Positive solutions for singular boundary value problem with fractional q-differences

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Abstract

In this paper, we deal with the following nonlinear singular m-point boundary value problem with fractional q-differences

$$(D_q^\alpha u)(t) + f(t, u(t)) = 0, \quad 0 < t < 1, \quad 2 < \alpha < 3,$$
$$u(0) = (D_q u)(0) = 0, \quad (D_q u)(1) = \sum_{i=1}^{m-2} \beta_i (D_q u)(\xi_i),$$

where $0 < \xi_1 < \xi_2 < \cdots < \xi_{m-2} < 1$ and $0 < \sum_{i=1}^{m-2} \beta_i \xi_i^{\alpha-2} < 1$, $0 < q < 1$. $f : (0,1] \times [0, +\infty) \to [0, +\infty)$ with $\lim_{t \to 0^+} f(t, \cdot) = \infty$, i.e., f is singular at $t = 0$. By using the fixed point theorem in partially ordered sets, some new existence and uniqueness of positive solutions to the above boundary value problem are established. As application, an example is presented to illustrate the main results.

Keywords: Fractional q-difference equations; Partially ordered sets; Fixed-point theorem; Positive solution

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1. Introduction

Recently, an increasing interest in studying the existence of solutions for boundary value problems of fractional order functional differential equations has been observed [2–6, 9–12, 15–17]. Fractional differential equations describe many phenomena in various fields of science and engineering such as physics, mechanics, chemistry, control, engineering, etc. For an extensive collection of such results, we refer the readers to the monographs by Samko et al [18], Podlubny [19] and Kilbas et al [20].

On the other hand, the q-difference calculus or quantum calculus is an old subject that was first developed by Jackson [21, 22]. It is rich in history and in applications as the reader

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can confirm in the paper [23].

The origin of the fractional q-difference calculus can be traced back to the works by Al-Salam [24] and Agarwal [25]. More recently, maybe due to the explosion in research within the fractional differential calculus setting, new developments in this theory of fractional q-difference calculus were made, e.g., q-analogues of the integral and differential fractional operators properties such as the q-Laplace transform, q-Taylor's formula [26, 27], just to mention some.

Recently, there are few works consider the existence of positive solutions for nonlinear q-fractional boundary value problem (see [8, 28, 29, 13]). As is well-known, the aim of finding positive solutions to boundary value problems is of main importance in various fields of applied mathematics (see the book [30] and references therein). In addition, since q-calculus has a tremendous potential for applications [23], we find it pertinent to investigate such a demand.

El-Shahed and Hassan [7] studied the existence of positive solutions of the q-difference boundary value problem

$$\begin{aligned}&D_q^2u(t)+a(t)f(u(t))=0, \quad 0\leq t\leq 1,\\ &\alpha u(0)-\beta D_q u(0)=0, \quad \gamma u(1)+\delta D_q u(1)=0.\end{aligned}$$

where $f:[0,1]\times[0,+\infty)\to[0,+\infty)$ is continuous and D_q^2 is the standard Riemann-Liouville q-derivative.

Liang and Zhang [14] considered the following nonlinear m-point fractional boundary value problem

$$\begin{aligned}&D_{0+}^{\alpha}u(t)+f(t,u(t))=0, \quad 0<t<1, \quad 2<\alpha\leq 3,\\ &u(0)=u'(0)=0, \quad u'(1)=\sum_{i=1}^{m-2}\beta_i u'(\xi_i),\end{aligned}$$

where $f:[0,1]\times[0,+\infty)\to[0,+\infty)$ is continuous and D_{0+}^{α} is the standard Riemann-Liouville fractional derivative. The uniqueness of positive solutions is established by using the fixed-point theorem in partially ordered sets.

Qiu and Bai [38] have proved the existence of a positive solution to boundary value problems of the nonlinear fractional differential equations

$$\begin{aligned}&D_{0+}^{\alpha}u(t)+f(t,u(t))=0, \quad 0<t<1, \quad 2<\alpha\leq 3,\\ &u(0)=u'(1)=u''(0)=0,\end{aligned}$$

where D_{0+}^{α} denotes Caputo derivative, and $f:(0,1]\times[0,+\infty)\to[0,+\infty)$ with $\lim_{t\to 0_+} f(t,\cdot)=\infty$ (i.e., f is singular at $t=0$). Their analysis relies on Krasnoselskii's fixed point theorem and nonlinear alternative of Leray-Schauder type in a cone.

More recently, Caballero Mena et al [32] have proved the existence and uniqueness of positive solutions for the singular fractional boundary value problem by using the fixed point theorem in partially ordered sets.

This work is motivated by the above references. In this paper, we deal with the following nonlinear singular m-point boundary value problem with fractional q-differences

$$(D_q^\alpha u)(t) + f(t, u(t)) = 0, \quad 0 < t < 1, \quad 2 < \alpha < 3, \tag{1.1}$$

$$u(0) = (D_q u)(0) = 0, \quad (D_q u)(1) = \sum_{i=1}^{m-2} \beta_i (D_q u)(\xi_i), \tag{1.2}$$

where $f : (0,1] \times [0,+\infty) \to [0,+\infty)$ with $\lim_{t\to 0^+} f(t,\cdot) = \infty$ (i.e., f is singular at $t = 0$), $0 < \xi_1 < \xi_2 < \cdots < \xi_{m-2} < 1$ and $0 < \sum_{i=1}^{m-2} \beta_i \xi_i^{\alpha-2} < 1$, $0 < q < 1$.

From the above works, we can see a fact, although the fractional boundary value problem have been investigated by some authors, the results dealing with the existence of positive solutions of multi-point boundary value problem with q-differences are relatively scarce, especially for the existence and uniqueness of a positive solution to singular fractional boundary value problem (1.1)-(1.2).

Motivated by the reasons above, in this paper we discuss singular fractional boundary value problem (1.1) and (1.2). Using a new fixed point theorem in partially ordered sets due to [1], we give some new existence and uniqueness criteria for singular boundary value problem (1.1) and (1.2). Finally, we present an example to demonstrate our results. Existence of fixed point in partially ordered sets has been considered recently in [31, 33–36].

2. Preliminaries

Let $q \in (0,1)$ and define

$$[a]_q = \frac{1-q^a}{1-q}, \quad a \in \mathbb{R}.$$

The q-analogue of the power function $(a-b)^n$ with $\mathbb{N}_0$ is

$$(a-b)^0 = 1, \quad (a-b)^n = \prod_{k=0}^{n-1} (a - bq^k), \quad n \in \mathbb{N}, \quad a, b \in \mathbb{R}.$$

More generally, if $\alpha \in \mathbb{R}$, then

$$(a-b)^{(\alpha)} = a^\alpha \prod_{n=0}^{\infty} \frac{a - bq^n}{a - bq^{\alpha+n}}.$$

Note that, if $b = 0$ then $a^{(\alpha)} = a^\alpha$. The q-gamma function is defined by

$$\Gamma_q(x) = \frac{(1-q)^{(x-1)}}{(1-q)^{x-1}}, \quad x \in \mathbb{R} \setminus \{0, -1, -2, \ldots\},$$

and satisfies $\Gamma_q(x+1) = [x]_q \Gamma_q(x)$.

The q-derivative of a function f is here defined by

$$(D_q f)(x) = \frac{f(x) - f(qx)}{(1-q)x}, \quad (D_q f)(0) = \lim_{x\to 0} (D_q f)(x),$$

and q-derivatives of higher order by

$$(D_q^0 f)(x) = f(x) \quad \text{and} \quad (D_q^n f)(x) = D_q(D_q^{n-1} f)(x), \quad n \in \mathbb{N}.$$

The q-integral of a function f defined in the interval $[0, b]$ is given by

$$(I_q f)(x) = \int_0^x f(t) d_q t = x(1-q) \sum_{n=0}^{\infty} f(xq^n) q^n, \quad x \in [0, b].$$

If $a \in [0, b]$ and f is defined in the interval $[0, b]$, its integral from a to b is defined by

$$\int_a^b f(t) d_q t = \int_0^b f(t) d_q t - \int_0^a f(t) d_q t.$$

Similarly as done for derivatives, an operator I_q^n can be defined, namely,

$$(I_q^0 f)(x) = f(x) \quad \text{and} \quad (I_q^n f)(x) = I_q(I_q^{n-1} f)(x), \quad n \in \mathbb{N}.$$

The fundamental theorem of calculus applies to these operators I_q and D_q, i.e.,

$$(D_q I_q f)(x) = f(x),$$

and if f is continuous at $x = 0$, then

$$(I_q D_q f)(x) = f(x) - f(0).$$

Basic properties of the two operators can be found in the book [37]. We now point out three formulas that will be used later (${}_i D_q$ denotes the derivative with respect to variable i)

$$[a(t-s)]^{(\alpha)} = a^{\alpha}(t-s)^{(\alpha)}, \tag{2.1}$$

$${}_t D_q (t-s)^{(\alpha)} = [\alpha]_q (t-s)^{(\alpha-1)}, \tag{2.2}$$

$$\left({}_x D_q \int_0^x f(x,t) d_q t \right)(x) = \int_0^x {}_x D_q f(x,t) d_q t + f(qx, x). \tag{2.3}$$

Remark 2.1. [28] We note that if $\alpha > 0$ and $a \leq b \leq t$, then $(t-a)^{(\alpha)} \geq (t-b)^{(\alpha)}$.

The following definition was considered first in [25].

Definition 2.1. Let $\alpha \geq 0$ and f be a function defined on $[0, 1]$. The fractional q-integral of the Riemann-Liouville type is $(I_q^0 f)(x) = f(x)$ and

$$(I_q^{\alpha} f)(x) = \frac{1}{\Gamma_q(\alpha)} \int_0^x (x - qt)^{(\alpha-1)} f(t) d_q t, \quad \alpha > 0, \quad x \in [0, 1].$$

Definition 2.2 ([26])**.** The fractional q-derivative of the Riemann-Liouville type of order $\alpha \geq 0$ is defined by $(D_q^0 f)(x) = f(x)$ and

$$(D_q^{\alpha} f)(x) = (D_q^m I_q^{m-\alpha} f)(x), \quad \alpha > 0,$$

where m is the smallest integer greater than or equal to α.

Next, we list some properties that are already known in the literature. Its proof can be found in [25, 26]

Lemma 2.1. *Let $\alpha, \beta \geq 0$ and f be a function defined on $[0,1]$. Then the next formulas hold:*
(1) $(I_q^{\beta} I_q^{\alpha} f)(x) = (I_q^{\alpha+\beta} f)(x)$,
(2) $(D_q^{\alpha} I_q^{\alpha} f)(x) = f(x)$.

Lemma 2.2 ([28])**.** *Let $\alpha > 0$ and p be a positive integer. Then the following equality holds:*

$$(I_q^{\alpha} D_q^{p} f)(x) = (D_q^{p} I_q^{\alpha} f)(x) - \sum_{k=0}^{p-1} \frac{x^{\alpha-p+k}}{\Gamma_q(\alpha+k-p+1)} (D_q^{k} f)(0).$$

In the sequel, we present the fixed point theorem which we will use later. This result appears in [1].

By Γ we denote the class of those functions $\chi : [0, +\infty) \to [0,1)$ satisfying the following condition

$$\chi(t_n) \to 1 \Rightarrow t_n \to 0.$$

Theorem 2.1 ([1])**.** *Let $(E, \leq)$ be a partially ordered set and suppose that there exists a metric d in E such that (E,d) is a complete metric space. Let $T : E \to E$ be nondecreasing mapping such that there exists an element $x_0 \in E$ with $x_0 \leq Tx_0$. Suppose that there exists $\chi \in \Gamma$ such that*

$$d(Tx, Ty) \leq \chi(d(x,y)) \cdot d(x,y), \quad for\ x, y \in E,\ with\ \ x \geq y,$$

Assume that either T is continuous or X is such that

$$if\ \{x_n\}\ is\ a\ nondecreasing\ sequence\ in\ E\ such\ that\ x_n \to x,\ then\ x_n \leq x,\ \forall\ n \in \mathbb{N} \tag{2.4}$$

Besides, if

$$for\ each\ x, y \in E\ there\ exists\ z \in E\ which\ is\ comparable\ to\ x\ and\ y, \tag{2.5}$$

then T has a unique fixed point.

3. Related lemmas

The basic space used in this paper is $E = C[0,1]$. Then E is a real Banach space with the norm $\|u\| = \max\limits_{0 \leq t \leq 1} |u(t)|$. Note that this space can be equipped with a partial order given by

$$x, y \in C[0,1], \quad x \leq y \Leftrightarrow x(t) \leq y(t), \quad t \in [0,1].$$

In [34] it is proved that $(C[0,1], \leq)$ with the classic metric given by

$$d(x,y) = \sup_{0 \leq t \leq 1} \{|x(t) - y(t)|\}$$

satisfied condition (2.4) of Theorem 2.1. Moreover, for $x, y \in C[0,1]$ as the function $\max\{x,y\} \in C[0,1]$, $(C[0,1], \leq)$ satisfies condition (2.5).

Lemma 3.1. *Let* $0 < \xi_1 < \xi_2 < \cdots < \xi_{m-2} < 1$ *and* $0 < \sum\limits_{i=1}^{m-2} \beta_i \xi_i^{\alpha-2} < 1$. *If* $h \in C[0,1]$, *then the boundary value problem*

$$(D_q^\alpha u)(t) + h(t) = 0, \quad 0 < t < 1, \quad 2 < \alpha < 3, \tag{3.1}$$

$$u(0) = (D_q u)(0) = 0, \quad (D_q u)(1) = \sum_{i=1}^{m-2} \beta_i (D_q u)(\xi_i) \tag{3.2}$$

has a unique solution

$$u(t) = \int_0^1 G(t,qs)h(s)d_qs + \frac{t^{\alpha-1}\sum\limits_{i=1}^{m-2}\beta_i\xi_i^{\alpha-2}}{[\alpha-1]_q(1-\sum\limits_{i=1}^{m-2}\beta_i\xi_i^{\alpha-2})}\int_0^1 H(\xi_i,qs)h(s)d_qs, \tag{3.3}$$

where

$$G(t,s) = \frac{1}{\Gamma_q(\alpha)}\begin{cases} (1-s)^{(\alpha-2)}t^{\alpha-1} - (t-s)^{(\alpha-1)}, & 0 \le s \le t \le 1, \\ (1-s)^{(\alpha-2)}t^{\alpha-1}, & 0 \le t \le s \le 1, \end{cases} \tag{3.4}$$

$$H(t,s) = {}_tD_qG(s,t) = \frac{[\alpha-1]_q}{\Gamma_q(\alpha)}\begin{cases} (1-s)^{(\alpha-2)}t^{\alpha-2} - (t-s)^{(\alpha-2)}, & 0 \le s \le t \le 1, \\ (1-s)^{(\alpha-2)}t^{\alpha-2}, & 0 \le t \le s \le 1. \end{cases} \tag{3.5}$$

Proof. In this case $p = 3$. In view of Lemma 2.1 and Lemma 2.2, from (3.1) we see that

$$(I_q^\alpha D_q^3 I_q^{3-\alpha}u)(x) = -I_q^\alpha f(t,u(t))$$

and

$$u(t) = c_1t^{\alpha-1} + c_2t^{\alpha-2} + c_3t^{\alpha-3} - \int_0^t \frac{(t-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)}h(s)d_qs. \tag{3.6}$$

From (3.2), we know that $c_3 = 0$. Let differentiating both sides of (3.7) one obtain, with the help of (2.1) and (2.2)

$$(D_qu)(t) = [\alpha-1]_qc_1t^{\alpha-2} + [\alpha-2]_qc_2t^{\alpha-3} - \frac{[\alpha-1]_q}{\Gamma_q(\alpha)}\int_0^t (t-qs)^{(\alpha-2)}h(s)d_qs.$$

Using the boundary condition (3.2), we have $c_2 = 0$ and

$$c_1 = \frac{1}{\Gamma_q(\alpha)(1-\sum\limits_{i=1}^{m-2}\beta_i\xi_i^{\alpha-2})}\left[\int_0^1 (1-qs)^{(\alpha-2)}h(s)d_qs - \sum_{i=1}^{m-2}\beta_i\int_0^{\xi_i}(\xi_i-qs)^{(\alpha-2)}h(s)d_qs\right].$$

Therefore, the unique solution of boundary value problem (3.1)-(3.2) is

$$\begin{aligned}
u(t) &= -\int_0^t \frac{(t-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} h(s) d_q s \\
&\quad + \frac{t^{\alpha-1}}{\Gamma_q(\alpha)(1-\sum\limits_{i=1}^{m-2}\beta_i\xi_i^{\alpha-2})}\left[\int_0^1 (1-qs)^{(\alpha-2)} h(s) d_q s - \sum_{i=1}^{m-2}\beta_i \int_0^{\xi_i} (\xi_i - qs)^{(\alpha-2)} h(s) d_q s\right] \\
&= -\int_0^t \frac{(t-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} h(s) d_q s - \frac{t^{\alpha-1}}{\Gamma_q(\alpha)(1-\sum\limits_{i=1}^{m-2}\beta_i\xi_i^{\alpha-2})} \sum_{i=1}^{m-2}\beta_i \int_0^{\xi_i} (\xi_i - qs)^{(\alpha-2)} h(s) d_q s \\
&\quad + \left(\frac{t^{\alpha-1}}{\Gamma_q(\alpha)} + \frac{\sum\limits_{i=1}^{m-2}\beta_i\xi_i^{\alpha-2}t^{\alpha-1}}{\Gamma_q(\alpha)(1-\sum\limits_{i=1}^{m-2}\beta_i\xi_i^{\alpha-2})}\right)\int_0^1 (1-qs)^{(\alpha-2)} h(s) d_q s \\
&= \frac{1}{\Gamma_q(\alpha)}\int_0^t ((1-qs)^{(\alpha-2)} t^{\alpha-1} - (t-qs)^{(\alpha-1)}) h(s) d_q s \\
&\quad + \frac{1}{\Gamma_q(\alpha)}\int_t^1 (1-qs)^{(\alpha-2)} t^{\alpha-1} h(s) d_q s + \frac{\sum\limits_{i=1}^{m-2}\beta_i\xi_i^{\alpha-2}t^{\alpha-1}}{\Gamma_q(\alpha)(1-\sum\limits_{i=1}^{m-2}\beta_i\xi_i^{\alpha-2})}\int_0^1 (1-qs)^{(\alpha-2)} h(s) d_q s \\
&\quad - \frac{t^{\alpha-1}}{\Gamma_q(\alpha)(1-\sum\limits_{i=1}^{m-2}\beta_i\xi_i^{\alpha-2})} \sum_{i=1}^{m-2}\beta_i \int_0^{\xi_i} (\xi_i - qs)^{(\alpha-2)} h(s) d_q s \\
&= \int_0^1 G(t,qs) h(s) d_q s + \frac{t^{\alpha-1}\sum\limits_{i=1}^{m-2}\beta_i\xi_i^{\alpha-2}}{[\alpha-1]_q(1-\sum\limits_{i=1}^{m-2}\beta_i\xi_i^{\alpha-2})}\int_0^1 H(\xi_i, qs) h(s) d_q s.
\end{aligned}$$

The proof is complete. □

Lemma 3.2 ([13])**.** *(i) $G(t,qs)$ is a continuous function on $[0,1]\times[0,1]$ and it satisfies $G(t,qs) > 0$ for $(t,s)\in(0,1)\times(0,1)$;*
(ii) $G(t,qs)$ is strictly increasing in the first variable;
(iii) $H(t,qs) > 0$ for $(t,s)\in(0,1)\times(0,1)$.

Lemma 3.3. *Let $0<\sigma<1$, $2<\alpha\le 3$ and $F:(0,1]\to\mathbb{R}$ is a continuous function with $\lim\limits_{t\to 0^+} F(t) = \infty$. Suppose that $t^\sigma F(t)$ is a continuous function on $[0,1]$. Then the function defined by*

$$Z(t) = \int_0^1 G(t,qs) F(s) d_q s$$

is continuous on $[0,1]$, where $G(t,qs)$ is the Green function defined by (3.4).

Proof. We split the proof in three steps.
Step I. $t_0 = 0$.

It is easily checked that $Z(0)=0$. Since $t^{\sigma}F(t)$ is continuous on $[0,1]$, we can find a constant $M>0$ such that $t^{\sigma}F(t)\leq M$ for any $t\in[0,1]$. Hence

$$\begin{aligned}
|Z(t)-Z(0)| &= |Z(t)| = \left|\int_0^1 G(t,qs)F(s)d_qs\right| \\
&= \left|\int_0^1 G(t,qs)s^{-\sigma}s^{\sigma}F(s)d_qs\right| \\
&= \left|\frac{1}{\Gamma_q(\alpha)}\int_0^t (t^{\alpha-1}(1-qs)^{(\alpha-2)}-(t-qs)^{(\alpha-1)})s^{-\sigma}s^{\sigma}F(s)d_qs\right. \\
&\quad \left. +\frac{1}{\Gamma_q(\alpha)}\int_t^1 t^{\alpha-1}(1-qs)^{(\alpha-2)}s^{-\sigma}s^{\sigma}F(s)d_qs\right| \\
&= \left|\frac{1}{\Gamma_q(\alpha)}\int_0^1 t^{\alpha-1}(1-qs)^{(\alpha-2)}s^{-\sigma}s^{\sigma}F(s)d_qs-\frac{1}{\Gamma_q(\alpha)}\int_0^t (t-qs)^{(\alpha-1)}s^{-\sigma}s^{\sigma}F(s)d_qs\right| \\
&\leq \left|\frac{1}{\Gamma_q(\alpha)}\int_0^1 t^{\alpha-1}(1-qs)^{(\alpha-2)}s^{-\sigma}s^{\sigma}F(s)d_qs\right|+\left|\frac{1}{\Gamma_q(\alpha)}\int_0^t (t-qs)^{(\alpha-1)}s^{-\sigma}s^{\sigma}F(s)d_qs\right| \\
&\leq \frac{M}{\Gamma_q(\alpha)}\int_0^1 t^{\alpha-1}(1-qs)^{(\alpha-2)}s^{-\sigma}d_qs+\frac{M}{\Gamma_q(\alpha)}\int_0^t (t-qs)^{(\alpha-1)}s^{-\sigma}d_qs \\
&\leq \frac{Mt^{\alpha-1}}{\Gamma_q(\alpha)}\int_0^1 (1-qs)^{(\alpha-2)}s^{-\sigma}d_qs+\frac{Mt^{\alpha-1}}{\Gamma_q(\alpha)}\int_0^t (1-\frac{qs}{t})^{(\alpha-1)}s^{-\sigma}d_qs. \qquad (3.7)
\end{aligned}$$

In the integral $\int_0^t(1-\frac{qs}{t})^{(\alpha-1)}s^{-\sigma}d_qs$ we make the change of variables $v=\frac{s}{t}$, then we obtain

$$\int_0^t (1-\frac{qs}{t})^{(\alpha-1)}s^{-\sigma}d_qs = t^{1-\sigma}\int_0^1 (1-qv)^{(\alpha-1)}v^{-\sigma}d_qv.$$

Taking into account (3.7), we have

$$\begin{aligned}
|Z(t)| &\leq \frac{Mt^{\alpha-1}}{\Gamma_q(\alpha)}\int_0^1 (1-qs)^{(\alpha-2)}s^{-\sigma}d_qs+\frac{Mt^{\alpha-1}}{\Gamma_q(\alpha)}t^{1-\sigma}\int_0^1 (1-qv)^{(\alpha-1)}v^{-\sigma}d_qv \\
&= \frac{Mt^{\alpha-1}}{\Gamma_q(\alpha)}B_q(1-\sigma,\alpha-1)+\frac{Mt^{\alpha-\sigma}}{\Gamma_q(\alpha)}B_q(1-\sigma,\alpha), \qquad (3.8)
\end{aligned}$$

where B_q denotes the Beta function defined by $B_q(t,s)=\displaystyle\int_0^1 x^{t-1}(1-qx)^{(s-1)}d_qs$. Therefore, by (3.8), we see that $Z(t)\to 0$ when $t\to 0$, this proves the continuity of Z at $t_0=0$.

Step II. $t_0\in(0,1)$.

We take $t_n\to t_0$ and we have to prove that $Z(t_n)\to Z(t_0)$. Without loss of generality we consider $t_n > t_0$ (the same argument works for $t_n < t_0$). In fact, we have

$$\begin{aligned}
|Z(t_n) - Z(t_0)| &= \frac{1}{\Gamma_q(\alpha)}\left|\int_0^{t_n}(t_n^{\alpha-1}(1-qs)^{(\alpha-2)} - (t_n-qs)^{(\alpha-1)})s^{-\sigma}s^{\sigma}F(s)d_qs\right. \\
&\quad + \int_{t_n}^1 t_n^{\alpha-1}(1-qs)^{(\alpha-2)}s^{-\sigma}s^{\sigma}F(s)d_qs - \int_{t_0}^1 t_0^{\alpha-1}(1-qs)^{(\alpha-2)}s^{-\sigma}s^{\sigma}F(s)d_qs \\
&\quad \left. - \int_0^{t_0}(t_0^{\alpha-1}(1-qs)^{(\alpha-2)} - (t_0-qs)^{(\alpha-1)})s^{-\sigma}s^{\sigma}F(s)d_qs\right| \\
&= \frac{1}{\Gamma_q(\alpha)}\left|\int_0^1 t_n^{\alpha-1}(1-qs)^{(\alpha-2)}s^{-\sigma}s^{\sigma}F(s)d_qs - \int_0^{t_n}(t_n-qs)^{(\alpha-1)}s^{-\sigma}s^{\sigma}F(s)d_qs\right. \\
&\quad \left. - \int_0^1 t_0^{\alpha-1}(1-qs)^{(\alpha-2)}s^{-\sigma}s^{\sigma}F(s)d_qs + \int_0^{t_0}(t_0-qs)^{(\alpha-1)}s^{-\sigma}s^{\sigma}F(s)d_qs\right| \\
&= \frac{1}{\Gamma_q(\alpha)}\left|\int_0^1 (t_n^{\alpha-1} - t_0^{\alpha-1})(1-qs)^{(\alpha-2)}s^{-\sigma}s^{\sigma}F(s)d_qs\right. \\
&\quad - \int_0^{t_0}((t_n-qs)^{(\alpha-1)} - (t_0-qs)^{(\alpha-1)})s^{-\sigma}s^{\sigma}F(s)d_qs \\
&\quad \left. - \int_{t_0}^{t_n}(t_n-qs)^{(\alpha-1)}s^{-\sigma}s^{\sigma}F(s)d_qs\right| \\
&\leq \frac{M(t_n^{\alpha-1} - t_0^{\alpha-1})}{\Gamma_q(\alpha)}\int_0^1 (1-qs)^{(\alpha-2)}s^{-\sigma}d_qs \\
&\quad + \frac{M}{\Gamma_q(\alpha)}\int_0^{t_0}((t_n-qs)^{(\alpha-1)} - (t_0-qs)^{(\alpha-1)})s^{-\sigma}d_qs \\
&\quad + \frac{M}{\Gamma_q(\alpha)} + \int_{t_0}^{t_n}(t_n-qs)^{(\alpha-1)}s^{-\sigma}d_qs \\
&= \frac{M(t_n^{\alpha-1} - t_0^{\alpha-1})}{\Gamma_q(\alpha)}B_q(1-\sigma, \alpha-1) + \frac{M}{\Gamma_q(\alpha)}J_n^1 + \frac{M}{\Gamma_q(\alpha)}J_n^2, \qquad (3.9)
\end{aligned}$$

where

$$\begin{aligned}
J_n^1 &= \int_0^{t_0}((t_n-qs)^{(\alpha-1)} - (t_0-qs)^{(\alpha-1)})s^{-\sigma}d_qs, \\
J_n^2 &= \int_{t_0}^{t_n}(t_n-s)^{(\alpha-1)}s^{-\sigma}d_qs.
\end{aligned}$$

We claim that $J_n^1 \to 0$ when $n \to 0$.

In fact, as $t_n \to t_0$, then

$$((t_n-qs)^{(\alpha-1)} - (t_0-qs)^{(\alpha-1)})s^{-\sigma} \to 0, \quad \text{when } n \to \infty.$$

Moreover,

$$((t_n-qs)^{(\alpha-1)} - (t_0-qs)^{(\alpha-1)})s^{-\sigma} \leq (|t_n-qs|^{(\alpha-1)} + |t_0-qs|^{(\alpha-1)})s^{-\sigma} \leq 2s^{-\sigma}$$

and as

$$\int_0^1 2s^{-\sigma}d_qs = \frac{2(1-q)}{1-q^{1-\sigma}} < \infty,$$

we have that the sequence $((t_n-qs)^{(\alpha-1)} - (t_0-qs)^{(\alpha-1)})s^{-\sigma}$ converges pointwise to the zero function and $|(t_n-qs)^{(\alpha-1)} - (t_0-qs)^{(\alpha-1)}|s^{-\sigma}$ is bounded by a function belonging to

$L^1[0,1]$, then by the Lebesgue's dominated convergence theorem, we have

$$J_n^1 \to 0 \quad \text{when } n \to \infty. \tag{3.10}$$

This proves the claim.

Now, we prove that $J_n^2 \to 0$, when $n \to \infty$.

In fact, as

$$\begin{aligned} J_n^2 &= \int_{t_0}^{t_n} (t_n - qs)^{(\alpha-1)} s^{-\sigma} d_q s \\ &\le \int_{t_0}^{t_n} s^{-\sigma} d_q s = \frac{1-q}{1-q^{1-\sigma}} (t_n^{1-\sigma} - t_0^{1-\sigma}) \end{aligned}$$

and taking into account that $t_n \to t_0$, from the last expression we get

$$J_n^2 \to 0 \quad \text{when } n \to \infty. \tag{3.11}$$

Finally, from (3.9), (3.10) and (3.11) we obtain

$$|Z(t_n) - Z(t_0)| \to 0 \quad \text{when } n \to \infty.$$

Step III. $t_0 = 1$.

It is easily checked that $H(1) = 0$. Following the same lines that in the proof of Step I, we can prove the continuity of Z at $t_0 = 1$. □

Lemma 3.4. *Suppose that $0 < \sigma < 1$. Then*

$$\sup_{t\in[0,1]} \int_0^1 G(t, qs) s^{-\sigma} d_q s = \frac{\rho^{\alpha-1} B_q(1-\sigma, \alpha-1) - \rho^{\alpha-\sigma} B_q(1-\sigma, \alpha)}{\Gamma_q(\alpha)},$$

where $G(t, s)$ is the Green's function appearing in of Lemma 3.1 and

$$\rho = \left(\frac{[\alpha-1]_q B_q(1-\sigma, \alpha-1)}{[\alpha-\sigma]_q B_q(1-\sigma, \alpha)} \right)^{\frac{1}{1-\sigma}}.$$

Proof. Since

$$\begin{aligned} \int_0^1 G(t, qs) s^{-\sigma} d_q s &= \frac{1}{\Gamma_q(\alpha)} \int_0^1 t^{\alpha-1} (1-qs)^{(\alpha-2)} s^{-\sigma} d_q s - \frac{1}{\Gamma_q(\alpha)} \int_0^t (t-qs)^{(\alpha-1)} s^{-\sigma} d_q s \\ &= \frac{t^{\alpha-1}}{\Gamma_q(\alpha)} \int_0^1 (1-qs)^{(\alpha-2)} s^{-\sigma} d_q s - \frac{1}{\Gamma_q(\alpha)} \int_0^t (t-qs)^{(\alpha-1)} s^{-\sigma} d_q s \\ &= \frac{t^{\alpha-1}}{\Gamma_q(\alpha)} B_q(1-\sigma, \alpha-1) - \frac{t^{\alpha-\sigma}}{\Gamma_q(\alpha)} B_q(1-\sigma, \alpha). \end{aligned}$$

Now, using elemental calculus we can prove that the function

$$g(t) = t^{\alpha-1} B_q(1-\sigma, \alpha-1) - t^{\alpha-\sigma} B_q(1-\sigma, \alpha)$$

has a maximum at the point $t_0 = \rho = \left(\frac{[\alpha-1]_q B_q(1-\sigma,\alpha-1)}{[\alpha-\sigma]_q B_q(1-\sigma,\alpha)} \right)^{\frac{1}{1-\sigma}}$. So we have

$$\sup_{t\in[0,1]} \int_0^1 G(t, qs) s^{-\sigma} d_q s = \frac{\rho^{\alpha-1} B_q(1-\sigma, \alpha-1) - \rho^{\alpha-\sigma} B_q(1-\sigma, \alpha)}{\Gamma_q(\alpha)}.$$

The proof is complete. □

Remark 3.1. Similar to the proof of Lemma 3.4, we have

$$\begin{aligned}\int_0^1 H(\xi_i, qs)s^{-\sigma}d_qs &= \frac{[\alpha-1]_q}{\Gamma_q(\alpha)}\left(\int_0^1 \xi_i^{\alpha-2}(1-s)^{(\alpha-2)}s^{-\sigma}d_qs - \int_0^{\xi_i}(\xi_i-s)^{(\alpha-2)}s^{-\sigma}d_qs\right)\\ &= \frac{[\alpha-1]_q}{\Gamma_q(\alpha)}\left(\xi_i^{\alpha-2}\int_0^1(1-qs)^{(\alpha-2)}s^{-\sigma}d_qs - \xi_i^{\alpha-\sigma-1}\int_0^1(1-qv)^{(\alpha-2)}v^{-\sigma}d_qv\right)\\ &= \frac{[\alpha-1]_q}{\Gamma_q(\alpha)}B_q(1-\sigma,\alpha-1)\left(\xi_i^{\alpha-2}-\xi_i^{\alpha-\sigma-1}\right).\end{aligned}$$

Now, we introduce the following class of functions. By $\mathcal{T}$ we denote the class of functions $\varphi : [0,+\infty) \to [0,+\infty)$ satisfying:
(i) φ is nondecreasing;
(ii) $\varphi(x) < x$, for any $x > 0$;
(iii) $\chi(x) = \frac{\varphi(x)}{x} \in \Gamma$, where Γ is the class of functions appearing in Theorem 2.1.

Remark 3.2. It is easily checked that examples of functions belonging to $\mathcal{T}$ are $\varphi(x) = \frac{x}{1+x}$ and $\varphi(x) = \ln(1+x)$ with $x \in [0,+\infty)$.

4. Main Result

The main result of this paper is the following.

Theorem 4.1. *Let $0 < \sigma < 1$, $2 < \alpha \leq 3$. The boundary value problem (1.1)-(1.2) has a unique positive and strictly increasing solution $u(t)$ (this means that $u(t) > 0$) if the following conditions are satisfied:*

(i) *$f : (0,1] \times [0,+\infty) \to [0,+\infty)$ is continuous and $\lim_{t\to 0^+} f(t,\cdot) = \infty$, $t^\sigma f(t,y)$ is a continuous on $[0,1] \times [0,+\infty)$;*

(ii) *There exists $0 < \lambda < L^{-1}$ such that for $u, v \in [0,+\infty)$ with $u \geq v$ and $t \in [0,1]$,*

$$0 \leq t^\sigma(f(t,u) - f(t,v)) \leq \lambda \cdot \varphi(u-v),$$

where $\varphi \in \mathcal{T}$ and

$$L = \frac{\rho^{\alpha-1}B_q(1-\sigma,\alpha-1) - \rho^{\alpha-\sigma}B_q(1-\sigma,\alpha)}{\Gamma_q(\alpha)} + \frac{\sum\limits_{i=1}^{m-2}\beta_i\left(\xi_i^{\alpha-2}-\xi_i^{\alpha-\sigma-1}\right)}{\Gamma_q(\alpha)(1-\sum\limits_{i=1}^{m-2}\beta_i\xi_i^{\alpha-2})}B_q(1-\sigma,\alpha-1),$$

ρ is the constant appearing in Lemma 3.3.

Proof. Consider the cone

$$K = \{u \in C[0,1] : u(t) \geq 0\}.$$

As K is a closed set of $C[0,1]$, K is a complete metric space with the distance given by $d(u,v) = \sup_{t\in[0,1]} |u(t) - v(t)|$. It is easily checked that K satisfies condition (2.1) and (2.2) of

Theorem 2.1.

Now, we consider the operator T defined by

$$(Tu)(t) = \int_0^1 G(t,qs)f(s,u(s))d_qs + \frac{t^{\alpha-1}\sum\limits_{i=1}^{m-2}\beta_i}{[\alpha-1]_q(1-\sum\limits_{i=1}^{m-2}\beta_i\xi_i^{\alpha-2})}\int_0^1 H(\xi_i,qs)f(s,u(s))d_qs,$$

By Lemma 3.3, we have that $Tu \in C[0,1]$. Moreover, in view of Lemma 3.2 and $t^\sigma f(t,y)$, for $u \in K$, we have $Tu \in K$. Hence, $T(K) \subset K$.

We now show that all the conditions of Theorem 2.1 are satisfied.

Firstly, by condition (ii), for $u, v \in K$ and $u \geq v$, we have

$$\begin{aligned}
(Tu)(t) &= \int_0^1 G(t,qs)s^{-\sigma}s^\sigma f(s,u(s))d_qs \\
&\quad + \frac{t^{\alpha-1}\sum\limits_{i=1}^{m-2}\beta_i}{[\alpha-1]_q(1-\sum\limits_{i=1}^{m-2}\beta_i\xi_i^{\alpha-2})}\int_0^1 H(\xi_i,qs)s^{-\sigma}s^\sigma f(s,u(s))d_qs \\
&\geq \int_0^1 G(t,qs)s^{-\sigma}s^\sigma f(s,v(s))d_qs \\
&\quad + \frac{t^{\alpha-1}\sum\limits_{i=1}^{m-2}\beta_i}{[\alpha-1]_q(1-\sum\limits_{i=1}^{m-2}\beta_i\xi_i^{\alpha-2})}\int_0^1 H(\xi_i,qs)s^{-\sigma}s^\sigma f(s,v(s))d_qs \\
&= Tv(t).
\end{aligned}$$

This proves that T is a nondecreasing operator.

On the other hand, for $u \geq v$ and by condition (ii) we have

$$\begin{aligned}
d(Tu,Tv) &= \sup_{0\leq t\leq 1}|(Tu)(t)-(Tv)(t)| = \sup_{0\leq t\leq 1}((Tu)(t)-(Tv)(t)) \\
&\leq \sup_{0\leq t\leq 1}\int_0^1 G(t,qs)s^{-\sigma}s^\sigma(f(s,u(s))-f(s,v(s)))d_qs \\
&\quad + \frac{\sum\limits_{i=1}^{m-2}\beta_i}{[\alpha-1]_q(1-\sum\limits_{i=1}^{m-2}\beta_i\xi_i^{\alpha-2})}\int_0^1 H(\xi_i,qs)s^{-\sigma}s^\sigma(f(s,u(s))-f(s,v(s)))d_qs \\
&\leq \sup_{0\leq t\leq 1}\int_0^1 G(t,qs)s^{-\sigma}\lambda\cdot\varphi(u(s)-v(s))d_qs \\
&\quad + \frac{\sum\limits_{i=1}^{m-2}\beta_i}{[\alpha-1]_q(1-\sum\limits_{i=1}^{m-2}\beta_i\xi_i^{\alpha-2})}\int_0^1 H(\xi_i,qs)s^{-\sigma}\lambda\cdot\varphi(u(s)-v(s))d_qs.
\end{aligned}$$

Since the function $\varphi(x)$ is nondecreasing, by Lemma 3.4 and Remark 3.1, we have

$$\begin{aligned} d(Tu,Tv) &\leq \sup_{0\leq t\leq 1}\int_0^1 G(t,s)s^{-\sigma}\lambda\cdot\varphi(d(u-v))ds \\ &\quad + \frac{\sum_{i=1}^{m-2}\beta_i}{(\alpha-1)(1-\sum_{i=1}^{m-2}\beta_i\xi_i^{\alpha-2})}\int_0^1 H(\xi_i,s)s^{-\sigma}\lambda\cdot\varphi(d(u-v))ds \\ &= \lambda\cdot\varphi(d(u-v))L. \end{aligned}$$

Thus the fact that $0<\lambda<L^{-1}$ give us

$$d(Tu,Tv) \leq \varphi(d(u-v)) = \frac{\varphi(d(u-v))}{d(u-v)}\cdot d(u-v) = \chi(d(u-v))\cdot d(u-v).$$

Obviously, the last inequality is satisfied for $u=v$.

Now, taking into account that the zero function satisfies $0\leq T0$, Theorem 2.1 says us that the operator T has a unique fixed point in K, or, equivalently, problem (1.1)-(1.2) has a unique nonnegative solution $u(t)\in C[0,1]$.

In the sequel, we will prove that $u(t)$ is a positive solution.

In contrary case, there exists $0<t_*<1$ such that $u(t_*)=0$. As the nonnegative solution $u(t)$ of problem (1.1)-(1.2) is fixed point of the operator T, this says us that

$$u(t) = \int_0^1 G(t,qs)f(s,u(s))d_qs + \frac{t^{\alpha-1}\sum_{i=1}^{m-2}\beta_i}{[\alpha-1]_q(1-\sum_{i=1}^{m-2}\beta_i\xi_i^{\alpha-2})}\int_0^1 H(\xi_i,qs)f(s,u(s))d_qs,$$

for $t\in(0,1)$, and particularly,

$$u(t_*) = \int_0^1 G(t_*,qs)f(s,u(s))d_qs + \frac{t_*^{\alpha-1}\sum_{i=1}^{m-2}\beta_i}{[\alpha-1]_q(1-\sum_{i=1}^{m-2}\beta_i\xi_i^{\alpha-2})}\int_0^1 H(\xi_i,qs)f(s,u(s))d_qs = 0.$$

The nonnegative character of $G(t,qs)$ and $f(s,u)$ and the last relation give

$$G(t_*,qs)f(s,u(s))=0, \quad a.e.(s). \tag{4.1}$$

Taking into account that $\lim_{t\to 0^+} f(t,0)=\infty$, this means that for $M>0$ we can find δ such that for $s\in[0,1]\cap(0,\delta)$ we have $f(s,0)>M$. Observe that $[0,1]\cap(0,\delta)\subset\{s\in[0,1]: f(s,u(s))>M\}$ and $\mu([0,1]\cap(0,\delta))>0$, where μ is the Lebesgue measure on $[0,1]$. This and (4.1) give us that

$$G(t_*,qs)=0, \quad a.e.(s)$$

and this is a contradiction because $G(t_*,qs)$ is a rational function in the variable s. Therefore $u(t)>0$ for $t\in(0,1)$.

Finally, we will prove that this solution $u(t)$ is strictly increasing function.

As $u(0)=\int_0^1 G(0,qs)f(s,u(s))ds$ and $G(0,qs)=0$ we have $u(0)=0$. Moreover, if we take $t_1,t_2\in[0,1]$ with $t_1<t_2$, we can consider the following cases.

Case 1: $t_1=0$, in this case, $u(t_1)=0$. On the other hand, by using the same reasoning as

above, we have $u(t_2) > 0 = u(t_1)$, for $0 = t_1 < t_2$.
Case 2: $0 < t_1$. In this case, let us take $t_2 \in [0,1]$ with $t_1 < t_2$, then

$$\begin{aligned} u(t_2) - u(t_1) &= (Tu)(t_2) - (Tu)(t_1) \\ &= \int_0^1 (G(t_2, qs) - G(t_1, qs)) f(s, u(s)) d_q s \\ &\quad + \frac{(t_2^{\alpha-1} - t_1^{\alpha-1}) \sum\limits_{i=1}^{m-2} \beta_i}{[\alpha-1]_q (1 - \sum\limits_{i=1}^{m-2} \beta_i \xi_i^{\alpha-2})} \int_0^1 H(\xi_i, qs) f(s, u(s)) d_q s. \end{aligned}$$

Taking into account Lemma 3.2 and the fact that $f \geq 0$, we get $u(t_2) - u(t_1) \geq 0$.

Suppose that $u(t_2) = u(t_1)$ then

$$\int_0^1 (G(t_2, qs) - G(t_1, qs)) f(s, u(s)) d_q s = 0$$

and this implies

$$(G(t_2, qs) - G(t_1, qs)) f(s, u(s)) = 0 \quad a.e.(s).$$

Again, the same reasoning as above gives us

$$f(s, u(s)) = 0 \quad a.e.(s)$$

this contradicts condition Lemma 3.2. Thus $u(t_1) < u(t_2)$. The proof is complete. □

5. Example

Example 5.1. *The fractional boundary value problem*

$$\begin{cases} D_q^{\frac{3}{2}} u(t) + \frac{(\frac{1}{10}t^2+1)\ln(2+u(t))}{\sqrt{t}} = 0, \quad 0 < t < 1, \\ u(0) = (D_q u)(0) = 0, \quad (D_q u)(1) = \frac{1}{4}(D_q u)(\frac{1}{4}) \end{cases} \tag{5.1}$$

has a unique and strictly increasing solution.

Proof. In this case, $\alpha = \frac{3}{2}$, $\sigma = \frac{1}{2}$. $f(t,u) = \frac{(\frac{1}{10}t^2+1)\ln(2+u(t))}{\sqrt{t}}$ for $(t,u) \in (0,1] \times [0,\infty)$. Now, we prove that $f(t,u)$ satisfies assumptions of Theorem 4.1. Note that $f : (0,1] \times [0,+\infty) \to [0,+\infty)$ is continuous and $\lim\limits_{t \to 0^+} f(t,\cdot) = \infty$, $t^{\frac{1}{2}} f(t,u) = (\frac{1}{10}t^2+1)\ln(2+u(t))$ is a continuous on $[0,1] \times [0,+\infty)$.

On the other hand, for $u \geq v$ and $t \in [0,1]$, we have

$$\begin{aligned} t^{\frac{1}{2}}(f(t,u) - f(t,v)) &= (\frac{1}{10}t^2+1)\ln(2+u) - (t^2+1)\ln(2+v) \\ &= (\frac{1}{10}t^2+1)\ln\left(\frac{2+u}{2+v}\right) \\ &= (\frac{1}{10}t^2+1)\ln\left(\frac{2+v+u-v}{2+v}\right) \\ &= (\frac{1}{10}t^2+1)\ln\left(1+\frac{u-v}{2+v}\right) \\ &\leq (\frac{1}{10}t^2+1)\ln(1+(u-v)) \leq \frac{11}{10}\ln(1+u-v). \end{aligned}$$

In this case, $\lambda=\frac{11}{10}, \xi_1=\frac{1}{4}, \beta_1=\frac{1}{4}, \alpha=\frac{3}{2}$. Then by direct calculation we can obtain that

$$\rho \quad = \quad \left(\frac{[\alpha-1]_q B_q(1-\sigma,\alpha-1)}{[\alpha-\sigma]_q B_q(1-\sigma,\alpha)}\right)^{\frac{1}{1-\sigma}} = \left(\frac{[\frac{1}{2}]_q B_q(\frac{1}{2},\frac{1}{2})}{[1]_q B_q(\frac{1}{2},\frac{3}{2})}\right)^2 = \frac{9(11-6\sqrt{2})}{49}$$

and

$$L=\frac{9\sqrt{2}-12}{9-4\sqrt{2}}+\frac{\Gamma_q(\frac{1}{2})}{4-\sqrt{2}}<\frac{10}{11}=\frac{1}{\lambda}.$$

Here we use the relations

$$B_q(s,t)=\frac{\Gamma_q(s)\Gamma_q(t)}{\Gamma_q(s+t)}, \quad [\frac{1}{2}]_q=2-\sqrt{2}, \quad [\frac{3}{2}]_q=2-\frac{\sqrt{2}}{2}, \quad [2]_q=\frac{3}{2}.$$

Thus Theorem 4.1 implies that boundary value problem (1.1)-(1.2) has a unique and strictly increasing solution. □

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