Some notes on CN rings *

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Abstract The main results: A ring R is CN if and only if for any $x \in N(R)$ and $y \in R$, $((1+x)y)^{n+k} = (1+x)^{n+k}y^{n+k}$, where n is a fixed positive integer and $k = 0, 1, 2$; (2) Let R be a CN ring and $n \geq 1$. If for any $x, y \in R \backslash N(R)$, $(xy)^{n+k} = x^{n+k}y^{n+k}$, where $k = 0, 1, 2$, then R is commutative; (3) Let R be a ring and $n \geq 1$. If for any $x \in R \backslash N(R)$ and $y \in R$, $(xy)^k = x^k y^k$, $k = n, n+1, n+2$, then R is commutative; (4) NLI exchange rings are clean.

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1 Introduction

Throughout this paper, all rings are associative with identity. Let R be a ring, we use $N(R)$, $J(R)$, $E(R)$, $Z(R)$ and $U(R)$ to denote the set of all nilpotent elements, the *Jacobson* radical, the set of all idempotent elements, the center and the set of all invertible elements of R, respectively. For any nonempty subset X of a ring R, $r(X) = r_R(X)$ and $l(X) = l_R(X)$ denote the right annihilator of X and the left annihilator of X, respectively.

Following [4], a ring R is called *CN* if $N(R) \subseteq Z(R)$. Clearly, commutative rings and reduced rings (thai is, a ring R with $N(R) = 0$) are CN.

A theorem of Herstein [8] stated that a ring R which satisfies the identity $(xy)^n = x^n y^n$, where n is a fixed positive integer greater than 1, must have nil commutator ideal. In [1], Bell proved that if R is an n-torsion-free ring with identity 1 and satisfies the two identities $(xy)^n = x^n y^n$ and

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$(xy)^{n+1} = x^{n+1}y^{n+1}$, then R is commutative. In [9], Khuzam proved that if R is $n(n-1)$-torsion-free ring with 1 and satisfies the identity $(xy)^n = x^n y^n$, then R is commutative. In [10], Khuzam proved that if R is a semiprime ring in which for each x in R there exists a positive integer $n = n(x) > 1$ such that $(xy)^n = x^n y^n$ for all $y \in R$, then R is commutative. In [11], Ligh and Richou proved that if R is a ring with 1 which satisfies the identities: $(xy)^k = x^k y^k$, $k = n, n+1, n+2$, where n is a positive integer, then R is commutative. The purpose of this note is to generalize these results.

2 Main Results

We begin with the following theorem which generalizes [4, Theorem 5].

Theorem 2.1 *The following conditions are equivalent for a ring R:*

(1) R is a CN ring;

(2) For any $a \in N(R)$, there exists $n = n(a) \geq 2$ such that $a - a^n \in Z(R)$;

(3) For any $a \in N(R)$ and $b \in R$, there exists $c = c(a, b) \in R$ such that $[a - a^2c, b] = 0$.

Proof (1) $\Longrightarrow$ $(i), i = 2, 3$ are trivial.

(2) $\Longrightarrow$ (1) Assume that $a \in N(R)$ with $a^m = 0$ for some $m \geq 2$. By (2), there exists $n_1 = n_1(a) \geq 2$ such that $a - a^{n_1} \in Z(R)$. Since $a^{n_1} \in N(R)$, by (2), there exists $n_2 = n_2(a^{n_1}) \geq 2$ such that $a^{n_1} - a^{n_1 n_2} \in Z(R)$. Continuing this process, there exists $n_s = n_s(a^{n_1 n_2 \cdots n_{s-1}}) \geq 2$ such that $a^{n_1 n_2 \cdots n_{s-1}} - a^{n_1 n_2 \cdots n_{s-1} n_s} \in Z(R)$ and $n_1 n_2 \cdots n_{s-1} n_s \geq m$. Hence $a^{n_1 n_2 \cdots n_{s-1} n_s} = 0$ and $a = a - a^{n_1 n_2 \cdots n_{s-1} n_s} = (a - a^{n_1}) + (a^{n_1} - a^{n_1 n_2}) + \cdots + (a^{n_1 n_2 \cdots n_{s-1}} - a^{n_1 n_2 \cdots n_{s-1} n_s}) \in Z(R)$.

(3) $\Longrightarrow$ (1) Assume that $a \in N(R)$ with $a^n = 0$ for some $n \geq 2$. By induction on n, we claim that $a \in Z(R)$. For each $x \in R$, by (3), there exists $c = c(a, x) \in R$ such that $[a - a^2c, x] = 0$. If $n = 2$, then $a \in Z(R)$, we are done. Now we assume that $n > 2$ and assume that for each $y \in N(R)$ with the index of nilpotence at most $n - 1$, we have $y \in Z(R)$. Since $(a^2)^{n-1} = 0$, by the induction hypothesis, $a^2 \in Z(R)$. For any $z \in (a^2) = a^2R$, we have $z^{n-1} \in a^{2(n-1)}R = 0$, so $z \in Z(R)$ by the induction hypothesis. This implies $a^2R \subseteq Z(R)$. Hence $0 = [a - a^2c, x] = [a, x]$ for any $x \in R$, so $a \in Z(R)$, this shows that R is CN. ∎

A ring R is called *NLI* if $N(R)$ is a *Lie*$-$ideal of R (that is, for any $a \in N(R)$ and $b \in R$, $ab - ba \in N(R)$ and $N(R)$ is an additive subgroup of R). Clearly, NI rings (that is, $N(R)$ forms an ideal of R) are NLI. A ring R is called *QCN* if for any $a \in N(R)$ and $b \in R$, there exist $n = n(a, b) > 1$ and $c \in R$ such that $ab - ba = (ab - ba)^n c$. Clearly, CN rings are QCN.

Theorem 2.2 *The following conditions are equivalent for a ring R:*

(1) R is a CN ring;

(2) R is a QCN NI ring;

(3) R is a QCN NLI ring.

Proof (1) $\Longrightarrow$ (2) $\Longrightarrow$ (3) is trivial.

(3) $\Longrightarrow$ (1) Assume that $a \in N(R)$ and $b \in R$. Since R is a QCN ring, $ab - ba = (ab - ba)^n c$ for some $n = n(a, b)$ and $c \in R$. Since R is an NLI ring, $ab - ba \in N(R)$. Let $m \geq 1$ such that $(ab - ba)^m = 0$. Clearly, $(n-1)m+1 \geq m$. Since $ab - ba = (ab - ba)^{(n-1)m+1} c^m$, $ab - ba = 0$. Hence $a \in Z(R)$ and R is a CN ring. ∎

In preparation for the proof of our next theorem, we first state the following known lemma ([12, Lemma 2]).

Lemma 2.3 *Let $x, y \in R$. Suppose that for some positive integer n, $xy^n = 0 = x(1+y)^n$. Then $x = 0$.*

Theorem 2.4 *The following conditions are equivalent for a ring R:*

(1) R is a CN ring;

(2) For any $x \in N(R)$ and $y \in R$, $((1+x)y)^{n+k} = (1+x)^{n+k} y^{n+k}$, where n is a fixed positive integer and $k = 0, 1, 2$;

(3) For any $x \in N(R)$ and $y \in R$, $((1+x)y)^{n+k} = y^{n+k}(1+x)^{n+k}$, where n is a fixed positive integer and $k = 0, 1, 2$.

Proof (1) $\Longrightarrow$ $(i), i = 2, 3$ are trivial.

(2) $\Longrightarrow$ (1) Assume that $x \in N(R)$ and $y \in R$. Then by the hypothesis,

$$(1+x)^{n+1} y^{n+1} = (1+x)^n y^n (1+x) y \tag{2.1}$$

$$(1+x)^{n+2} y^{n+2} = (1+x)^{n+1} y^{n+1} (1+x) y \tag{2.2}$$

Since $1 + x$ is invertible in R, (2.1) gives

$$(xy^n - y^n x) y = 0 \tag{2.3}$$

(2.2) gives

$$(xy^{n+1} - y^{n+1} x) y = 0 \tag{2.4}$$

Multiply (2.3) on the left by y, one gets

$$(yxy^n - y^{n+1} x) y = 0 \tag{2.5}$$

From (2.4) and (2.5) we have

$$(xy - yx)y^{n+1} = 0 \tag{2.6}$$

Since (2.6) holds for all $y \in R$, substitute $y + 1$ for y, to get

$$(xy - yx)(1 + y)^{n+1} = 0 \tag{2.7}$$

From (2.6), (2.7) and Lemma 2.3, we have

$$xy = yx \tag{2.8}$$

Hence R is CN.

(3) $\Longrightarrow$ (1) Suppose that $x \in N(R)$ and $y \in R$. Since $((1 + x)y)^{n+1} = (1 + x)y((1 + x)y)^n$, by the hypothesis, we have

$$y^{n+1}(1 + x)^{n+1} = (1 + x)y^{n+1}(1 + x)^n \tag{2.9}$$

$$y^{n+2}(1 + x)^{n+2} = (1 + x)y^{n+2}(1 + x)^{n+1} \tag{2.10}$$

Since $1 + x$ is invertible in R, (2.9) gives

$$xy^{n+1} = y^{n+1}x \tag{2.11}$$

(2.10) gives

$$xy^{n+2} = y^{n+2}x \tag{2.12}$$

Multiply (2.11) on the left by y, from (2.12), one gets

$$(xy - yx)y^{n+1} = 0 \tag{2.13}$$

Similar to the proof of (2) $\Longrightarrow$ (1), we have

$$xy = yx \tag{2.14}$$

Hence R is CN. ∎

Let R be a CN ring. Then for any $n \geq 2$ and any $a \in N(R)$ and $b \in R$, we have $(ab)^n = a^n b^n = b^n a^n$. But the converse is not true in general.

Example 2.5 *Let D be a division ring and $R = \begin{pmatrix} D & D \\ 0 & D \end{pmatrix}$. Then $N(R) = \begin{pmatrix} 0 & D \\ 0 & 0 \end{pmatrix}$ with $N(R)^2 = 0$. Since $N(R)$ is an ideal of R, for any $n \geq 2$, any $A \in N(R)$ and $B \in R$, we have $(AB)^n = A^n B^n = B^n A^n = 0$. But R is not CN.*

Theorem 2.6 *Let R be a CN ring and $n \geq 1$. If for any $x, y \in R \backslash N(R)$, $(xy)^{n+k} = x^{n+k}y^{n+k}$, where $k = 0, 1, 2$, then R is commutative.*

Proof It follows immediately from the result in [11]. ∎

Lemma 2.7 *Let R be a semiprime ring and $n \geq 2$. If for any $a \in N(R)$ and $b \in R$, $(ab)^n = a^n b^n$, then R is reduced.*

Proof Let $a \in R$ with $a^n = 0$. Then $(ax)^n = 0$ for each $x \in R$. If $a \neq 0$, then aR is a nonzero nil right ideal of R satisfying the identity $z^n = 0$ for all $z \in aR$. Now by [6, Lemma 1.1], R has a nonzero nilpotent ideal which is a contradiction since R is semiprime. Thus $a = 0$, this implies R is reduced. ∎

Theorem 2.8 *Let R be a semiprime ring and $n \geq 1$. If for any $x \in R \backslash J(R)$ and $y \in R$, $(xy)^{n+k} = x^{n+k}y^{n+k}$, where $k = 0, 1$, then R is commutative.*

Proof If $N(R) \cap J(R) = 0$, then by Lemma 2.7, R is reduced. If $N(R) \cap J(R) \neq 0$, then there exists $0 \neq a \in N(R) \cap J(R)$ with $a^2 = 0$. By the hypothesis, for any $y \in R$, we have

$$(1+a)^{n+k}(ya)^{n+k} = ((1+a)ya)^{n+k} \tag{2.15}$$

Clearly, for any $i \geq 1$, one gets

$$((1+a)ya)^i = (1+a)(ya)^i \tag{2.16}$$

Hence

$$(1+a)^{n+k}(ya)^{n+k} = (1+a)(ya)^{n+k} \tag{2.17}$$

Since $1+a$ is invertible in R and $(1+a)^i = 1+ia$ for each $i \geq 1$, we have

$$(n+k-1)a(ya)^{n+k} = 0, k = 0, 1 \tag{2.18}$$

This implies

$$a(ya)^{n+1} = 0 \tag{2.19}$$

Hence

$$(ay)^{n+2} = 0 \tag{2.20}$$

This leads to aR is a nonzero nil right ideal of R satisfying the identity $z^{n+2} = 0$ for all $z \in aR$. Now by [6, Lemma 1.1], R has a nonzero nilpotent ideal which is a contradiction since R is semiprime. Thus $N(R) \cap J(R) = 0$ and so R is reduced.

Now suppose $x, y \in R$. If $x, 1 + x \notin J(R)$, then by the hypothesis, we have

$$x^{n+1}y^{n+1} = x^n y^n xy \tag{2.21}$$

$$(x+1)^{n+1}y^{n+1} = (x+1)^n y^n (x+1)y \tag{2.22}$$

They give

$$x^n(xy^n - y^n x)y = 0 \tag{2.23}$$

$$(x+1)^n(xy^n - y^n x)y = 0 \tag{2.24}$$

From Lemma 2.3, (2.23) and (2.24), one gets

$$(xy^n - y^n x)y = 0 \tag{2.25}$$

If $x \in J(R)$, then $x+1$ is invertible in R, so

$$(x+1)^{n+1}y^{n+1} = (x+1)^n y^n (x+1)y \tag{2.26}$$

This gives

$$(xy^n - y^n x)y = 0 \tag{2.27}$$

If $x \notin J(R)$ and $1+x \in J(R)$, then x is invertible in R, so (2.21) implies

$$(xy^n - y^n x)y = 0 \tag{2.28}$$

Hence, in any case, we have

$$(xy^n - y^n x)y = 0 \tag{2.29}$$

$$(y(xy^n - y^n x))^2 = 0 \tag{2.30}$$

Since R is reduced, one gets

$$y(xy^n - y^n x) = 0 \tag{2.31}$$

Clearly, for any $r \in R$, we have

$$((xy^n - y^n x)ry)^2 = 0 \tag{2.32}$$

Hence, for any $r \in R$, we have

$$(xy^n - y^n x)ry = 0 \tag{2.33}$$

that is,

$$(xy^n - y^n x)Ry = 0 \tag{2.34}$$

Thus

$$(xy^n - y^n x)R(xy^n - y^n x) = 0 \tag{2.35}$$

Since R is semiprime, one gets

$$xy^n = y^n x \tag{2.36}$$

Since R has no nonzero nil ideals, by [7, Theorem], R is commutative.∎

Corollary 2.9 *Let R be a primitive ring and $n \geq 1$. If for any $x, y \in R$, $(xy)^k = x^k y^k$, where $k = n, n+1$, then R is a field.*

Proof By Theorem 2.8, R is commutative. We claim that R is a division ring. If not, there exists a subring S of R such that S is isomorphic to 2×2 full matrix ring $M_2(D)$ over a division ring D. Clearly, for any $x, y \in S$, $(xy)^k = x^k y^k$, where $k = n, n+1$, hence $M_2(D)$ satisfies the same conditions. Now let $A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, B = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$, $(AB)^{n+1} \neq A^{n+1}B^{n+1}$, which is a contradiction. Hence R is a division ring, and so R is a field.

Theorem 2.10 *R is a CN ring if and only if for some positive integer $n \geq 1, m > 1$ and any $x \in R$ and $y \in N(R)$, $[xy - x^n y^m, x] = 0$.*

Proof One direction is clear.

Now assume that $n \geq 1, m > 1$ such that for any $x \in R$ and $y \in N(R)$, we have $[xy - x^n y^m, x] = 0$. Since $y \in N(R)$, there exists $p \geq 1$ such that $y^{m^p} = 0$. The equation $[xy - x^n y^m, x] = 0$ gives

$$x[x, y] = x^n [x, y^m] \tag{2.37}$$

Since $y^m \in N(R)$, substitute y^m for y in (2.37), one gets

$$x^2[x, y] = x^{n+1}[x, y^m] = x^n(x[x, y^m]) = x^{2n}[x, y^{m^2}] \tag{2.38}$$

Hence

$$x^p[x, y] = x^{np}[x, y^{m^p}] \tag{2.39}$$

This implies

$$x^p[x, y] = 0 \tag{2.40}$$

Since (2.40) holds for all $x \in R$, substitute $x + 1$ for x and use Lemma 2.3, we have $[x, y] = 0$. Hence R is CN. ∎

Theorem 2.11 *Let R be a ring and $n \geq 1$. If for any $x \in R \backslash N(R)$ and $y \in R$, $(xy)^k = x^k y^k$, $k = n, n+1, n+2$, then R is commutative.*

Proof Suppose that $x, y \in R$. If $x \in N(R)$, then $1+x$ is invertible. By the hypothesis,

$$(1+x)^k y^k = ((1+x)y)^k, k = n, n+1, n+2 \tag{2.41}$$

Hence, one gets

$$(1+x)^{n-1} y^n = y((1+x)y)^{n-1} \tag{2.42}$$

$$(1+x)^n y^{n+1} = y((1+x)y)^n \tag{2.43}$$

$$(1+x)^{n+1} y^{n+2} = y((1+x)y)^{n+1} \tag{2.44}$$

Multiply (2.42) on the right by $(1+x)y$, from (2.43), one gets

$$y^n xy = xy^{n+1} \tag{2.45}$$

Multiply (2.43) on the right by $(1+x)y$, from (2.44), one gets

$$y^{n+1} xy = xy^{n+2} \tag{2.46}$$

Multiply (2.45) on the left by y, from (2.46), one gets

$$(xy - yx)y^{n+1} = 0 \tag{2.47}$$

If $x \notin N(R)$, then by the hypothesis, one gets

$$(xy)^k = x^k y^k, k = n, n+1, n+2 \tag{2.48}$$

If $1+x \in N(R)$, then x is invertible in R. Similar to the proof of above, (2.48) implies

$$(xy - yx)y^{n+1} = 0 \tag{2.49}$$

If $1+x \notin N(R)$, then one has

$$((1+x)y)^k = (1+x)^k y^k, k = n, n+1, n+2 \tag{2.50}$$

Similar to the proof of Theorem 2.6, (2.48) and (2.50) imply

$$(xy - yx)y^{n+1} = 0 \tag{2.51}$$

Hence, (2.47), (2.49) and (2.51) imply that in any case, one has

$$(xy - yx)y^{n+1} = 0 \tag{2.52}$$

Substitute $y+1$ for y in (2.52), one gets

$$(xy - yx)(y + 1)^{n+1} = 0 \tag{2.53}$$

By Lemma 2.3, (2.52) and (2.53), one obtains $xy = yx$. Thus R is commutative. ∎

Following [3], an element x of R is called weakly clean if $x = u + e$ or $x = u - e$ for some $u \in U(R)$ and $e \in E(R)$. The ring R is said to be weakly clean if all of its elements are weakly clean. Clean rings are weakly clean. But the converse is not true because of the example $Z_{(3)} \cap Z_{(5)}$ where $Z_{(p)} = \{\frac{r}{s} | p$ does not divide $s\}$. An element x of R is called weakly exchange if there exists $e \in E(R)$ such that $e \in xR$ and $1-e \in (1-x)R$ or $1-e \in (1+x)R$. The ring R is said to be weakly exchange if all of its elements are weakly exchange. Clearly, exchange elements are weakly exchange. Checking carefully the proof of [3, Theorem 2.1], we find that weakly clean elements are weakly exchange, so weakly clean rings and exchange rings are all weakly exchange. In fact, [3, Theorem 2.1] showed that Abel weakly exchange rings are weakly clean. In this paper, we obtain that NLI weakly exchange rings are weakly clean.

Theorem 2.12 *Let R be an NLI ring and $x \in R$. (1) If x is weakly exchange, then x is weakly clean.*

(2) If x is exchange, then x is clean.

(3) If R is a weakly exchange ring, then R is a weakly clean ring.

(4) If R is an exchange ring, then R is a clean ring.

Proof (1) Let $e \in E(R)$ such that $e \in xR$ and $1 - e \in (1 - x)R$ or $1-e \in (1+x)R$. Write $e = xy$ for some $y = ye \in R$. If $1-e \in (1-x)R$, then let $1-e = (1-x)z$ for some $z = z(1-e) \in R$. By computing, we have $(x-(1-e))(y-z) = 1-(1-e)y-ez$. Since R is a NLI ring and $(1-e)y = (1-e)ye \in N(R)$, $(1-e)yez-ez(1-e)y \in N(R)$, that is, $(1-e)yz-ezy \in N(R)$. Hence there exists $n \geq 1$ such that $((1-e)yz - ezy)^n = 0$. By computing, we have $((1-e)yz)^n+(-1)^n(ezy)^n = 0$, this leads to $((1-e)yz)^n = (ezy)^n = 0$. Since $(ez+(1-e)y)^2 = ezy+(1-e)yz$, $(ez+(1-e)y)^{2n} = (ezy)^n+((1-e)yz)^n = 0$. Hence $1-ez-(1-e)y \in U(R)$, that is, $(x-(1-e))(y-z) \in U(R)$. Let $u \in R$ such that $((x - (1 - e))(y - z))u = 1$. Let $g = ((y - z)u)(x - (1 - e))$. Then $(x-(1-e))g = x-(1-e)$ and $g^2 = g$. Let $h = (x-(1-e)) - g(x-(1-e))$. Then $hg = h$, $gh = 0$ and $h^2 = 0$. Since R is an NLI ring, $(y - z))uh - h(y - z))u \in N(R)$, that is, $(y - z))uh - (1 - g) \in N(R)$. Hence there exists $n \geq 1$ such that $((y - z))uh - (1 - g))^n = 0$, this gives $1 - g = d(y - z)uh$ for some $d \in R$. Thus $1 - g = d(y - z))uh = d(y - z))uhg = (1 - g)g = 0$, so $((y-z)u)(x-(1-e)) = g = 1$. Hence $x-(1-e) \in U(R)$. If $1-e \in (1+x)R$, then let $1 - e = (1 + x)w$ for some $w = w(1 - e) \in R$. By computing, we have $(x + (1 - e))(y + w) = 1 + (1 - e)y - ew$. Similarly, we obtain $1 + (1 - e)y - ew \in U(R)$, this gives $x + (1 - e) \in U(R)$. We are done.

(2) It has been have shown in (1).

(3) and (4) are immediate corollaries of (1) and (2), respectively. ∎

A ring R is called a *generalized CN ring* if for any $a \in N(R)$ and $b \in R$, $ab = 0$ implies $aRb = 0$ or there exists $c \in R$ such that $0 \neq acb \in Z(R)$. Clearly, CN rings are generalized CN. But the converse is not true. For example, the ring R in Example 2.5 is a generalized CN ring, but R is not CN.

A ring R is called *left SF* if every simple left $R-$module is flat. In [13, Remark 3.13], it is shown that if R is a reduced left SF ring, then R is strongly regular. We can generalize this result as follows.

Proposition 2.13 *Let R be a generalized CN ring. If R is a left SF ring, then R is a strongly regular ring.*

Proof Let $a \in R$ with $aRa = 0$. If $a \neq 0$, then there exists a maximal left ideal M of R such that $r(aR) \subseteq M$. Since R is a left SF ring, R/M is flat as left $R-$module. Since $a \in r(aR) \subseteq M$, $a = am$ for some $m \in M$. Since R is a generalized CN ring and $a(1-m) = 0$, $aR(1-m) = 0$, or there exists $c \in R$ such that $0 \neq ac(1-m) \in Z(R)$. If $aR(1-m) = 0$, then $1-m \in r(aR) \subseteq M$, this gives $1 = (1-m)+m \in M$, a contradiction. Thus there exists $c \in R$ such that $0 \neq ac(1-m) \in Z(R)$. Since $(ac(1-m))^2 = 0$, there exists a maximal left ideal N of R such that $l(ac(1-m)) \subseteq N$. Since R is a left SF ring, R/N is flat as left $R-$module. Then since $ac(1-m) \in N$, $ac(1-m) = ac(1-m)n$ for some $n \in N$. Since $ac(1-m) \in Z(R)$, $ac(1-m) = nac(1-m)$, this leads to $1-n \in l(ac(1-m)) \subseteq N$, which implies $1 = (1-n)+n \in N$, a contradiction. Hence $a = 0$, which implies R is a semiprime ring. Now let $b \in R$ with $b^2 = 0$. Since R is a generalized CN ring, either $bRb = 0$ or there exists $c \in R$ such that $0 \neq bab \in Z(R)$. If there exists $c \in R$ such that $0 \neq bcb \in Z(R)$, then $bcbRbcb = bcbbcbR = 0$, so $bcb = 0$ because R is semiprime, which is a contradiction. Hence $bRb = 0$, also, the semiprimeness of R implies $b = 0$. Hence R is a reduced ring. By [13, Remark 3.13], R is a strongly regular ring. ∎

Following [2], a ring R is said to be *semiperiodic* if for each $x \in R \backslash (J(R) \cup Z(R))$, there exist $m, n \in \mathbb{Z}$, of opposite parity, such that $x^n - x^m \in N(R)$. Clearly, the class of semiperiodic rings contains all commutative rings, all Jacobson radical rings, and certain non-nil periodic rings.

Lemma 2.14 *Let R be a generalized CN ring. If R is a semiperiodic ring, then $N(R) \subseteq J(R)$*

Proof Let $a \in N(R)$ with $a^k = 0$, and let $x \in R$. If $ax \in J(R)$, then ax is right quasiregular; and if $ax \in Z(R)$, then ax is nilpotent and again ax is right quasi-regular. Suppose, then, that $ax \notin J(R) \cup Z(R)$, in which case [2, Lemma 2.3(iii)] gives $q \in \mathbb{Z}^+$ and an idempotent e of form axy such that $(ax)^q = (ax)^q e$. Since $e = axy = eaxy = ea(1-e)xy + eaexy = ea(1-e)xy + ea^2(xy)^2 = ea(1-e)xy + ea^2(1-e)(xy)^2 + ea^2e(xy)^2 = ea(1-e)xy + ea^2(1-e)(xy)^2 + ea^3(xy)^3 = \cdots = \Sigma_{i=1}^{k-1} ea^i(1-e)(xy)^i + ea^k(xy)^k = \Sigma_{i=1}^{k-1} ea^i(1-e)(xy)^i$. For any $z \in R$, $ez(1-e) \in N(R)$ and $(ez(1-e))^2 = 0$. Since R is a generalized CN ring, either $ez(1-e)Rez(1-e) = 0$ or there exists $c \in R$ such that $0 \neq ez(1-e)cez(1-e) \in Z(R)$. If there exists $c \in R$ such that $0 \neq ez(1-e)cez(1-e) \in Z(R)$, then $ez(1-e)cez(1-e) = (ez(1-e)cez(1-e))(1-e) = (1-e)ez(1-e)cez(1-e) = 0$, which is a contradiction. Hence $ez(1-e)Rez(1-e) = 0$, which implies $ez(1-e) \in J(R)$ for any $z \in R$. Therefore $e = \Sigma_{i=1}^{k-1} ea^i(1-e)(xy)^i \in J(R)$, this leads to $e = 0$ and $(ax)^q = 0$, which shows that ax is right quasi-regular. Thus $a \in J(R)$. ∎

Theorem 2.15 *If R is a generalized CN semiperiodic ring, then $R/J(R)$ is commutative.*

Proof By [2, Theorem 4.3], R is either commutative or periodic, so we may assume R is periodic. Since $J(R)$ contains no nonzero idempotents, $J(R)$ is contained in $N(R)$ and hence $J(R) = N(R)$ by Lemma 2.14. Thus $R/J(R) = R/N(R)$ is reduced; and since $R/N(R)$ is also semiperiodic, it is commutative by [2, Theorem 4.4]. ∎

Theorem 2.16 *Let R be a generalized CN semiperiodic ring. Then*
(1) $N(R)$ is an ideal of R.
(2) If $J(R) \neq N(R)$, then R is commutative.

Proof In the proof of Theorem 2.15, we obtain that if R is not commutative, then $J(R) = N(R)$. Hence (2) holds and (1) also holds for noncommutative ring R. But also if R is commutaive, $N(R)$ is an ideal; hence (1) holds in any case. ∎

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References

[1] H. E. Bell, On the power map and ring commutativity, Canad. Math. Bull., 21(1978):399-404.

[2] H. E. Bell, and A. Yaqub, On commutativity of semiperiodic rings. Result Math., Online first, 2008 Birkhauser Verlag, Doi: 10.1007/s00025-008-0305-5.

[3] A. Y. M. Chin and K. T. Qua, A note on weakly clean rings. Acta. Math. Hungar., 132(1-2)(2011): 113-116.

[4] M. P. Drazin, Rings with central idempotent or nilpotent elements, Proc. Edinb. Math. Soc., 9(4)(1958): 157-165.

[5] I. N. Herstein, A generalization of a theorem of Jacobson III. Amer. J. Math., 75(1953): 105-111.

[6] I. N. Herstein, Rings with involution, Univ. Chicago Press, Chicago; London, 1976.

[7] I. N. Herstein, A commutativity theorem, J. Algebra, 38(1976): 112-118.

[8] I. N. Herstein, Power maps in rings, Michigan Math. J., 8(1961): 29-32.

[9] H. Abu-Khuzam, A connnutativity theorem for rings, Math. Japan., 25(1980): 593-595.

[10] H. Abu-Khuzam, A connnutativity theorem for semiprime rings, Bull. Austral. Math. Soc., 27(1983): 221-224.

[11] S. Ligh and A. Richoux, A commutativity theorem for rings, Bull. Austral. Math. Soc., 16(1977): 75-77.

[12] W. K. Nicholson and A. Yaqub, A commutativity theorem, Algebra Universalis, 10(1980): 260-263. 113-116.

[13] M. B. Rege, On von Neumann regular rings and $SF-$rings. Math. Japonica, 31(6)(1986): 927-936.