ON EQUALITY OF ORDER OF A FINITE p-GROUP AND ORDER OF ITS AUTOMORPHISM GROUP

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Abstract. Let G be a group and let $Aut(G)$ be the full automorphism of G. The purpose of this paper to consider finite p-groups G for which $|G| = |Aut(G)|$. We classify groups satisfying this condition among those in certain classes of finite p-groups.

1. Introduction

Let G be a group. We denote by $G^{'}$, $\Phi(G)$, $Z(G)$, $Aut(G)$ and $Inn(G)$, respectively the commutator subgroup, Frattini subgroup, the centre, the automorphism group and the inner automorphism group of G. An automorphism α of G is called a central automorphism if $x^{-1}\alpha(x) \in Z(G)$ for each $x \in G$. The central automorphisms of G, denoted by $\mathrm{Aut}_c(G)$, fix G' elementwise and form a normal subgroup of the full automorphism group of G. The group of central automorphisms of a finite group G is of great importance in investigating of $Aut(G)$, and has been studied by several authors (see, for example, [1, 17, 19, 21]). It is conjectured that if G is a finite noncyclic p-group of order greater than p^2, then $|G|$ divides $|Aut(G)|$. A finite p-group satisfying this conjecture is called a LA-group. A. D. Otto [19] first showed that an abelian finite p-group is a LA-group. He also showed that if a p-group G is the direct product of a purely non-abelian group B and an abelian group P and $|B| \mid |Aut(B)|$, then $|G| \mid |Aut(G)|$. Finite p-groups of class 2 and finite p-abelian p-groups are LA-groups, as was shown by R. Faudree and R. M. Davitt respectively in [11] and [6]. A. D. Otto and R. M. Davitt also showed that a finite metacyclic p-group ($p > 2$), a finite p-group ($p > 2$) with the central quotient metacyclic, a finite modular p-group ($p > 2$) and a finite p-group G which satisfies $[G : Z(G)] \leq p^4$ are all LA-groups ([8], [7], [9], [5]). T. Exarchakos [10] showed that any p-group of maximal class and any p-group with cyclic Frattini subgroup is a LA-groups. S. Fouladi, A. R. Jamali and R. Orfi [13] proved that finite p-groups of coclass 2, are LA-groups. This conjecture in full generality is still open ([16], Problem 12.77). Let $|Aut(G)|_p$ be the order of a Sylow p-subgroup of $Aut(G)$. I. Malinowska [15], characterized the finite p-groups G of maximal class for which $|Aut(G)|_p = |G|$, in response to a problem posed by Berkovich in [2]. A similar description has been given by S. Fouladi, A. R. Jamali and R. Orfi [12] for the finite non-abelian p-groups with cyclic Frattini subgroup. The purpose of this paper is to consider p-groups G for which $|G| = |Aut(G)|$. M. F. Newman and E. A. O'Brien [18] gave (without proof) three infinite families of 2-groups for which $|G| = |Aut(G)|$. M. J. Curran [3] showed that for each $m \geq 3$, there is a 2-group P with $|P| = 2^m = |Aut(P)|$. For p odd, no such examples are known. In this paper we describe completely the p-groups G such that $|G| = |Aut(G)|$ and such that G is of maximal class or with cyclic Frattini subgroup. We show that in every finite non-abelian p-group G such that $|Z(G)| = p$ and $|G| = |Aut(G)|$, all non-abelian maximal subgroups are characteristic. Also we prove that if G is a finite p-group of class 2 with cyclic centre such that $|Aut(G)| = |G|$, then

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$p = 2$ and there exists a cyclic subgroup Σ of $Aut(G)$ such that $[Aut(G) : Aut_c(G)\Sigma] = 2$ and $|Aut_c(G) \cap \Sigma| = 2$.

2. Proofs

Definition 2.1. A finite non-abelian p-group G is called purely non-abelian if it has no non-trivial abelian direct factor.

Proposition 2.2. *Let G be a finite non-abelian p-group such that $|G| = |Aut(G)|$. Then*

(1) *If p is odd, then G is purely non-abelian.*
(2) *If $p = 2$ and $|Z(G)| = 4$, then G is purely non-abelian.*
(3) *If $p = 2$, then G cannot have a homocyclic direct factor of rank 2.*

Proof. (1) Suppose, for a contradiction, that $G = A \times B$ where A is a non-trivial abelian group. Since $\exp(A) > 2$, A has an automorphism of order 2 (the inverting map) and hence G has an automorphism of order 2, which is a contradiction.

(2) Suppose, for a contradiction, that $G = A \times B$ where A is a non-trivial abelian group. Since B is non-abelian, $|Hom(B, A)| = |Hom(B/B', A)| \geq 2^2$ where $Hom(B, A)$ is the set of all group homomorphisms from B to A. For each non-trivial element α of $Hom(B, A)$, the map α^* defined by $\alpha^*(ab) = ab\alpha(b)$ for all $a \in A$, $b \in B$ defines a non-inner automorphism of G. Hence $[Aut(G) : Inn(G)] \geq 2^2$. Also for each non-trivial element β of $Hom(A, Z(B))$, the map β^* defined by $\beta^*(ab) = a\beta(a)b$ for all $a \in A$, $b \in B$ defines a non-inner automorphism of G. Therefore $[Aut(G) : Inn(G)] \geq 2^2 + 1$ and hence $[Aut(G) : Inn(G)] \geq 2^3$. Since $|Z(G)| = 4$, we have $|Aut(G)| > |G|$ which is a contradiction.

(3) Suppose that G has a direct factor of the form $\langle a\rangle \times \langle b\rangle$, where a and b have the same order (not 1). Hence G has an automorphism of order 3 defined by the mapping a to b and b to $a^{-1}b^{-1}$ which is impossible.

□

I. Malinowska [15] described completely the p-groups G such that $|Aut(G)|_p = |G|$ and such that G is either abelian or of maximal class. In the following we classify the p-groups G such that $|G| = |Aut(G)|$ and such that G is either abelian or of maximal class.

Proposition 2.3. *Let G be an abelian p-group of order p^n ($n > 2$). Then $|G| = |Aut(G)|$ if and only if $G \simeq C_{2^{n-1}} \times C_2$.*

Proof. Suppose first that G is an abelian p-group of order p^n and $|G| = |Aut(G)|$. By Proposition 2.2, we have $p = 2$. Let $n = 3$. Then $G \simeq C_8$, $G \simeq C_4 \times C_2$ or $G \simeq C_2 \times C_2 \times C_2$. If $G \simeq C_8$, then $|Aut(G)| = 4 \neq |G|$. If $G \simeq C_4 \times C_2$, then $|Aut(G)| = |G| = 2^3$. If $G \simeq C_2 \times C_2 \times C_2$, then by part (3) of Proposition 2.2, $|Aut(G)| \neq |G|$. Now let $n \geq 4$. Since $|G| = |Aut(G)|$, we have $|G| = |Aut(G)|_p$ and hence by [15, Theorem 2.3] $G \simeq C_{2^{n-1}} \times C_2$.

□

The following proposition is a consequence of [22].

Proposition 2.4. *Let G be an extra-special p-group. Then $|G| = |Aut(G)|$ if and only if $G \simeq D_8$.*

Proof. Suppose that G is an extra-special p-group and $|G| = |Aut(G)|$. If p is odd, then $p - 1$ divides $|G|$ which is impossible. Therefore $p = 2$. If G is isomorphic to the central product of $n - 1$ copies of D_8 and one copy of Q_8, then $2^n + 1$ divides $|Aut(G)|$ and so $|G|$ which is impossible. Thus G is isomorphic to the central product of n copies of D_8. Let $n > 1$. Then $2^n - 1$ divides $|Aut(G)|$ and so $|G|$ which is a contradiction. Hence $n = 1$ and $G \simeq D_8$. □

Proposition 2.5. *Let G be a finite p-group of maximal class. Then $|G| = |Aut(G)|$ if and only if $G \simeq D_8$ or $G \simeq S_{16} = \langle x, y \mid x^8 = y^2 = 1,\ [x, y] = x^2\rangle$.*

Proof. Since $|G| = |Aut(G)|$, we have $|G| = |Aut(G)|_p$. Hence by [15, Theorem 3.4] G is a non-abelian group of order p^3 or G is isomorphic to one of the following groups of order p^4:

(1) $\langle x, y, z \mid x^9 = y^3 = z^3 = 1,\ [x, y] = x^3,\ [x, z] = y,\ [y, z] = 1\rangle$;
(2) $\langle x, y, z \mid x^{p^2} = y^p = [y, z] = 1,\ z^p = x^p,\ [x, z] = y,\ [x, y] = x^p\rangle$, where $p > 2$;
(3) $\langle x, y, z \mid x^{p^2} = y^p = [y, z] = 1,\ [x, y] = x^p,\ [x, z] = y,\ z^p = x^{\alpha p}\rangle$, where $p > 3$ and α is a quadratic non-residue for p.
(4) $\langle x, y \mid x^8 = y^2 = 1,\ [x, y] = x^2\rangle$.

If G is a non-abelian group of order p^3, then, by Proposition 2.4, $G \simeq D_8$.
If G is as (1), (2) or (3), then the map x to x^{-1}, y to yx^p and z to z^{-1} ($p = 3$ for the group in (1)) can be extended to an automorphism of order 2, which is a contradiction. Therefore $G \simeq S_{16} = \langle x, y \mid x^8 = y^2 = 1,\ [x, y] = x^2\rangle$. □

Theorem 2.6. *Let G be a finite non-abelian p-group with cyclic Frattini subgroup. Then $|G| = |Aut(G)|$ if and only if G is isomorphic to S_{16}, D_8, or M_{2^n}, where*

$$M_{2^n} = \langle x, y \mid x^{2^{n-1}} = y^2 = 1, y^{-1}xy = x^{1+2^{n-2}}\rangle,\ (n \geq 4).$$

Proof. By the hypothesis $|G| = |\mathrm{Aut}(G)|_p$ and hence by [12, Theorem 1.1] we have $G \simeq S_{16}$ or $Z(G)$ is cyclic and $|G/Z(G)| = p^2$. If $p > 2$, then by [4, Lemma 2.3] there exist elements $a, b \in G$ such that, after setting $B = \langle b\rangle Z(G)$, we have $G = B\langle a\rangle$ and $|B \cap \langle a\rangle| = 1$. Hence the map α defined on G by $\alpha(xa^i) = x^{-1}a^i$ for every $x \in B$ and every $0 \leq i < |a|$ is an automorphism of order 2 which is a contradiction. Therefore $p = 2$ and hence by [12, Theorem 1.1] G is one of the groups S_{16}, D_8, M_{2^n} or $L_{2^{n+2}}$, where

$$L_{2^{n+2}} = \langle x, y, z \mid x^2 = (xy)^2 = 1, z^{2^n} = 1, [x, z] = [y, z] = 1, z^{2^{n-1}} = y^2\rangle (n > 1).$$

If G is isomorphic to one of the groups S_{16}, D_8, then $|G| = |Aut(G)|$. If $G \simeq M_{2^n} = \langle x, y \mid x^{2^{n-1}} = y^2 = 1, y^{-1}xy = x^{1+2^{n-2}}\rangle$, $(n \geq 4)$, then the set of all elements of order 2^{n-1} in G is $\{x^iy^j \mid 0 \leq j \leq 1,\ 1 \leq i < 2^{n-1},\ (i, 2) = 1\}$ and the set of all noncentral elements of order 2 in G is $\{y,\ x^{2^{n-2}}y\}$. The map $x \longmapsto x^iy^j$, $y \longmapsto y$ for all $0 \leq j \leq 1$ and $1 \leq i < 2^{n-1}$, $(i, 2) = 1$ can be extended to an automorphism of G. Also the map $x \longmapsto x^iy^j$, $y \longmapsto x^{2^{n-2}}y$ for all $0 \leq j \leq 1$ and $1 \leq i < 2^{n-1}$, $(i, 2) = 1$ can be extended to an automorphism of G. Hence $|Aut(G)| = 2^n = |G|$. Now let G be isomorphic to $L_{2^{n+2}}$. Set $t = xz^{2^{n-2}}$. We have $y^4 = 1$, $y^2 = t^2$ and $t^{-1}yt = y^{-1}$. Thus $\langle y, t\rangle \simeq Q_8$. Hence G is the nondirect central product $\langle y, t\rangle$ and $\langle z\rangle$. Therefore G has an automorphism of order 3 defined by the mapping y to t, t to y^3t and z to z which is impossible. □

Proposition 2.7. *Let G be a finite non-abelian p-group and $|Z(G)| = p$. If $|G| = |Aut(G)|$, then every non-abelian maximal subgroup of G is characteristic.*

Proof. Suppose that M is a non-abelian maximal subgroup of G. It follows from $Z(G) \subseteq \Phi(G)$ that $C_G(M) = Z(M)$. Suppose first that $Z(M) = Z(G)$. Let $x \notin M$ and z be a generator of $Z(G)$. Then the map θ defined on G by $\theta(mx^k) = mx^kz^k$ for every $m \in M$ and every $k \in \{0, 1, \ldots, p-1\}$ is an automorphism of G which is not inner. For, otherwise, if $\theta = \gamma_a$ for some $a \in G$, then for all $m \in M$, we have $m = \theta(m) = \gamma_a(m) = a^{-1}ma$ whence $a \in C_G(M) = Z(M) = Z(G)$. Therefore $xz = \theta(x) = \gamma_a(x) = a^{-1}xa = x$ and so $z = 1$, which is a contradiction. Thus $\theta \in Aut(G) \setminus Inn(G)$. Since $Inn(G)$ is a maximal subgroup of $Aut(G)$, we have $Aut(G) = \langle\theta\rangle Inn(G)$. Since θ fixes M elementwise and M is normal in G, M is characteristic in G. Now let $Z(M) \neq Z(G)$. Since M is non-abelian, $Z(G) < Z(M) = C_G(M) < M$. Then by [4, Lemma 4.1] G has an outer automorphism β such that $M^\beta = M$. Since $Inn(G)$ is a maximal subgroup of $Aut(G)$, we have $Aut(G) = \langle\beta\rangle Inn(G)$ and therefore M is characteristic in G. □

Theorem 2.8. *Let G be a finite p-group of class 2 such that $Z(G)$ is cyclic and $|G| = |Aut(G)|$. Then $p = 2$ and there exists a cyclic subgroup Σ of $Aut(G)$ such that $[Aut(G) : Aut_c(G)\Sigma] = 2$ and $|Aut_c(G) \cap \Sigma| = 2$.*

Proof. Let G be a finite p-group of class 2 such that $Z(G)$ is cyclic and $|G| = |Aut(G)|$. Since $Z(G)$ is cyclic, G is a central product $\prod_{i=1}^n G_i$, where each subgroup G_i can be written as $\langle a_i, b_i\rangle Z(G)$ for suitable a_i and b_i and $G_i \cap G_j = Z(G)$ if $i \neq j$. By [4, Lemma 2.3], we may also choose a_i and b_i such that $G_i = (\langle b_i\rangle Z(G))\langle a_i\rangle$ and $|\langle b_i\rangle Z(G) \cap \langle a_i\rangle| \leq 2$ for each i. Then G has an automorphism ϕ that fixes all a_i and inverts every b_i and all elements of $Z(G)$. Thus $p = 2$ (see the proof of Theorem 2.4 of [4]). Suppose that $G' = \langle u\rangle$ and $Z(G) = \langle z\rangle$, where $|u| = 2^b$ and $|z| = 2^a$. We prove that there are $g, h \in G$ such that $u = [g, h]$ and $h^{2^{b+1}} = 1$. Since G is a finite 2-group and G' is cyclic, we have $u = [g, h_1]$ for some g, $h_1 \in G$. Then $g^{2^b} = z^s$ and $h_1^{2^b} = z^t$ for some t, $s \in \mathbb{Z}$. If $\langle z^t\rangle \subseteq \langle z^s\rangle$, then $z^t = z^{rs}$ for some r. Put $h = g^{-r}h_1$. So we have $[g, h] = [g, h_1] = u$ and $h^{2^{b+1}} = (g^{-r}h_1)^{2^{b+1}} = (g^{-r})^{2^{b+1}}(h_1)^{2^{b+1}}[h_1, g^{-r}]^{2^{b+1}(2^{b+1}-1)/2} = 1$. If $\langle z^s\rangle \subseteq \langle z^t\rangle$ then $z^s = z^{r_1 t}$ for some r_1. Since $u = [g, h_1]$ is a generator of G', $u^{-1} = [h_1, g]$ is a generator of G'. Put $g_1 = g{h_1}^{-r_1}$. So we have $[h_1, g_1] = [h_1, g] = u^{-1}$ and $g_1^{2^{b+1}} = 1$. Let $H = \langle g, h\rangle$. Then $G = HC_G(\langle g, h\rangle)$. Hence by [11, Lemma 2], the correspondence $g \to gh^2$, $h \to h$, $x \to x$ for all $x \in C_G(\langle g, h\rangle)$ defines an automorphism σ of G which leaves the elements of $Z(G)$ fixed and $|\sigma Aut_c(G)| = 2^{b-1}$. Now we show that $exp(G/G') \leq exp(Z(G)) = 2^a$. Let $x \in G$. Then $x^{2^b} \in Z(G)$ and hence $x^{2^a} = (x^{2^b})^{2^{a-b}} \in G'$ since, $|Z(G)/G'| = 2^{a-b}$. Therefore $exp(G/G') \leq exp(Z(G)) = 2^a$. Since $Z(G)$ is cyclic, G is purely non-abelian and hence $|Aut_c(G)| = |Hom(G/G', Z(G)| = |G/G'|$. Set $\Sigma = \langle\sigma\rangle$. Then $|\Sigma| = |\langle\sigma\rangle| = 2^b = |G'|$ and $|\Sigma \cap Aut_c(G)| = 2$. Also $|Aut_c(G)\Sigma| = |Aut_c(G)||\Sigma|/|\Sigma \cap Aut_c(G)| = |G/G'||G'|/2 = |G|/2 = |Aut(G)|/2$ and so $[Aut(G) : Aut_c(G)\Sigma] = 2$. □

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References

[1] J. E. Adney, T. Yen, *Automorphisms of a p-group*, Illinois J. Math. **9** (1965), 137-143.
[2] Y. Berkovich, *On subgroups of finite p-groups*, Journal of Algebra **224** (2000), 198-240.
[3] M. J. Curran, *A note on p-groups that are automorphism groups*, Suppl. Rend. Circ. Mat. Palermo, II. Ser. **23** (1990), 57-61.
[4] G. Cutolo, *On a question about automorphisms of finite p-groups*, J. Group Theory **9** (2006), 231-250.
[5] R. M. Davitt, *On the automorphism group of a finite p-group with a small central quotient*, Can. J. Math., **32** (1980), 1168-1176.
[6] R. M. Davitt, *The automorphism group of finite p-abelian p-groups*, Illinois J. Math **16** (1972), 76-85.
[7] R. M. Davitt, A. D. Otto, *On the automorphism group of a finite p-group with the central quotient metacyclic*, Proc. Amer. Math. Soc., **30** (1971), 467-472.
[8] R. M. Davitt, *The automorphism group of a finite metacyclic p-group*, Proc. Amer. Math. Soc., **25** (1970), 876-879.
[9] R. M. Davitt, A. D. Otto, *On the automorphism group of a finite modular p-group* , Proc. Amer. Math. Soc., **35** (1972), 399-404.
[10] T. Exarchakos, *LA-groups*, J. Math. Soc. Japan, **32**, 2 (1981), 185-190.

[11] R. Faudree, *A note on the automorphism group of a group*, Proc. Amer. Math. Soc., **19** (1968), 1379-1382.
[12] S. Fouladi, A. R. Jamali and R. Orfi, *On the automorphism group of a finite p-group with cyclic Frattini subgroup*, Proc. Roy. Irish Acad, **108**A (2), (2008), 165-175.
[13] S. Fouladi, A. R. Jamali and R. Orfi, *Automorphism groups of finite p-groups of coclass 2*, J. Group Theory **10** (2007), 437-440.
[14] B. Huppert, *Endliche Gruppen I*, Springer-Verlag, New York, 1967.
[15] I. Malinowska, *Finite p-groups with few p-automorphisms*, J. Group Theory **4** (2001), 395-400.
[16] V. D. Mazurov, E. I. Khukhro (Eds.), *Unsolved problems in group theory*, The Kourovka Notebook, vol. **16**, Russian Academy of Sciences, Siberian Division, Institue of Mathematics, Novosibirisk, 2006.
[17] H. Mousavi, A. Shomali, *Central automorphisms of semidirect products*, To appear in Bull. Malays. Math. Sci. Soc.
[18] M. F. Newman, E. A. O'Brien, *A CAYLEY library for the groups of order dividing* 128, Group theory (Singapore, 1987), de Gruyter, Berlin-New York, (1989), 437-442.
[19] A. D. Otto, *Central automorphisms of a finite p-group*, Trans. Amer. Math. Soc **125** (1966), 280-287.
[20] D. J. S. Robinson, *A course in the theory of groups*, Springer-Verlag, New York, 1982.
[21] M. Shabani-Attar, *C-Characteristically simple groups*, Bull. Malays. Math. Sci. Soc. (2)**35**(1) (2012), 147-154.
[22] D. L. Winter, *The automorphism group of an extraspecial p-group*, Rocky Mountain J. Math. **2** (1972), no. 2, 159-168.

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