

CONVERGENCE THEOREMS FOR A HYBRID PAIR OF GENERALIZED NONEXPANSIVE MAPPINGS IN BANACH SPACES

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ABSTRACT. In this paper, we study convergence of modified Ishikawa iteration process involving a pair of hybrid mappings satisfying a generalized nonexpansive condition on Banach spaces. In process, several relevant results are generalized and improved.

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1. INTRODUCTION

Let X be a Banach space and K be a nonempty subset of X . Let $CB(K)$ be the family of nonempty closed bounded subsets of K while $KC(K)$ be the family of nonempty compact convex subsets of K . A subset K of X is called proximal if for each $x \in X$, there exists an element $k \in K$ such that

$$d(x, k) = \text{dist}(x, K) = \inf\{\|x - y\| : y \in K\}.$$

It is well known that every closed convex subset of a uniformly convex Banach space is proximal. We shall denote by $PB(K)$, the family of nonempty bounded proximal subsets of K . The Hausdorff metric H on $CB(K)$ is defined as

$$H(A, B) = \max\left\{\sup_{x \in A} d(x, B), \sup_{y \in B} d(y, A)\right\} \text{ for } A, B \in CB(K).$$

A mapping $f : K \rightarrow K$ is said to be nonexpansive if

$$\|fx - fy\| \leq \|x - y\|, \text{ for all } x, y \in K,$$

while a multivalued mapping $T : K \rightarrow CB(K)$ is said to be nonexpansive if

$$H(T(x), T(y)) \leq \|x - y\|, \text{ for all } x, y \in K.$$

We use the notation $Fix(T)$ for the set of fixed points of the mapping T while $Fix(f) \cap Fix(T)$ denotes the set of common fixed points of f and T , i.e. a point x is said to be a common fixed point of f and T if $x = fx \in Tx$.

In 2008, Suzuki [1] introduced a new class of mappings which is larger than the class of nonexpansive mappings and name the defining condition as Condition (C) which runs follows:

A mapping T defined on a subset K of a Banach space X is said to satisfy condition (C) if

$$\frac{1}{2}\|x - Tx\| \leq \|x - y\| \Rightarrow \|Tx - Ty\| \leq \|x - y\|, x, y \in K.$$

He then proved some fixed point as well as convergence theorems for such mappings. Inspired from this, recently García-Falsat et al. [2] defined two new generalizations of Condition (C) and term their

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conditions as Condition (E) and Condition (C_λ) and studied the existence of fixed point besides their asymptotic behaviour for both the classes of mappings in Banach spaces.

In 2010, Sokhuma and Kaewkhao [3] introduced a modified Ishikawa iterative process involving a pair of single valued and multivalued nonexpansive mappings in Banach spaces and proved strong convergence theorems.

The purpose of this paper is to study modified Ishikawa iterative method for generalized nonexpansive mappings introduced by García-Falsat et al. [2].

2. PRELIMINARIES

With a view to make, our presentation self contained, we collect some basic definitions, needed results and some iterative methods which will be used frequently in the text later. We start with the following definitions:

Definition 2.1. [2] *Let K be a nonempty subset of a Banach space X and $S : K \rightarrow X$ be a single valued mapping. Then S is said to satisfy the condition (E_μ) for some $\mu \geq 1$ if*

$$\|x - Sy\| \leq \mu\|x - Sx\| + \|x - y\|, \text{ for } x, y \in K.$$

We say that S satisfies condition (E) whenever S satisfies the condition (E_μ) for some $\mu \geq 1$.

Definition 2.2. [2] *Let K be a nonempty subset of a Banach space X and $S : K \rightarrow X$ be a single valued mapping. Then S is said to satisfy the condition (C_λ) for some $\lambda \in (0, 1)$ if*

$$\lambda\|x - Sx\| \leq \|x - y\| \Rightarrow \|Sx - Sy\| \leq \|x - y\|, \text{ for } x, y \in K.$$

From the foregoing definitions, one can infer the following observations:

Lemma 2.1. [2] *If $S : K \rightarrow X$ is a nonexpansive mapping, then S satisfies the condition (E_1). But the converse statement is not true in general.*

Lemma 2.2. [2] *From Lemma 1 in [1], one can see that if $S : K \rightarrow K$ satisfies the condition (C), then S satisfies condition E_3 . But the converse statement is not true in general.*

The following example supports two preceding facts.

Example 2.1. [2] *Consider $X = C[0, 1]$ and the nonempty subset K as follows:*

$$K = \{f \in C[0, 1] : 0 = f(0) \leq f(x) \leq f(1) = 1\}.$$

For any $g \in K$, we construct the function $F_g : K \rightarrow K$

$$F_g h(t) = (g \circ h)(t) = g(h(t)).$$

It is easy to verify that F_g satisfies the condition (E_1) and but does not satisfy the condition (C).

By choosing $\lambda = \frac{1}{2}$ we can retrieve the class of Suzuki's generalized nonexpansive mappings. From the definition of condition (C_λ), one can easily see that condition (C_{λ_1}) implies the condition (C_{λ_2}) for $0 < \lambda_1 < \lambda_2 < 1$. The following example shows that the converse fails.

Example 2.2. [2] *For a given $\lambda \in (0, 1)$ define a mapping S on $[0, 1]$ by*

$$S(x) = \begin{cases} \frac{x}{2} & \text{if } x \neq 1 \\ \frac{1+\lambda}{2+\lambda} & \text{if } x = 1. \end{cases}$$

Then the mapping S satisfies condition (C_λ) but it fails condition (C_{λ_1}) whenever $0 < \lambda_1 < \lambda$. Moreover S satisfies condition (E_μ) for $\mu = \frac{2+\lambda}{2}$.

In the same paper, García-Falsat et al. used the concept of strongly demiclosed to prove a weaker version of famous principle of demiclosedness for generalized nonexpansive mappings in which weak convergence has been replaced by strong convergence. Definition run as follows:

Definition 2.3. [2] *Given a mapping $S : K \rightarrow X$, we say that $(I - S)$ is strongly demiclosed at 0 if for every sequence $\{x_n\}$ in K strongly convergent to $z \in K$ and such that $x_n - Sx_n \rightarrow 0$ we have that $z = Sz$.*

Lemma 2.3. [2] *Let K be a nonempty subset of a Banach space X . If $S : K \rightarrow X$ satisfies condition (E) on K , then $(I - S)$ is strongly demiclosed at 0.*

Kaewcharoen and Panyanak [4] gave the multivalued version of Condition (C_λ) .

Definition 2.4. [2] *Let $T : K \rightarrow CB(X)$ be a multivalued mapping. Then T is said to satisfy condition (C_λ) for some $\lambda \in (0, 1)$ if each $x, y \in K$,*

$$\lambda \operatorname{dist}(x, Tx) \leq \|x - y\| \text{ implies } H(Tx, Ty) \leq \|x - y\|.$$

As the singlevalued case, one can easily verify that condition for $0 < \lambda_1 < \lambda_2 < 1$, (C_{λ_1}) implies condition (C_{λ_2}) .

Lemma 2.4. [4] *Let K be a nonempty subset of a Banach space X and $T : K \rightarrow CB(X)$ be a mapping. If T is a nonexpansive, then T satisfies condition (C_λ) .*

The following example exhibits that the converse statement need not be true.

Example 2.3. [4] *Define a mapping $T : [0, 3] \rightarrow CB([0, 3])$ by*

$$T(x) = \begin{cases} \{0\}, & \text{if } x \neq 3, \\ [0.5, 1], & \text{if } x = 3. \end{cases}$$

T satisfies condition (C_λ) for $\lambda = \frac{1}{2}$ but T is not nonexpansive.

Now we list the following important property of a uniformly convex Banach space due to Schu [5] and an important lemma due to Khaewkhao and Sokhuma [3].

Lemma 2.5. *Let X be a uniformly convex Banach space, let $\{u_n\}$ be a sequence of real numbers such that $0 < b \leq u_n \leq c < 1$ for all $n \geq 1$, and let $\{x_n\}$ and $\{y_n\}$ be sequences in X such that $\limsup_{n \rightarrow \infty} \|x_n\| \leq a$, $\limsup_{n \rightarrow \infty} \|y_n\| \leq a$, and $\lim_{n \rightarrow \infty} \|u_n x_n + (1 - u_n) y_n\| = a$ for some $a > 0$. Then, $\lim_{n \rightarrow \infty} \|x_n - y_n\| = 0$.*

Lemma 2.6. *Let X be a Banach space, and let K be a nonempty closed convex subset of X . Then,*

$$\operatorname{dist}(y, Ty) \leq \|y - x\| + \operatorname{dist}(x, Tx) + H(Tx, Ty),$$

where $x, y \in K$ and T is a multivalued nonexpansive mapping from K into $CB(K)$.

In 1974, Ishikawa [6] introduced an iterative procedure for approximating fixed point as follows: For $x_0 \in K$, the sequence $\{x_n\}$ of Ishikawa iteration defined by

$$\begin{aligned} y_n &= (1 - \beta_n)x_n + \beta_n Sx_n, \\ x_{n+1} &= (1 - \alpha_n)x_n + \alpha_n S y_n, \quad n \geq 0 \end{aligned}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are real sequences.

In 2005, Sastry and Babu [7] defined Ishikawa iteration scheme for multivalued mappings. Let $T : K \rightarrow PB(K)$ a multivalued mapping and fix $p \in F(T)$. Then the sequence of Ishikawa iteration is defined as follows:

Choose $x_0 \in K$,

$$y_n = \beta_n z_n + (1 - \beta_n)x_n, \quad \beta_n \in [0, 1], \quad n \geq 0,$$

where $z_n \in T(x_n)$ such that $\|z_n - p\| = d(p, T(x_n))$ and

$$x_{n+1} = \alpha_n z'_n + (1 - \alpha_n)x_n, \quad \alpha_n \in [0, 1], \quad n \geq 0,$$

where $z'_n \in T(y_n)$ such that $\|z'_n - p\| = d(p, T(y_n))$.

Sastry and Babu [7] proved that Ishikawa iteration scheme for a multivalued nonexpansive mapping T with a fixed point p converges to a fixed point q of T under certain conditions. In 2007, Panyanak [8] extended the results of Sastry and Babu to uniformly convex Banach space for multivalued nonexpansive mappings. Panyanak also modified the iteration scheme of Sastry and Babu and imposed the question of convergence of this scheme. He introduced the following modified Ishikawa iteration method,

Choose $x_0 \in K$, then

$$y_n = \beta_n z_n + (1 - \beta_n)x_n, \quad \beta_n \in [a, b], \quad 0 < a < b < 1, \quad n \geq 0,$$

where $z_n \in Tx_n$ is such that $\|z_n - u_n\| = \text{dist}(u_n, Tx_n)$, and $u_n \in F(T)$ such that $\|x_n - u_n\| = \text{dist}(x_n, F(T))$, and

$$x_{n+1} = \alpha_n z'_n + (1 - \alpha_n)x_n \quad \alpha_n \in [a, b],$$

where $z'_n \in T(y_n)$ such that $\|z'_n - v_n\| = \text{dist}(v_n, Ty_n)$, and $v_n \in F(T)$ such that $\|y_n - v_n\| = \text{dist}(y_n, F(T))$.

In 2009, Song and Wang [9] pointed out the gap in the result of Panyanak [8]. They solved/revised the gap and gave the partial answer to the question raised by Panyanak by using the following iteration scheme.

Let $\alpha_n, \beta_n \in [0, 1]$ and $\gamma_n \in (0, \infty)$ such that $\lim_{n \rightarrow \infty} \gamma_n = 0$. Choose $x_0 \in K$, then

$$\begin{aligned} y_n &= \beta_n z_n + (1 - \beta_n)x_n, \\ x_{n+1} &= \alpha_n z'_n + (1 - \alpha_n)x_n, \end{aligned}$$

where $\|z_n - z'_n\| \leq H(Tx_n, Ty_n) + \gamma_n$ and $\|z_{n+1} - z'_n\| \leq H(Tx_{n+1}, Ty_n) + \gamma_n$ for $z_n \in Tx_n$ and $z'_n \in Ty_n$.

Simultaneously, Shahzad and Zegeye [11] extended the results of Sastry and Babu, Panyanak, and Song and Wang to quasi nonexpansive multivalued mappings and also relaxed the end point condition and compactness of the domain by using the following modified iteration scheme and gave the affirmative answer to the Panyanak question in a more general setting.

$$\begin{aligned} y_n &= \beta_n z_n + (1 - \beta_n)x_n, \quad \beta_n \in [0, 1], \quad n \geq 0, \\ x_{n+1} &= \alpha_n z'_n + (1 - \alpha_n)x_n, \quad \alpha_n \in [0, 1], \quad n \geq 0, \end{aligned}$$

where $z_n \in Tx_n$ and $z'_n \in Ty_n$.

Recently, Sokhuma and Kaewkhao [3] introduced the following modified Ishikawa iteration scheme for a pair of single valued and multivalued mapping.

Let K be a nonempty closed and bounded convex subset of Banach space X , let $S : K \rightarrow K$ be a single valued nonexpansive mapping and let $T : K \rightarrow CB(K)$ be a multivalued nonexpansive mapping. The sequence $\{x_n\}$ of the modified Ishikawa iteration is defined by

$$\begin{cases} y_n = \beta_n z_n + (1 - \beta_n)x_n, \\ x_{n+1} = \alpha_n S y_n + (1 - \alpha_n)x_n, \end{cases} \quad (1)$$

where $x_0 \in K$, $z_n \in Tx_n$ and $0 < a \leq \alpha_n$, $\beta_n \leq b < 1$.

Furthermore, they proved the following strong convergence theorem:

Theorem 2.7. *Let K be a nonempty compact convex subset of a uniformly convex Banach space X , and let $S : K \rightarrow K$ and $T : K \rightarrow CB(K)$ be a single valued and a multivalued nonexpansive mapping, respectively, and $\text{Fix}(S) \cap \text{Fix}(T) \neq \emptyset$ satisfying $Tw = \{w\}$ for all $w \in \text{Fix}(S) \cap \text{Fix}(T)$. Let $\{x_n\}$ be the sequence of the modified Ishikawa iteration defined by (1) with $0 < a \leq \alpha_n$, $\beta_n \leq b < 1$. Then $\{x_n\}$ converges strongly to a common fixed point of S and T .*

3. MAIN RESULTS

Before establishing our main results we prove following lemmas. These results generalize the results of [3, 10] and will be used to accomplish the main Theorem 3.6.

Lemma 3.1. *Let K be a non-empty compact convex subset of a uniformly convex Banach space X , and let $S : K \rightarrow K$ and $T : K \rightarrow CB(K)$ be a single valued and a multivalued mapping such that both satisfy the condition (C_λ) , respectively, and $\text{Fix}(S) \cap \text{Fix}(T) \neq \emptyset$ satisfying $Tw = \{w\}$ for all $w \in \text{Fix}(S) \cap \text{Fix}(T)$. Let $\{x_n\}$ be the sequence of the modified Ishikawa iteration defined by (1). Then, $\lim_{n \rightarrow \infty} \|x_n - w\|$ exists for all $w \in \text{Fix}(S) \cap \text{Fix}(T)$.*

Proof. Letting $w \in \text{Fix}(S) \cap \text{Fix}(T)$. Since for $\lambda \in (0, 1)$,

$$\lambda \|w - Sw\| = 0 \leq \|((1 - \beta_n)x_n + \beta_n z_n) - w\|,$$

by using definition of Condition (C_λ) , we get

$$\|S((1 - \beta_n)x_n + \beta_n z_n) - Sw\| \leq \|(1 - \beta_n)x_n + \beta_n z_n - w\|.$$

Similarly, since $\lambda \operatorname{dist}(w, Tw) = 0 \leq \|x_n - w\|$, then we get, $H(Tx_n, Tw) \leq \|x_n - w\|$. Consider

$$\begin{aligned} \|x_{n+1} - w\| &= \|(1 - \alpha_n)x_n + \alpha_n S((1 - \beta_n)x_n + \beta_n z_n) - w\| \\ &= \|(1 - \alpha_n)x_n + \alpha_n S((1 - \beta_n)x_n + \beta_n z_n) - (1 - \alpha_n)w - \alpha_n w\| \\ &\leq (1 - \alpha_n)\|x_n - w\| + \alpha_n \|S((1 - \beta_n)x_n + \beta_n z_n) - w\| \\ &= (1 - \alpha_n)\|x_n - w\| + \alpha_n \|S((1 - \beta_n)x_n + \beta_n z_n) - Sw\| \\ &\leq (1 - \alpha_n)\|x_n - w\| + \alpha_n \|(1 - \beta_n)x_n + \beta_n z_n - w\| \\ &= (1 - \alpha_n)\|x_n - w\| + \alpha_n \|(1 - \beta_n)x_n + \beta_n z_n - (1 - \beta_n)w - \beta_n w\| \\ &\leq (1 - \alpha_n)\|x_n - w\| + \alpha_n \{(1 - \beta_n)\|x_n - w\| + \beta_n \|z_n - w\|\} \\ &= (1 - \alpha_n)\|x_n - w\| + \alpha_n \{(1 - \beta_n)\|x_n - w\| + \beta_n \operatorname{dist}(z_n, Tw)\} \\ &\leq (1 - \alpha_n)\|x_n - w\| + \alpha_n \{(1 - \beta_n)\|x_n - w\| + \beta_n H(Tx_n, Tw)\} \\ &\leq (1 - \alpha_n)\|x_n - w\| + \alpha_n \{(1 - \beta_n)\|x_n - w\| + \beta_n \|x_n - w\|\} \\ &= \|x_n - w\|. \end{aligned}$$

Since $\{\|x_n - w\|\}$ is a bounded and decreasing sequence, hence the limit of $\{\|x_n - w\|\}$ exists.

By using the Lemma 2.5 we get the following results:

Lemma 3.2. *Let K be a nonempty compact convex subset of a uniformly convex Banach space X , and let $S : K \rightarrow K$ and $T : K \rightarrow CB(K)$ be a single valued and a multivalued mapping such that both satisfy the condition (C_λ) , respectively, and $\operatorname{Fix}(S) \cap \operatorname{Fix}(T) \neq \emptyset$ satisfying $Tw = \{w\}$ for all $w \in \operatorname{Fix}(S) \cap \operatorname{Fix}(T)$. Let $\{x_n\}$ be the sequence of the modified Ishikawa iteration defined by (1). If $0 < a \leq \alpha_n, \beta_n \leq b < 1$ for some $a, b \in \mathbb{R}$, then $\lim_{n \rightarrow \infty} \|Sy_n - x_n\| = 0$.*

Proof. Let $w \in \operatorname{Fix}(S) \cap \operatorname{Fix}(T)$.

Since for any $\lambda \in (0, 1)$, $\lambda \|w - Sw\| = 0 \leq \|y_n - w\|$,

therefore by using the definition of condition (C_λ) , we get

$$\|Sy_n - Sw\| \leq \|y_n - w\|.$$

Similarly, since $\lambda \operatorname{dist}(w, Tw) = 0 \leq \|x_n - w\|$ then we get

$$H(Tx_n, Tw) \leq \|x_n - w\|.$$

By Lemma 3.1, $\lim_{n \rightarrow \infty} \|x_n - w\|$ exists, say c , and consider,

$$\begin{aligned} \|Sy_n - w\| &= \|Sy_n - Sw\| \\ &\leq \|y_n - w\| \\ &= \|(1 - \beta_n)x_n + \beta_n z_n - w\| \\ &\leq (1 - \beta_n)\|x_n - w\| + \beta_n \operatorname{dist}(z_n, Tw) \\ &\leq (1 - \beta_n)\|x_n - w\| + \beta_n H(Tx_n, Tw) \\ &\leq (1 - \beta_n)\|x_n - w\| + \beta_n \|x_n - w\| \\ &= \|x_n - w\|. \end{aligned}$$

Therefore, we have

$$\limsup_{n \rightarrow \infty} \|Sy_n - w\| \leq \limsup_{n \rightarrow \infty} \|y_n - w\| \leq \limsup_{n \rightarrow \infty} \|x_n - w\| = c. \quad (2)$$

Further, we have

$$\begin{aligned}
c &= \lim_{n \rightarrow \infty} \|x_{n+1} - w\| \\
&= \lim_{n \rightarrow \infty} \|(1 - \alpha_n)x_n + \alpha_n S y_n - w\| \\
&= \lim_{n \rightarrow \infty} \|\alpha_n S y_n - \alpha_n w + x_n - \alpha_n x_n + \alpha_n w - w\| \\
&= \lim_{n \rightarrow \infty} \|\alpha_n(S y_n - w) + (1 - \alpha_n)(x_n - w)\|.
\end{aligned}$$

By Lemma 2.5, we obtain $\lim_{n \rightarrow \infty} \|(S y_n - w) - (x_n - w)\| = \lim_{n \rightarrow \infty} \|S y_n - x_n\| = 0$.

Lemma 3.3. *Let K be a nonempty compact convex subset of a uniformly convex Banach space X , and let $S : K \rightarrow K$ and $T : K \rightarrow CB(K)$ be a single valued and a multivalued mapping such that both satisfy the condition (C_λ) , respectively, and $Fix(S) \cap Fix(T) \neq \emptyset$ satisfying $Tw = \{w\}$ for all $w \in Fix(S) \cap Fix(T)$. Let $\{x_n\}$ be the sequence of the modified Ishikawa iteration defined by (1). If $0 < a \leq \alpha_n$, $\beta_n \leq b < 1$ for some $a, b \in \mathbb{R}$, then $\lim_{n \rightarrow \infty} \|x_n - z_n\| = 0$.*

Proof. Let $w \in Fix(S) \cap Fix(T)$.

Since $\lambda \|w - Sw\| = 0 \leq \|y_n - w\|$, we get

$$\|S y_n - Sw\| \leq \|y_n - w\|.$$

Similarly, since $\lambda dist(w, Tw) = 0 \leq \|x_n - w\|$ then we get

$$H(Tx_n, Tw) \leq \|x_n - w\|.$$

We put $\lim_{n \rightarrow \infty} \|x_n - w\| = c$. For $n \geq 0$, we have

$$\begin{aligned}
\|x_{n+1} - w\| &= \|(1 - \alpha_n)x_n + \alpha_n S y_n - w\| \\
&= \|(1 - \alpha_n)x_n + \alpha_n S y_n - (1 - \alpha_n)w - \alpha_n w\| \\
&\leq (1 - \alpha_n)\|x_n - w\| + \alpha_n \|S y_n - w\| \\
&= (1 - \alpha_n)\|x_n - w\| + \alpha_n \|S y_n - Sw\| \\
&\leq (1 - \alpha_n)\|x_n - w\| + \alpha_n \|y_n - w\|
\end{aligned}$$

and therefore

$$\begin{aligned}
\|x_{n+1} - w\| - \|x_n - w\| &\leq \alpha_n (\|y_n - w\| - \|x_n - w\|), \\
\frac{\|x_{n+1} - w\| - \|x_n - w\|}{\alpha_n} &\leq \|y_n - w\| - \|x_n - w\|.
\end{aligned}$$

Since $0 < a \leq \alpha_n \leq b < 1$, we have

$$\liminf_{n \rightarrow \infty} \left\{ \left(\frac{\|x_{n+1} - w\| - \|x_n - w\|}{\alpha_n} \right) + \|x_n - w\| \right\} \leq \liminf_{n \rightarrow \infty} \|y_n - w\|.$$

It follows that

$$c \leq \liminf_{n \rightarrow \infty} \|y_n - w\|.$$

Since, from (2) $\limsup_{n \rightarrow \infty} \|y_n - w\| \leq c$, hence we have

$$\begin{aligned}
c &= \lim_{n \rightarrow \infty} \|y_n - w\| \\
&= \lim_{n \rightarrow \infty} \|(1 - \beta_n)x_n + \beta_n z_n - w\| \\
&= \lim_{n \rightarrow \infty} \|(1 - \beta_n)(x_n - w) + \beta_n(z_n - w)\|. \tag{3}
\end{aligned}$$

Recall that $\|z_n - w\| = dist(z_n, Tw) \leq H(Tx_n, Tw) \leq \|x_n - w\|$. Thus, we have

$$\limsup_{n \rightarrow \infty} \|z_n - w\| \leq \limsup_{n \rightarrow \infty} \|x_n - w\| = c.$$

Since $0 < a \leq \beta_n \leq b < 1$ and by Lemma 2.5 and (3), we obtain $\lim_{n \rightarrow \infty} \|x_n - z_n\| = 0$.

Lemma 3.4. *Let K be a nonempty compact convex subset of a uniformly convex Banach space X . Let $S : K \rightarrow K$ be a single valued mapping which satisfies conditions (C_λ) and (E) while $T : K \rightarrow CB(K)$ be a multivalued mapping which satisfies condition (C_λ) . Also, $Fix(S) \cap Fix(T) \neq \emptyset$ and $Tw = \{w\}$ for $w \in Fix(S) \cap Fix(T)$. Let $\{x_n\}$ be the sequence of the modified Ishikawa iteration defined by (1).*

If $0 < a \leq \alpha_n, \beta_n \leq b < 1$, then $\lim_{n \rightarrow \infty} \|Sx_n - x_n\| = 0$.

Proof. As S satisfies the condition (E), there exists $\mu \geq 1$, such that

$$\begin{aligned} \|Sx_n - x_n\| &= \|Sx_n - y_n + y_n - x_n\| \\ &\leq \|Sx_n - y_n\| + \|y_n - x_n\| \\ &\leq \mu \|Sy_n - y_n\| + 2\|y_n - x_n\| \\ &\leq \mu \|Sy_n - x_n\| + (\mu + 2)\|y_n - x_n\| \\ &= \mu \|Sy_n - x_n\| + (\mu + 2)\|(1 - \beta_n)x_n + \beta_n z_n - x_n\| \\ &= \mu \|Sy_n - x_n\| + (\mu + 2)\beta_n \|x_n - z_n\|, \end{aligned}$$

therefore,

$$\lim_{n \rightarrow \infty} \|Sx_n - x_n\| \leq \lim_{n \rightarrow \infty} \mu \|Sy_n - x_n\| + \lim_{n \rightarrow \infty} (\mu + 2)\beta_n \|x_n - z_n\|.$$

Thus, by Lemma 3.2 and Lemma 3.3,

$$\lim_{n \rightarrow \infty} \|Sx_n - x_n\| = 0.$$

Now, we are equipped to prove the our main results of this section.

Theorem 3.5. Let K be a nonempty compact convex subset of a uniformly convex Banach space X . Let $S : K \rightarrow K$ be a single valued mapping which satisfies conditions (C_λ) and (E) while $T : K \rightarrow CB(K)$ be a multivalued mapping which satisfies condition (C_λ) . Also, $Fix(S) \cap Fix(T) \neq \emptyset$ and $Tw = \{w\}$ for $w \in Fix(S) \cap Fix(T)$. Let $\{x_n\}$ be the sequence of the modified Ishikawa iteration defined by (1). If $0 < a \leq \alpha_n, \beta_n \leq b < 1$, then $\{x_{n_i}\} \rightarrow y$ for some subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that $\{x_{n_i}\} \in Tx_{n_i}$ for all n_i implies $y \in Fix(S) \cap Fix(T)$.

Proof. Assume that $\lim_{i \rightarrow \infty} \|x_{n_i} - y\| = 0$. By Lemma 3.4, we obtain $0 = \lim_{i \rightarrow \infty} \|Sx_{n_i} - x_{n_i}\| = \lim_{i \rightarrow \infty} \|(I - S)(x_{n_i})\|$. Since $I - S$ is strongly demiclosed at 0, we have $(I - S)(y) = 0$. Thus $y = Sy$, i.e., $y \in Fix(S)$. Since $\lambda dist(x_{n_i}, Tx_{n_i}) = 0 \leq \|x_{n_i} - y\|$, then we have $H(Tx_{n_i}, Ty) \leq \|x_{n_i} - y\|$. By Lemma 2.6, we have

$$\begin{aligned} dist(y, Ty) &\leq \|y - x_{n_i}\| + dist(x_{n_i}, Tx_{n_i}) + H(Tx_{n_i}, Ty) \\ &\leq \|y - x_{n_i}\| + dist(x_{n_i}, Tx_{n_i}) + \|x_{n_i} - y\| \rightarrow 0 \end{aligned}$$

as $i \rightarrow \infty$. It follows that $y \in Fix(T)$. Thus $y \in Fix(S) \cap Fix(T)$.

Theorem 3.5. Let K be a nonempty compact convex subset of a uniformly convex Banach space X . Let $S : K \rightarrow K$ be a single valued mapping which satisfies conditions (C_λ) and (E) while $T : K \rightarrow CB(K)$ be a multivalued mapping which satisfies condition (C_λ) . Also, $Fix(S) \cap Fix(T) \neq \emptyset$ and $Tw = \{w\}$ for $w \in Fix(S) \cap Fix(T)$. Let $\{x_n\}$ be the sequence of the modified Ishikawa iteration defined by (1) with $0 < a \leq \alpha_n, \beta_n \leq b < 1$. Then $\{x_n\}$ converges strongly to a common fixed point of S and T .

Proof. Since $\{x_n\}$ is contained in K which is compact, there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that $\{x_{n_i}\}$ converges strongly to some point $y \in K$, that is, $\lim_{i \rightarrow \infty} \|x_{n_i} - y\| = 0$. By Theorem 3.5, we get $y \in Fix(S) \cap Fix(T)$, and by Lemma 3.1, we have that $\lim_{n \rightarrow \infty} \|x_n - y\|$ exists. It must be the case in which $\lim_{n \rightarrow \infty} \|x_n - y\| = \lim_{i \rightarrow \infty} \|x_{n_i} - y\| = 0$. Thus, $\{x_n\}$ converges strongly to $y \in Fix(S) \cap Fix(T)$.

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