1}\Lambda_2^{\beta_1}M_2^{\alpha_1+\beta_1},\ \Lambda_2^{n(q-1)} \ge \Lambda_1^{\alpha_2}\Lambda_2^{\beta_2}M_2^{\alpha_2+\beta_2},\\
&\overline{u}(x,t)\mid_{\partial\Omega} \ge 0,\ \overline{v}(x,t)\mid_{\partial\Omega} \ge 0,\ \overline{u}(x,0) = u_0(x),\ \overline{v}(x,0) = v_0(x).
\end{aligned} \tag{3.5}$$

Then (3.5) holds if we choose Λ_1, Λ_2 large enough such that

$$\begin{aligned}
\Lambda_1 &> \max\{\max_{x\in\overline{\Omega}} u_0(x), \big(M_2^{\alpha_1+\beta_1+\frac{(\alpha_2+\beta_2)\beta_1}{n(q-1)-\beta_2}}\big)^{\frac{1}{m(p-1)-\alpha_1-\frac{\alpha_2\beta_1}{n(q-1)-\beta_2}}}\},\\
\Lambda_2 &> \max\{\max_{x\in\overline{\Omega}} v_0(x), \big(M_2^{\alpha_2+\beta_2+\frac{(\alpha_1+\beta_1)\alpha_2}{m(p-1)-\alpha_1}}\big)^{\frac{1}{n(q-1)-\beta_2-\frac{\alpha_2\beta_1}{m(p-1)-\alpha_1}}}\}.
\end{aligned}$$

(c) For $\mu_1 = r - \alpha_1$, $\mu_2 = n(q-1) - \beta_2$, we then have $\beta_1\alpha_2 < (r-\alpha_1)[n(q-1)-\beta_2]$. Choose $\Lambda_1 \ge \max_{x\in\overline{\Omega}} u_0(x)$ and $\Lambda_2 \ge \max_{x\in\overline{\Omega}} v_0(x)$ satisfy

$$(\Lambda_1^{\alpha_2}M_2^{\beta_2})^{\frac{1}{n(q-1)-\beta_2}} \le \Lambda_2 \le (a\Lambda_1^{r-\alpha_1}M_2^{-\beta_1})^{\frac{1}{\beta_1}}.$$

Let $(\overline{u}, \overline{v}) = (\Lambda_1, \Lambda_2\psi(x))$ with $\psi(x)$ defined by (3.2). By direct computation, we arrive at

$$\begin{aligned}
\overline{u}_t - \Delta_{m,p}\overline{u} - \overline{u}^{\alpha_1}\overline{v}^{\beta_1} + a\overline{u}^r &\ge a\Lambda_1^r - \Lambda_1^{\alpha_1}\Lambda_2^{\beta_1}M_2^{\beta_1} \ge 0,\\
\overline{v}_t - \Delta_{n,q}\overline{v} - \overline{u}^{\alpha_2}\overline{v}^{\beta_2} + b\overline{v}^s &\ge \Lambda_2^{n(q-1)} - \Lambda_1^{\alpha_2}\Lambda_2^{\beta_2}M_2^{\beta_2} \ge 0.
\end{aligned} \tag{3.6}$$

(d) For $\mu_1 = m(p-1) - \alpha_1$, $\mu_2 = s - \beta_2$, we then have $\beta_1\alpha_2 < [m(p-1)-\alpha_1](s-\beta_2)$. Let $(\overline{u}, \overline{v}) = (\Lambda_1\varphi(x), \Lambda_2)$ with $\varphi(x)$ defined by (3.2), where $\Lambda_1 \ge \max_{x\in\overline{\Omega}} u_0(x)$ and $\Lambda_2 \ge \max_{x\in\overline{\Omega}} v_0(x)$. Then, (3.6) hold if

$$(\Lambda_2^{\alpha_1}M_2^{\beta_1})^{\frac{1}{m(p-1)-\beta_1}} \le \Lambda_1 \le (b\Lambda_2^{s-\alpha_2}M_2^{-\beta_2})^{\frac{1}{\beta_2}}.$$

The proof of Theorem 1.1 is complete. □

4 Proof of Theorem 1.2

In this section, we investigate the blow-up property of the solutions to problem (1.1) and prove Theorem 1.2. The main method is constructing a blowing-up lower solution and using the comparison principle to achieve our purpose.
Proof of Theorem 1.2. Observe that $(1/\tau, 1/\theta) > (0,0)$ implies

$$\beta_1\alpha_2 > \mu_1\mu_2 = \max\{m(p-1)-\alpha_1, r-\alpha_1\}\max\{n(q-1)-\beta_2, s-\beta_2\}.$$

For $\mu_1 = r-\alpha_1$, $\mu_2 = s-\beta_2$. Choosing

$$\Lambda_1 = \frac{1}{2}\left[(\frac{1}{a})^{\frac{1}{r-\alpha_1}}\Lambda_2^{\frac{\beta_1}{r-\alpha_1}} + b^{\frac{1}{\alpha_2}}\Lambda_2^{\frac{s-\beta_2}{\alpha_2}}\right], \Lambda_2 = \left(a^{\alpha_2}b^{r-\alpha_1}\right)^{\frac{1}{\beta_1\alpha_2-(r-\alpha_1)(s-\beta_2)}},$$

then $(\overline{u},\overline{v}) = (\Lambda_1,\Lambda_2)$ is a global supersolution for problem (1.1) provided that $\Lambda_1 \geq \max_{x\in\overline{\Omega}} u_0(x)$ and $\Lambda_2 \geq \max_{x\in\overline{\Omega}} v_0(x)$.

For $\mu_1 = m(p-1)-\alpha_1$, $\mu_2 = n(q-1)-\beta_2$. Let $(\overline{u},\overline{v}) = (\Lambda_1\varphi(x), \Lambda_2\psi(x))$, where $\varphi(x)$ and $\psi(x)$ satisfying (3.2), respectively. Choosing

$$\Lambda_1 = \frac{1}{2}\left(M_2^{\frac{\alpha_1+\beta_1}{m(p-1)-\alpha_1}}\Lambda_2^{\frac{\beta_1}{m(p-1)-\alpha_1}} + M_2^{-\frac{\alpha_2+\beta_2}{\alpha_2}}\Lambda_2^{\frac{n(q-1)-\beta_2}{\alpha_2}}\right),$$

$$\Lambda_2 = \left(M_2^{\alpha_2+\beta_2+\frac{(\alpha_1+\beta_1)\alpha_2}{m(p-1)-\alpha_1}}\right)^{\frac{1}{n(q-1)-\beta_2-\frac{\alpha_2\beta_1}{m(p-1)-\alpha_1}}},$$

therefore, $(\overline{u},\overline{v})$ is a global supersolution for system (1.1) if $\Lambda_1 \geq \max_{x\in\overline{\Omega}} u_0(x)$ and $\Lambda_2 \geq \max_{x\in\overline{\Omega}} v_0(x)$.

For other cases, the solutions of (1.1) should be global due to the above discussion.

Next, we begin to prove our blow-up conclusion under large enough initial data. Due to the requirement of the comparison principle, we will construct blow-up subsolutions in some subdomain of Ω in which $u, v > 0$. We use an idea from Souplet [14] and apply it to degenerate equations. Since problem (1.1) does not make sense for negative values of (u,v), we actually consider the following problem

$$\begin{aligned}
&Pu(x,t) \equiv u_t - \Delta_{m,p}u - u_+^{\alpha_1}v_+^{\beta_1} + au_+^r, && x\in\Omega, t>0,\\
&Qv(x,t) \equiv v_t - \Delta_{n,q}v - u_+^{\alpha_2}v_+^{\beta_2} + bv_+^s, && x\in\Omega, t>0,\\
&u(x,t) = v(x,t) = 0, && x\in\partial\Omega, t>0,\\
&u(x,0) = u_0(x), \quad v(x,0) = v_0(x), && x\in\overline{\Omega},
\end{aligned}$$

where $u_+ = \max\{0,u\}$, $v_+ = \max\{0,v\}$. Let $\varpi(x)$ be a nontrivial nonnegative continuous function and vanish on $\partial\Omega$. Without loss of generality, we may assume that $0 \in \Omega$ and $\varpi(0) > 0$. We shall construct a self-similar blow-up subsolution to complete our proof.

Set

$$\underline{u}(x,t) = (\tau - t)^{-\gamma_1} V_1(\xi), \xi = |x|(\tau - t)^{-\sigma_1}, V_1(\xi) = (1 + \frac{A}{2} - \frac{\xi^2}{2A})_+^{1/m},$$
$$\underline{v}(x,t) = (\tau - t)^{-\gamma_2} V_2(\eta), \eta = |x|(\tau - t)^{-\sigma_2}, V_2(\eta) = (1 + \frac{A}{2} - \frac{\eta^2}{2A})_+^{1/n},$$

where $\gamma_i, \sigma_i > 0 (i = 1, 2)$, $A > 1$ and $0 < \tau < 1$ are parameters to be determined. It is easy to see that $\underline{u}(x,t), \underline{v}(x,t)$ blow up at time τ, so it is enough to prove that $(\underline{u}(x,t), \underline{v}(x,t))$ is a lower solution of problem (1.1). If we choose τ small enough such that

$$\mathrm{supp}\underline{u}(\cdot,t) = \overline{B(0, R(\tau - t)^{\sigma_1})} \subset \overline{B(0, R\tau^{\sigma_1})} \subset \Omega,$$
$$\mathrm{supp}\underline{v}(\cdot,t) = \overline{B(0, R(\tau - t)^{\sigma_2})} \subset \overline{B(0, R\tau^{\sigma_2})} \subset \Omega,$$

where $R = (A(2+A))^{1/2}$, then $\underline{u}(x,t) \mid_{\partial\Omega} = 0, \underline{v}(x,t) \mid_{\partial\Omega} = 0$. Next if we choose the initial data large enough such that

$$u_0(x) \geq \frac{1}{\tau^{\gamma_1}} V_1(\frac{|x|}{\tau^{\sigma_1}}), \; v_0(x) \geq \frac{1}{\tau^{\gamma_2}} V_2(\frac{|x|}{\tau^{\sigma_2}}),$$

then $(\underline{u}(x,t), \underline{v}(x,t))$ is a lower solution of problem (1.1) if for any $(x,t) \in \Omega \times (0, \tau]$,

$$\underline{u}_t - \Delta_{m,p}\underline{u} \leq a\underline{u}^{\alpha_1}\underline{v}^{\beta_1}, \tag{4.1}$$
$$\underline{v}_t - \Delta_{n,q}\underline{v} \leq b\underline{u}^{\alpha_2}\underline{v}^{\beta_2}. \tag{4.2}$$

After a direct computation, we obtain

$$\begin{aligned}
\underline{u}_t &= \frac{\gamma_1 V_1(\xi) + \sigma_1 \xi V_1'(\xi)}{(\tau - t)^{\gamma_1 + 1}}, \quad \nabla \underline{u}^m = \frac{x}{A(\tau - t)^{m\gamma_1 + 2\sigma_1}}, -\Delta \underline{u}^m = \frac{N}{A(\tau - t)^{m\gamma_1 + 2\sigma_1}}, \\
\underline{v}_t &= \frac{\gamma_2 V_2(\eta) + \sigma_2 \eta V_2'(\eta)}{(\tau - t)^{\gamma_2 + 1}}, \nabla \underline{v}^n = \frac{x}{A(\tau - t)^{n\gamma_2 + 2\sigma_2}}, -\Delta \underline{v}^n = \frac{N}{A(\tau - t)^{n\gamma_2 + 2\sigma_2}},
\end{aligned} \tag{4.3}$$

$$\begin{aligned}
&- \Delta_{m,p}\underline{u} = |\nabla \underline{u}^m|^{p-2} \Delta \underline{u}^m + (p-2)|\nabla \underline{u}^m|^{p-4} (\nabla \underline{u}^m)^\tau \cdot (H_x(\underline{u}^m)) \cdot \nabla \underline{u}^m \\
&= |\nabla \underline{u}^m|^{p-2} \Delta \underline{u}^m + (p-2)|\nabla \underline{u}^m|^{p-4} \sum_{j=1}^N \sum_{i=1}^N \frac{\partial \underline{u}^m}{\partial x_i} \frac{\partial^2 \underline{u}^m}{\partial x_i \partial x_j} \frac{\partial \underline{u}^m}{\partial x_j},
\end{aligned} \tag{4.4}$$

$$\begin{aligned}
&- \Delta_{n,q}\underline{v} = |\nabla \underline{v}^n|^{q-2} \Delta \underline{v}^n + (q-2)|\nabla \underline{v}^n|^{q-4} (\nabla \underline{v}^n)^\tau \cdot (H_x(\underline{v}^n)) \cdot \nabla \underline{v}^n \\
&= |\nabla \underline{v}^n|^{q-2} \Delta \underline{v}^n + (q-2)|\nabla \underline{v}^n|^{q-4} \sum_{j=1}^N \sum_{i=1}^N \frac{\partial \underline{v}^n}{\partial x_i} \frac{\partial^2 \underline{v}^n}{\partial x_i \partial x_j} \frac{\partial \underline{v}^n}{\partial x_j},
\end{aligned} \tag{4.5}$$

where $H_x(\underline{u}^m)$, $H_x(\underline{v}^n)$ denote the Hessian matrix of $\underline{u}^m(x,t)$, $\underline{v}^n(x,t)$, respectively.

Use the notation $d(\Omega)=\mathrm{diam}(\Omega)$, then from (4.4) and (4.5), we obtain

$$\begin{aligned}|\Delta_{m,p}\underline{u}| \le & \frac{N}{A(\tau-t)^{m\gamma_1+2\sigma_1}}\big(\frac{d(\Omega)}{(\tau-t)^{m\gamma_1+2\sigma_1}}\big)^{p-2}\\ &+(p-2)\big(\frac{d(\Omega)}{(\tau-t)^{m\gamma_1+2\sigma_1}}\big)^{p-4}\big(\frac{d(\Omega)}{(\tau-t)^{m\gamma_1+2\sigma_1}}\big)^{2}\frac{N}{A(\tau-t)^{m\gamma_1+2\sigma_1}}\\ =&\frac{N(p-1)(d(\Omega))^{p-2}}{A(\tau-t)^{(m\gamma_1+2\sigma_1)(p-1)}}.\end{aligned} \tag{4.6}$$

Similarly, from (4.4) and (4.5) we obtain

$$\begin{aligned}|\Delta_{n,q}\underline{v}| \le & \frac{N}{A(\tau-t)^{n\gamma_2+2\sigma_2}}\big(\frac{d(\Omega)}{(\tau-t)^{n\gamma_2+2\sigma_2}}\big)^{q-2}\\ &+(q-2)\big(\frac{d(\Omega)}{(\tau-t)^{n\gamma_2+2\sigma_2}}\big)^{q-4}\big(\frac{d(\Omega)}{(\tau-t)^{n\gamma_2+2\sigma_2}}\big)^{2}\frac{N}{A(\tau-t)^{n\gamma_2+2\sigma_2}}\\ =&\frac{N(q-1)(d(\Omega))^{q-2}}{A(\tau-t)^{(n\gamma_2+2\sigma_2)(q-1)}}.\end{aligned} \tag{4.7}$$

Next, we compute the local term of (4.1)

$$\begin{aligned}\underline{u}^{\alpha_1}\underline{v}^{\beta_1} &= \frac{1}{(\tau-t)^{\gamma_1\alpha_1+\gamma_2\beta_1}}V_1^{\alpha_1}\big(\frac{|x|}{(\tau-t)^{\sigma_1}}\big)V_2^{\beta_1}\big(\frac{|x|}{(\tau-t)^{\sigma_2}}\big),\\ \underline{u}^{\alpha_2}\underline{v}^{\beta_2} &= \frac{1}{(\tau-t)^{\gamma_1\alpha_2+\gamma_2\beta_2}}V_1^{\alpha_2}\big(\frac{|x|}{(\tau-t)^{\sigma_1}}\big)V_2^{\beta_2}\big(\frac{|x|}{(\tau-t)^{\sigma_2}}\big).\end{aligned} \tag{4.8}$$

If $0\le \xi,\eta\le A$, then $1\le V_1(\xi)\le (1+A/2)^{1/m}$, $1\le V_2(\eta)\le(1+A/2)^{1/n}$ and $V_1'(\xi)\le 0$, $V_2'(\eta)\le 0$. Combining the above inequalities, we obtain

$$\begin{aligned}P\underline{u}(x,t)\le & \frac{\gamma_1(1+\frac{A}{2})^{1/m}}{(\tau-t)^{\gamma_1+1}}+\frac{N(p-1)(d(\Omega))^{p-2}}{A(\tau-t)^{(m\gamma_1+2\sigma_1)(p-1)}}+\frac{a(1+\frac{A}{2})^{r/m}}{(\tau-t)^{r\gamma_1}}\\ &-\frac{1}{(\tau-t)^{\gamma_1\alpha_1+\gamma_2\beta_1}},\end{aligned} \tag{4.9}$$

$$\begin{aligned}Q\underline{v}(x,t)\le & \frac{\gamma_2(1+\frac{A}{2})^{1/n}}{(\tau-t)^{\gamma_2+1}}+\frac{N(q-1)(d(\Omega))^{q-2}}{A(\tau-t)^{(n\gamma_2+2\sigma_2)(q-1)}}+\frac{b(1+\frac{A}{2})^{r/n}}{(\tau-t)^{s\gamma_2}}\\ &-\frac{1}{(\tau-t)^{\gamma_1\alpha_2+\gamma_2\beta_2}}.\end{aligned} \tag{4.10}$$

If $\xi,\eta\ge A$, since $m,n\ge 1$, we obtain $V_1(\xi)\le 1$, $V_2(\eta)\le 1$ and $V_1'(\xi)\le -1/m$, $V_2'(\eta)\le -1/n$. Combining the above inequalities (4.3)-(4.8), we obtain

$$P\underline{u}(x,t)\le \frac{\gamma_1-\frac{1}{m}\sigma_1 A}{(\tau-t)^{\gamma_1+1}}+\frac{N(p-1)(d(\Omega))^{p-2}}{A(\tau-t)^{(m\gamma_1+2\sigma_1)(p-1)}}+\frac{a}{(\tau-t)^{r\gamma_1}}, \tag{4.11}$$

$$Q\underline{v}(x,t)\le \frac{\gamma_2-\frac{1}{n}\sigma_2 A}{(\tau-t)^{\gamma_2+1}}+\frac{N(q-1)(d(\Omega))^{q-2}}{A(\tau-t)^{(n\gamma_2+2\sigma_2)(q-1)}}+\frac{b}{(\tau-t)^{s\gamma_2}}. \tag{4.12}$$

If $0 \le \xi \le A$ and $\eta \ge A$, we have that (4.9) and (4.12) hold. If $\xi \ge A$ and $0 \le \eta \le A$, we have that (4.10) and (4.11) hold.

So, from the above discussions, (4.1) hold if the right-hand sides of (4.9)-(4.12) are nonpositive.

Since $1/\tau, 1/\theta < 0$, we see that $\beta_1\alpha_2 > \mu_1\mu_2$. In addition, it is clear that

$$\frac{\mu_1}{\beta_1} < \frac{\alpha_2+1}{\beta_1+1} \text{ or } \frac{\mu_2}{\alpha_2} < \frac{\beta_1+1}{\alpha_2+1}. \tag{4.13}$$

For $\mu_1/\beta_1 < (\alpha_2+1)/(\beta_1+1)$, we choose γ_1 and γ_2 such that

$$\begin{aligned} \frac{\mu_1}{\beta_1} < \frac{\gamma_2}{\gamma_1} &< \min\{\frac{\alpha_2+1}{\beta_1+1}, \frac{\alpha_2}{\mu_2}\}, \\ \alpha_1+\mu_1 < \frac{1+\gamma_1}{\gamma_1} &< \min\{\frac{r}{\gamma_1(r-1)}, \frac{\beta_1\gamma_2+\alpha_1\gamma_1}{\gamma_1}\}. \end{aligned} \tag{4.14}$$

Recall that $\mu_1 = \max\{m(p-1)-\alpha_1, r-\alpha_1\}$ and $\mu_2 = \max\{n(q-1)-\beta_2, s-\beta_2\}$, then (4.14) implies

$$\begin{aligned} &\beta_1\gamma_2+\alpha_1\gamma_1 > r\gamma_1, \beta_1\gamma_2+\alpha_1\gamma_1 > m(p-1)\gamma_1, \beta_1\gamma_2+\alpha_1\gamma_1 > \gamma_1+1 > r\gamma_1, \\ &\beta_2\gamma_2+\alpha_2\gamma_1 > s\gamma_2, \beta_2\gamma_2+\alpha_2\gamma_1 > n(q-1)\gamma_2, \beta_2\gamma_2+\alpha_2\gamma_1 > \gamma_2+1 > s\gamma_2. \end{aligned}$$

Next, we can choose positive constants σ_1, σ_2 sufficiently small such that

$$\begin{aligned} \sigma_1 = \sigma_2 < \min\Big\{ &\frac{\beta_1\gamma_2+\alpha_1\gamma_1-\gamma_1-1}{2N}, \frac{\beta_1\gamma_2+\alpha_1\gamma_1-m(p-1)\gamma_1}{2(N+p-1)}, \\ &\frac{\beta_1\gamma_2+\alpha_1\gamma_1-r\gamma_1}{2N}, \frac{\gamma_1+1+m(p-1)\gamma_1}{2(p-1)}, \frac{\beta_2\gamma_2+\alpha_2\gamma_1-\gamma_2-1}{2N}, \\ &\frac{\beta_2\gamma_2+\alpha_2\gamma_1-n(q-1)\gamma_2}{2(N+q-1)}, \frac{\beta_2\gamma_2+\alpha_2\gamma_1-s\gamma_2}{2N}, \frac{\gamma_2+1+n(q-1)\gamma_2}{2(q-1)} \Big\}, \end{aligned}$$

consequently, we have

$$\begin{aligned} \beta_1\gamma_2+\alpha_1\gamma_1 &> \max\Big\{\gamma_1+1, (m\gamma_1+2\sigma_1)(p-1), r\gamma_1\Big\}, \\ \beta_2\gamma_2+\alpha_2\gamma_1 &> \max\Big\{\gamma_2+1, (n\gamma_2+2\sigma_2)(q-1), s\gamma_2\Big\}, \\ \gamma_1+1 &> \max\Big\{r\gamma_1, (m\gamma_1+2\sigma_1)(p-1)\Big\}, \\ \gamma_2+1 &> \max\Big\{s\gamma_2, (m\gamma_2+2\sigma_2)(q-1)\Big\}. \end{aligned} \tag{4.15}$$

For $\mu_2/\alpha_2 < (\beta_1+1)/(\alpha_2+1)$, we fix γ_1 and γ_2 to satisfy

$$\begin{aligned} \frac{\mu_2}{\alpha_2} < \frac{\gamma_1}{\gamma_2} &< \min\{\frac{\beta_1+1}{\alpha_2+1}, \frac{\beta_1}{\mu_1}\}, \\ \beta_2+\mu_2 < \frac{1+\gamma_2}{\gamma_2} &< \min\{\frac{s}{\gamma_2(s-1)}, \frac{\beta_2\gamma_2+\alpha_2\gamma_1}{\gamma_2}\}, \end{aligned} \tag{4.16}$$

then we can also select σ_1, σ_2 small enough such that (4.15) holds.

Furthermore, if we choose $A > \max\{1, m\gamma_1/\sigma_1, n\gamma_2/\sigma_2\}$, then for $\tau > 0$ sufficiently small, the right-hand sides of (4.9)-(4.12) are nonpositive, so (4.1) and (4.2) holds, and we obtain Theorem 1.2.

5 Proof of Theorem 1.3

Proof of Theorem 1.3. In the critical case of $(1/\tau, 1/\theta) = (0, 0)$, we have

$$\beta_1\alpha_2 = \mu_1\mu_2 = \max\{m(p-1) - \alpha_1, r - \alpha_1\} \max\{n(q-1) - \beta_2, s - \beta_2\}.$$

(i) For $r > m(p-1)$, $s > n(q-1)$, we know $\beta_1\alpha_2 = (r - \alpha_1)(s - \beta_2)$. Thanks to $a^{\alpha_2} b^{r-\alpha_1} \geq 1$, we can choose Λ_1 and Λ_2 sufficiently large such that $\Lambda_1 \geq \max\limits_{x\in\overline{\Omega}} u_0(x)$, $\Lambda_2 \geq \max\limits_{x\in\overline{\Omega}} v_0(x)$ and

$$\Lambda_2^{\frac{\beta_1}{r-\alpha_1}} (\frac{1}{a})^{\frac{1}{r-\alpha_1}} \leq \Lambda_1 \leq \Lambda_2^{\frac{s-\beta_2}{\alpha_2}} b^{\frac{1}{\alpha_2}}.$$

Clearly, $(\overline{u}, \overline{v}) = (\Lambda_1, \Lambda_2)$ is a supersolution of problem (1.1), then by comparison principle, the solution of (1.1) should be global.

Next, we begin to prove our blow-up conclusion.

Since $\beta_1\alpha_2 = \mu_1\mu_2$, we can choose constants $l_1, l_2 > 1$ such that

$$\frac{n(q-1) - \beta_2 - 1}{r - \alpha_1 - 1} < \frac{s - \beta_2}{\alpha_2} = \frac{l_1}{l_2} = \frac{\beta_1}{r - \alpha_1} < \frac{s - \beta_2 - 1}{m(p-1) - \alpha_1 - 1}. \tag{5.1}$$

According to Proposition 2.1, we only need to construct a suitable blow-up subsolution of problem (1.1) on $\overline{\Omega}_T \times \overline{\Omega}_T$. Let $\gamma(t)$ be the solution of the following ordinary differential equation

$$\gamma'(t) = c_1\gamma^{\delta_1} - c_2\gamma^{\delta_2},\ \gamma(0) = \gamma_0 > 0,\ t > 0,$$

where

$$c_1 = \min\{\frac{\zeta^{\alpha_1}\vartheta^{\beta_1} - a\zeta^r}{l_1\zeta}, \frac{\zeta^{\alpha_2}\vartheta^{\beta_2} - b\vartheta^s}{l_2\vartheta}\}, c_2 = \max\{\frac{1}{l_1\zeta}, \frac{1}{l_2\vartheta}\},$$
$$\delta_1 = \min\{l_1(r-1) + 1, (s-1)l_2 + 1\},$$
$$\delta_2 = \max\{[m(p-1) - 1]l_1 + 1, [n(q-1) - 1]l_2 + 1\}.$$

Since $\vartheta^{\beta_1} > a\zeta^{r-\alpha_1}$ and $\zeta^{\alpha_2} > b\vartheta^{s-\beta_2}$, we have $c_1 > 0$. On the other hand, by virtue of (5.1), it is easy to see that $\delta_1 > \delta_2$. Then, it is obvious that there exists a constant $0 < T^\star < +\infty$ such that

$$\lim_{t\to T^\star} \gamma(t) = +\infty.$$

Construct

$$(\underline{u}(x,t),\underline{v}(x,t)) = (\gamma^{l_1}(t)\zeta(x),\gamma^{l_2}(t)\vartheta(x)),$$

where $\zeta(x),\vartheta(x)$ satisfying (1.6). Moreover, by the assumptions on initial data, we can take small enough constant γ_0 such that

$$u_0(x) \geq \gamma_0^{l_1}M_1 \text{ and } v_0(x) \geq \gamma_0^{l_2}M_2 \text{ for all } x\in\Omega, \tag{5.2}$$

where $M_1 = \max\limits_{x\in\Omega}\zeta(x)$, $M_2 = \max\limits_{x\in\Omega}\vartheta(x)$.

Now, we begin to verify that $(\overline{u}(x,t),\overline{v}(x,t))$ is a blow-up subsolution of the problem (1.1) on $\overline{\Omega}_T\times\overline{\Omega}_T$, $T<T^{\star}$. In fact, $\forall (x,t)\in\Omega_T\times(0,T)$, a series of computations show

$$\begin{aligned} P\underline{u}(x,t) &\equiv \underline{u}_t - \Delta_{m,p}\underline{u} - \underline{u}^{\alpha_1}\underline{v}^{\beta_1} + a\underline{u}^r \\ &= l_1\zeta\gamma^{l_1-1}\gamma'(t) + \gamma^{m(p-1)l_1} - \gamma^{l_1\alpha_1+l_2\beta_1}\zeta^{\alpha_1}\vartheta^{\beta_1} + a\gamma^{rl_1}\zeta^r \qquad (5.3)\\ &= l_1\zeta\gamma^{l_1-1}\big(\gamma'(t) + \frac{1}{l_1\zeta}\gamma^{m(p-1)l_1-l_1+1} - \frac{\zeta^{\alpha_1}\vartheta^{\beta_1}-a\zeta^r}{l_1\zeta}\gamma^{l_1(r-1)+1}\big) \\ &\leq 0. \end{aligned}$$

Similarly, we also have

$$Q\underline{v}(x,t) \equiv \underline{v}_t - \Delta_{n,q}\underline{v} - \underline{u}^{\alpha_2}\underline{v}^{\beta_2} + b\underline{v}^s \leq 0. \tag{5.4}$$

On the other hand, $\forall t\in[0,T]$, we have

$$\underline{u}(x,t)|_{x\in\partial\Omega} = \gamma^{l_1}(t)\zeta(x)|_{x\in\partial\Omega} = 0, \quad \underline{v}(x,t)|_{x\in\partial\Omega} = \gamma^{l_2}(t)\vartheta(x)|_{x\in\partial\Omega} = 0. \tag{5.5}$$

Combining now (5.2)-(5.5), we see that $(\underline{u},\underline{v})$ is a subsolution of (1.1) and $(\underline{u},\underline{v}) < (u,v)$ on $\overline{\Omega}_T\times\overline{\Omega}_T$ by comparison principle, thus (u,v) must blow up in finite time since $(\underline{u},\underline{v})$ does.

(ii) For $r<m(p-1)$, $s<n(q-1)$, we know $\beta_1\alpha_2 = [m(p-1)-\alpha_1][n(q-1)-\beta_2]$. Under the assumption $(\zeta^{\alpha_2}\vartheta^{\beta_2})^{1/[n(q-1)-\beta_2]}(\zeta^{\alpha_1}\vartheta^{\beta_1})^{1/\beta_1}\leq 1$, we can choose Λ_1,Λ_2 such that

$$\Lambda_1^{\frac{\alpha_2}{n(q-1)-\beta_2}}(\zeta^{\alpha_2}\vartheta^{\beta_2})^{\frac{1}{n(q-1)-\beta_2}} \leq \Lambda_2 \leq \Lambda_1^{\frac{m(p-1)-\alpha_1}{\beta_1}}(\zeta^{\alpha_1}\vartheta^{\beta_1})^{-\frac{1}{\beta_1}}.$$

Then $(\overline{u},\overline{v}) = (\Lambda_1,\Lambda_2)$ is a global supersolution of (1.1).

Since $\beta_1\alpha_2 = [m(p-1)-\alpha_1][n(q-1)-\beta_2]$, we can choose constants $l_1,l_2>1$ such that

$$\frac{s-1}{m(p-1)-1} < \frac{n(q-1)-\beta_2}{\alpha_2} = \frac{l_1}{l_2} = \frac{\beta_1}{m(p-1)-\alpha_1} < \frac{n(q-1)-1}{r-1}. \tag{5.6}$$

Next, we consider the following ordinary differential equation

$$\gamma'(t) = c_1\gamma^{\delta_1} - c_2\gamma^{\delta_2},\ \gamma(0) = \gamma_0 > 0,\ t > 0,$$

where

$$c_1 = \min\{\zeta^{\alpha_1}\vartheta^{\beta_1} - 1, \zeta^{\alpha_2}\vartheta^{\beta_2} - 1\},\quad c_2 = \max\{\frac{a\zeta^{r-1}}{l_1}, \frac{b\vartheta^{s-1}}{l_2}\},$$
$$\delta_1 = \min\{[m(p-1)-1]l_1 + 1, [n(q-1)-1]l_2 + 1\},$$
$$\delta_2 = \max\{l_1(r-1) + 1, (s-1)l_2 + 1\}.$$

Since $\zeta^{\alpha_1}\vartheta^{\beta_1} > 1$, $\zeta^{\alpha_2}\vartheta^{\beta_2} > 1$, we have $c_1 > 0$. On the other hand, in light of (5.6), it is easy to show that $\delta_1 > \delta_2$. Then, it is clear that $\gamma(t)$ will become infinite in a finite time $T^\star < +\infty$.

Let

$$(\underline{u}(x,t), \underline{v}(x,t)) = (\gamma^{l_1}(t)\zeta(x), \gamma^{l_2}(t)\vartheta(x)),$$

where $\zeta(x), \vartheta(x)$ satisfying (1.6). Similar to the arguments for the case $r > m(p-1)$, $s > n(q-1)$, we can prove that $(\underline{u}(x,t), \underline{v}(x,t))$ is a blow-up subsolution of the problem (1.1) on $\overline{\Omega}_T \times \overline{\Omega}_T$, $T < T^\star$. Then, the solution (u,v) of (1.1) blows up in finite time.

(iii) For $r < m(p-1)$, $s > n(q-1)$, we know $\beta_1\alpha_2 = [m(p-1)-\alpha_1][s-\beta_2]$. Since $(\zeta^{\alpha_2})(\zeta^{\alpha_1})^{(s-\beta_2)/\beta_1} \le b$, we can choose Λ_1, Λ_2, such that

$$b^{-\frac{1}{s-\beta_2}}\Lambda_1^{\frac{\alpha_2}{s-\beta_2}}(\zeta^{\alpha_2})^{\frac{1}{s-\beta_2}} \le \Lambda_2 \le \Lambda_1^{\frac{m(p-1)-\alpha_1}{\beta_1}}(\zeta^{\alpha_1})^{-\frac{1}{\beta_1}}.$$

We can check $(\overline{u}, \overline{v}) = (\Lambda_1\zeta, \Lambda_2)$ is a global supersolution of (1.1).

Thanks to $\beta_1\alpha_2 = [m(p-1)-\alpha_1][s-\beta_2]$, we can choose constants $l_1, l_2 > 1$ such that

$$\frac{n(q-1)-\beta_2}{\alpha_2} < \frac{s-\beta_2}{\alpha_2} = \frac{l_1}{l_2} = \frac{\beta_1}{m(p-1)-\alpha_1} < \frac{\beta_1}{r-\alpha_1}.$$

Let

$$(\underline{u}(x,t), \underline{v}(x,t)) = (\gamma^{l_1}(t)\zeta(x), \gamma^{l_2}(t)\vartheta(x)),$$

where $\zeta(x), \vartheta(x)$ are defined in (1.6), and $\Gamma(t)$ satisfies the following Cauchy problem

$$\gamma'(t) = c_1\gamma^{\delta_1} - c_2\gamma^{\delta_2},\ \gamma(0) = \gamma_0 > 0,\ t > 0,$$

where

$$c_1 = \min\{\zeta^{\alpha_1}\vartheta^{\beta_1} - 1, \frac{\zeta^{\alpha_2}\vartheta^{\beta_2} - b\vartheta^s}{l_2\vartheta}\}, c_2 = \max\{\frac{a\zeta^{r-1}}{l_1}, \frac{1}{l_2\vartheta}\},$$

$$\delta_1 = \min\{[m(p-1)-1]l_1+1, (s-1)l_2+1\}, \delta_2 = \max\{l_1(r-1)+1, [n(q-1)-1]l_2+1\}.$$

Then, the left arguments are the same as those for the case $r > m(p-1)$, $s > n(q-1)$, so we omit them.

(iv) The proof of this case is parallel to (iii). The proof of Theorem 1.3 is complete.

6 Acknowledgements

The author expresses his deepest thanks to the reviewers and the editor for their careful reading and valuable suggestions. This work was partially supported by Chinese Universities Scientific Fund of Ocean University of China (201113008).

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