Asymptotic profile of solutions to the Degasperis-Procesi equation

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Abstract: It is shown that a strong solution of the Degasperis-Procesi equation, initially decaying exponentially together with its spatial derivative, must be identically equal to zero if it also decays exponentially at a later time. The decay rate of the corresponding strong solution at infinity is also given for some kinds of initial data with exponential decay.

Key words: The Degasperis-Procesi equation, asymptotic description

1 Introduction

Recently, Degasperis and Procesi [20] considered the following family of third order dispersive conservation laws,

$$u_t + c_0 u_x + \gamma u_{xxx} - \alpha^2 u_{xxt} = \left(c_1 u^2 + c_2 u_x^2 + c_3 u u_{xx}\right)_x, \tag{1.1}$$

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where α, γ, c_0, c_1, c_2 and c_3 are real constants. Within this family, only three equations that satisfy asymptotic integrability condition up to third order are singled out, namely the KdV equation,

$$u_t + u_x + uu_x + u_{xxx} = 0,$$

the Camassa-Holm equation,

$$u_t - u_{xxt} + 3uu_x = 2u_x u_{xx} + uu_{xxx},$$

and a new equation (the Degasperis-Procesi equation, the DP equation, for simplicity) which can be written as (after rescaling) the dispersionless form [20],

$$u_t - u_{xxt} + 4uu_x = 3u_x u_{xx} + uu_{xxx}. \tag{1.2}$$

It is worth noting that in [21], both the Camassa-Holm and DP equations are derived as members of a one-parameter family of asymptotic shallow water approximations to the Euler equations: this is important because it shows that (after the addition of linear dispersion terms) both the Camassa-Holm and DP equations are physically relevant, otherwise the DP equation would be of purely theoretical interest.

When $\alpha = c_2 = c_3 = 0$ in (1.1), it becomes the well-known KdV equation, which has been extensively studied by [3, 4, 26, 27, 33].

When $c_1 = -3c_3/2\alpha^2$ and $c_2 = c_3/2$ in (1.1), we recover the Camassa-Holm equation derived physically by Camassa and Holm in [7] by approximating directly the Hamiltonian for Euler's equations in the shallow water regime, where $u(x,t)$ represents the free surface above a flat bottom. Recently, the alternative derivations of the Camassa-Holm equation as a model for water waves, respectively as the equation for geodesic flow on the diffeomorphism group of the circle were presented by Johnson [25] and respectively by Constantin and Kolev [13]. The geometric interpretation is important because it can be used to prove that the Least Action Principle holds for the Camassa-Holm equation, cf. [14]. It is worth to point out that a fundamental aspect of the Camassa-Holm equation, the fact that it is a completely integrable system, was shown in [15] for the periodic case and [1, 12] for the nonperiodic case. Some satisfactory results have been obtained for this shallow water equation recently. Local well-posedness for the initial datum $u_0(x) \in H^s$ with $s > 3/2$ was proved by several authors, see [28, 31], and global existence was establish for some kind of initial datum in [9, 11]. For the initial data with lower regularity, we refer to Molinet's paper [32] and also the recent paper [5]. Moreover, necessary and sufficient condition for wave breaking is established in [29, 30, 37, 38]. However, in [34], global existence of weak solutions is proved but uniqueness is obtained only under an a priori assumption that is known to hold only for initial data $u_0(x) \in H^1$ such that $u_0 - u_{0xx}$ is a sign-definite Radon measure (under this condition, global existence and uniqueness was shown in [16] also). Also it is worth to note that global conservative solutions are constructed for any initial data in H^1 by Bressan and Constantin [5] recently. In [2] and [18], it was proved that all solitary waves (peaked when $c_0 = 0$ or smooth when $c_0 \neq 0$) are solitons. The stabilities of the solitons are proved in [17] and [18] respectively. Recently, in [22], among others, Himonas, Misiołek, Ponce and Zhou showed the infinite propagation speed for the Camassa-Holm equation in the sense that a strong solution of the Cauchy problem with compact initial profile can not be compactly supported at any later time unless it is the zero solution, which is an improvement of previous results in this direction obtained in [10, 24].

Although, the DP equation (1.2) has a similar form to the Camassa-Holm equation and admits exact peakon solutions analogous to the Camassa-Holm peakons [19], these two equations are pretty different. The isospectral problem for equation (1.2) is

$$\Psi_x - \Psi_{xxx} - \lambda y \Psi = 0,$$

while it for Camassa-Holm equation is

$$\Psi_{xx} - \frac{1}{4}\Psi - \lambda y \Psi = 0,$$

where $y = u - u_{xx}$ for both cases. This implies the inside structures of the DP equation (1.2) and the Camassa-Holm equation are truly different. However, we also have some similar results on the DP equation, see [6, 23, 36]

Analogous to the Camassa-Holm equation, (1.2) can be written in Hamiltonian form and have infinitely many conservation laws. Here we list some of the simplest conserved quantities [19]:

$$H_{-1}=\int_{\mathbb{R}}u^3dx,\quad H_0=\int_{\mathbb{R}}ydx,\quad H_1=\int_{\mathbb{R}}yvdx,$$
$$H_5=\int_{\mathbb{R}}y^{1/3}dx,\quad H_7=\int_{\mathbb{R}}\left(y_x^2y^{-7/3}+9y^{-1/3}\right)dx,$$

where $v=(4-\partial_x^2)^{-1}u$. So they are different from the invariants of the Camassa-Holm equation,

$$E(u)=\int_{\mathbb{R}}(u^2+u_x^2)dx,\quad F(u)=\int_{\mathbb{R}}(u^3+uu_x^2)dx.$$

Set $Q=(1-\partial_x^2)$, then the operator Q^{-1} in $\mathbb{R}$ can be expressed by

$$Q^{-1}f=G*f=\frac{1}{2}\int_{\mathbb{R}}e^{-|x-y|}f(y)dy.$$

Equation (1.2) can be written as

$$u_t+uu_x+\partial_xG*\left(\frac{3}{2}u^2\right)=0, \tag{1.3}$$

while the Camassa-Holm equation can be written as

$$u_t+uu_x+\partial_xG*\left(u^2+\frac{1}{2}u_x^2\right)=0. \tag{1.4}$$

Due to the similarity of (1.3) and (1.4), just by following the argument for the Camassa-Holm equation, it is easy to establish the following well-posedness theorem for (1.3).

Theorem 1.1. [35, 36] *Given $u(x,t=0)=u_0\in H^s(\mathbb{R})$, $s>3/2$, then there exists a T and a unique solution u to (1.2) (also (1.3)) such that*

$$u(x,t)\in C([0,T);H^s(\mathbb{R}))\cap C^1([0,T);H^{s-1}(\mathbb{R})).$$

It should be mentioned that due to the form of (1.3) (no derivative appears in the convolution term), Coclite and Karlsen [8] established global existence and uniqueness result for entropy weak solutions belonging to the class $L^1(\mathbb{R})\cap BV(\mathbb{R})$.

2 Unique continuation

We will formulate decay conditions on a solution, at two distinct times, which guarantee that $u\equiv 0$ is the unique solution of (1.3). The idea of proving unique continuation results for nonlinear dispersive equations under decay assumptions of the solution at two different times was motivated by the recent works on the nonlinear Schrödinger and the κ-generalized Korteweg-de Vries equations.

In order to prove the result, we have the following theorem.

Theorem 2.1. *Assume that for some $T > 0$, and $s > 3/2$, $u \in C([0,T]; H^s(\mathbb{R}))$ is a strong solution of the initial value problem associated to equation (1.3), and that $u_0(x) = u(x,0)$ satisfies that for some $\theta \in (0,1)$,*

$$|u_0(x)|, \quad |\partial_x u_0(x)| = O(e^{-\theta x}) \quad as \quad x \uparrow \infty. \tag{2.1}$$

Then

$$|u(x,t)|, \quad |\partial_x u(x,t)| = O(e^{-\theta x}) \quad as \quad x \uparrow \infty, \tag{2.2}$$

uniformly in the time interval $[0,T]$.

Notation. *We shall say that*

$$|f(x)| = O(e^{\alpha x}) \; as \; x \uparrow \infty \quad if \; \lim_{x\to\infty} \frac{|f(x)|}{e^{\alpha x}} = L,$$

and

$$|f(x)| = o(e^{\alpha x}) \; as \; x \uparrow \infty \quad if \; \lim_{x\to\infty} \frac{|f(x)|}{e^{\alpha x}} = 0.$$

for some constant L.

Proof: We introduce the following notations

$$F(u) = \frac{3}{2}u^2 \tag{2.3}$$

and

$$M = \sup_{t\in[0,T]} \|u(t)\|_{H^s}. \tag{2.4}$$

Multiplying equation (1.3) by u^{2p-1} with $p \in Z^+$, and integrating the result in the x-variable, one gets

$$\int_{-\infty}^{\infty} u^{2p-1} u_t dx + \int_{-\infty}^{\infty} u^{2p-1} u u_x dx + \int_{-\infty}^{\infty} u^{2p-1} \partial_x G * F(u) dx = 0. \tag{2.5}$$

The first term in (2.5) is

$$\begin{aligned} \int_{-\infty}^{\infty} u^{2p-1} u_t dx &= \int_{-\infty}^{\infty} \frac{1}{2p} \frac{du^{2p}}{dt} dx \\ &= \frac{1}{2p} \frac{d}{dt} \int_{-\infty}^{\infty} u^{2p} dx = \|u(t)\|_{2p}^{2p-1} \frac{d\|u(t)\|_{2p}}{dt}, \end{aligned} \tag{2.6}$$

the rest are

$$\left|\int_{-\infty}^{\infty} u^{2p-1} u u_x dx\right| = \left|\int_{-\infty}^{\infty} u^{2p} u_x dx\right| \leq \|u_x(t)\|_\infty \|u(t)\|_{2p}^{2p}, \tag{2.7}$$

and

$$\left|\int_{-\infty}^{\infty} u^{2p-1} \partial_x G * F(u) dx\right| \leq \|u(t)\|_{2p}^{2p-1} \cdot \|\partial_x G * F(u)(t)\|_{2p}. \tag{2.8}$$

From above inequalities, we get

$$\frac{d}{dt}\|u(t)\|_{2p} \leq \|u_x(t)\|_\infty \|u(t)\|_{2p} + \|\partial_x G * F(u)(t)\|_{2p}, \tag{2.9}$$

and therefore, by Gronwall's inequality

$$\|u(t)\|_{2p} \leq \left(\|u(0)\|_{2p} + \int_0^t \|\partial_x G * F(u)(\tau)\|_{2p} d\tau\right) e^{Mt}. \tag{2.10}$$

Since $f \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$, implies

$$\lim_{q\to\infty} \|f\|_q = \|f\|_\infty, \tag{2.11}$$

taking the limits in (2.10)(note that $\partial_x G \in L^1$ and $F(u) \in L^1 \cap L^\infty$), from (2.11) we get

$$\|u(t)\|_\infty \leq \left(\|u(0)\|_\infty + \int_0^t \|\partial_x G * F(u)(\tau)\|_\infty d\tau\right) e^{Mt}. \tag{2.12}$$

We shall now repeat the above arguments using the weight

$$\varphi_N(x) = \begin{cases} 1, & x \leq 0, \\ e^{\theta x}, & x \in (0, N), \\ e^{\theta N}, & x \geq N, \end{cases} \tag{2.13}$$

where $N \in \mathbb{Z}^+$. Observe that for all N we have

$$0 \leq \varphi_N^{'}(x) \leq \varphi_N(x) \quad a.e. \;\; x \in \mathbb{R}. \tag{2.14}$$

Using notation in (2.3), from equation (1.3) we obtain

$$(u\varphi_N)_t + (u\varphi_N)u_x + \varphi_N \partial_x G * F(u) = 0. \tag{2.15}$$

Hence, as in the weightless case (2.12), we get

$$\|u(t)\varphi_N\|_\infty \leq \left(\|u(0)\varphi_N\|_\infty + \int_0^t \|\varphi_N \partial_x G * F(u)(\tau)\|_\infty d\tau\right) e^{Mt}. \tag{2.16}$$

A simple calculation shows that there exists $C_0 > 0$ depending only on $\theta \in (0,1)$ such that for any $N \in \mathbb{Z}^+$,

$$\frac{1}{2}\varphi_N(x)\int_{-\infty}^{\infty} e^{-|x-y|}\frac{1}{\varphi_N(y)}dy \leq C_0 = \frac{2}{1-\theta}. \tag{2.17}$$

Thus, for any appropriate function f one sees that

$$\begin{aligned} |\varphi_N \partial_x G * f^2(x)| &= \left|\frac{1}{2}\varphi_N(x)\int_{-\infty}^{\infty} sgn(x-y)e^{-|x-y|}f^2(y)dy\right| \\ &\leq \frac{1}{2}\varphi_N(x)\int_{\infty}^{\infty} e^{-|x-y|}\frac{1}{\varphi_N(y)}\varphi_N(y)f(y)f(y)dy \\ &\leq \left(\frac{\varphi_N(x)}{2}\int_{-\infty}^{\infty} e^{-|x-y|}\frac{1}{\varphi_N(y)}dy\right)\|\varphi_N f\|_\infty \|f\|_\infty \\ &\leq C_0\|\varphi_N f\|_\infty\|f\|_\infty. \end{aligned} \tag{2.18}$$

Combining (2.16), we get

$$\begin{aligned} \|u(t)\varphi_N\|_\infty &\leq e^{Mt}\left(\|u_0\varphi_N\|_\infty + \int_0^t \frac{3C_0}{2}\|\varphi_N u\|_\infty\|u\|_\infty d\tau\right) \\ &\leq e^{Mt}\left(\|u_0\varphi_N\|_\infty + \int_0^t \frac{3C_0}{2}M\|\varphi_N u\|_\infty d\tau\right) \\ &\leq C_1\left(\|u_0\varphi_N\|_\infty + \int_0^t \|\varphi_N u\|_\infty d\tau\right) \end{aligned}$$

where $C_1 = C_1(M;T,) > 0$. By Gronwall's inequality, there exists a constant $\tilde{C} = \tilde{C}(M;T)$ for any $N \in \mathbb{Z}^+$, and any $t \in [0,T]$ such that

$$\|\varphi_N u\|_\infty \leq \tilde{C}\|u_0\varphi_N\|_\infty \leq \tilde{C}\|u_0 \cdot \max(1, e^{\theta x})\|_\infty. \tag{2.19}$$

Finally, taking the limit as N goes to infinity in (2.19) we find that for any $t \in [0,T]$

$$|u(x,t)e^{\theta x}| \leq \tilde{C}\|u_0 \cdot \max(1, e^{\theta x})\|_\infty. \tag{2.20}$$

From (2.1), we get $|u(x,t)| = O(e^{-\theta x}) \quad \text{as} \quad x \uparrow \infty$.

Next, differentiating (1.3) in the x-variable produces the equation

$$u_{xt} + uu_{xx} + u_x^2 + \partial_x^2 G * (\frac{3}{2}u^2) = 0. \tag{2.21}$$

Again, multiplying equation (2.21) by u_x^{2p-1}, $(p \in \mathbb{Z}^+)$, integrating the result in the x-variable, and using integration by parts,

$$\int_{\infty}^{\infty} uu_{xx}(u_x)^{2p-1}dx = \int_{\infty}^{\infty} u\frac{(u_x)^{2p}}{2p}dx = -\frac{1}{2p}\int_{\infty}^{\infty} u_x(u_x)^{2p}dx, \tag{2.22}$$

one gets the inequality

$$\frac{d}{dt}\|u_x(t)\|_{2p} \le 2\|u_x(t)\|_\infty \|u_x(t)\|_{2p} + \|\partial_x^2 G * F(u)(t)\|_{2p}, \tag{2.23}$$

and therefore as before

$$\|u_x(t)\|_{2p} \le \left(\|u_x(0)\|_{2p} + \int_0^t \|\partial_x^2 G * F(u)(\tau)\|_{2p} d\tau \right) e^{2Mt}. \tag{2.24}$$

Since $\partial_x^2 G = G - \delta$, we can use (2.11) and pass to the limit in (2.24) to obtain

$$\|u_x(t)\|_\infty \le \left(\|u_x(0)\|_\infty + \int_0^t \|\partial_x^2 G * F(u)(\tau)\|_\infty d\tau \right) e^{2Mt}, \tag{2.25}$$

from (2.21) we get

$$\partial_t(u_x\varphi_N) + uu_{xx}\varphi_N + (u_x\varphi_N)u_x + \varphi_N \partial_x^2 G * F(u) = 0. \tag{2.26}$$

We need to eliminate the second derivatives in the second term in (2.26). Thus, combining integration by parts and (2.14), we find

$$\begin{aligned}
& \left| \int_{-\infty}^{\infty} uu_{xx}\varphi_N (u_x\varphi_N)^{2p-1} dx \right| \\
= & \left| \int_{-\infty}^{\infty} u (u_x\varphi_N)^{2p-1} (\partial_x(u_x\varphi_N) - u_x\varphi_N^{'}) dx \right| \\
= & \left| \int_{-\infty}^{\infty} u\partial_x (\frac{(u_x\varphi_N)^{2p}}{2p}) dx - \int_{-\infty}^{\infty} uu_x\varphi_N^{'} (u_x\varphi_N)^{2p-1} dx \right| \\
\le & \left| \int_{-\infty}^{\infty} u\partial_x (\frac{(u_x\varphi_N)^{2p}}{2p}) dx \right| + \left| \int_{-\infty}^{\infty} uu_x\varphi_N (u_x\varphi_N)^{2p-1} dx \right| \\
\le & \ (\|u(t)\|_\infty + \|\partial_x u(t)\|_\infty) \|\partial_x u\varphi_N\|_{2p}^{2p}.
\end{aligned} \tag{2.27}$$

Since $\partial_x^2 G = G - \delta$, the argument in (2.18) also shows that

$$|\varphi_N \partial_x^2 G * f^2(x)| \le C_0 \|\varphi_N f\|_\infty \|f\|_\infty. \tag{2.28}$$

Similarly, we get

$$\|u_x(t)\varphi_N\|_\infty \le C_2 \left(\|u_x(0)\varphi_N\|_\infty + \int_0^t \|u(\tau)\varphi_N\|_\infty d\tau \right)$$

where $C_2 = C_2(M;T)$

Then taking the limit as N goes to infinity, we find that for any $t \in [0,T]$

$$|u_x(t)e^{\theta x}| \le C_2 \left(\|u_x(0)e^{\theta x}\|_\infty + \int_0^t \|u(\tau)e^{\theta x}\|_\infty d\tau \right).$$

Since $|u(x,t)| = O(e^{-\theta x}) \quad x \uparrow \infty$ and (2.1), we get

$$|\partial_x u(x)| = O(e^{-\theta x}) \quad x \uparrow \infty.$$

□

Theorem 2.2. *Assume that for some $T > 0$, and $s > 3/2$, $u \in C([0,T]; H^s(\mathbb{R}))$ is a strong solution of the initial value problem associated to equation (1.3). If $u_0(x) = u(x,0)$ satisfies that for some $\alpha \in (1/2, 1)$*

$$|u_0(x)| = o(e^{-x}), \quad |\partial_x u_0(x)| = O(e^{-\alpha x}) \quad as \quad x \uparrow \infty, \tag{2.29}$$

and there exists $t_1 \in (0,T]$ such that

$$|u(x,t_1)| \sim o(e^{-x}) \quad as \quad x \uparrow \infty. \tag{2.30}$$

Then $u \equiv 0$.

Proof: Integrating equation (1.3) over the time interval $[0, t_1]$, we get

$$u(x,t_1) - u(x,0) + \int_0^{t_1} u u_x(x,\tau) d\tau + \int_0^{t_1} \partial_x G * (\frac{3}{2}u^2)(x,\tau) d\tau = 0. \tag{2.31}$$

By hypothesis (2.29) and (2.30), we have

$$u(x,t_1) - u(x,0) = o(e^{-x}) \quad as\ x \uparrow \infty. \tag{2.32}$$

From (2.29) and Theorem 2.1, it follows that

$$\int_0^{t_1} u u_x(x,\tau) d\tau = O(e^{-2\alpha x}) \quad as\ x \uparrow \infty, \tag{2.33}$$

and so

$$\int_0^{t_1} u u_x(x,\tau) d\tau = o(e^{-x}) \quad as\ x \uparrow \infty. \tag{2.34}$$

We shall show that if $u \neq 0$, then the last term in (2.31) is $O(e^{-x})$ but not $o(e^{-x})$. Thus, we have

$$\int_0^{t_1} \partial_x G * (\frac{3}{2}u^2)(x,\tau) d\tau = \partial_x G * \int_0^{t_1} \frac{3}{2} u^2 d\tau = \partial_x G * \rho(x), \tag{2.35}$$

where by (2.29) and Theorem 2.1

$$0 \leq \rho(x) = O(e^{-2\alpha x}), \; so \quad that \quad \rho(x) = o(e^{-x}) \quad as \quad x \uparrow \infty. \tag{2.36}$$

Therefore

$$\partial_x G * \rho(x) = -\frac{1}{2}e^{-x}\int_{-\infty}^{x} e^{y}\rho(y)dy + \frac{1}{2}e^{x}\int_{x}^{\infty} e^{-y}\rho(y)dy. \tag{2.37}$$

From (2.36) it follows that

$$e^{x}\int_{x}^{\infty} e^{-y}\rho(y)dy = o(1)e^{x}\int_{x}^{\infty} e^{-2y}dy = o(1)e^{-x} = o(e^{-x}),$$

and if $\rho \neq 0$, one has that

$$\int_{-\infty}^{x} e^{y}\rho(y)dy \geq c_0 > 0, \quad for \quad x \gg 1. \tag{2.38}$$

Hence the last term in (2.35) and (2.37) satisfies

$$\partial_x G * \rho(x) \leq -\frac{1}{2}e^{-x}, \quad for \quad x \gg 1, \tag{2.39}$$

which combined with (2.31)-(2.34) yields a contradiction.

Thus, $\rho(x) \equiv 0$ and consequently $u \equiv 0$, see (2.35). □

Remark 2.1. *Theorem 2.2 holds with the corresponding decay hypothesis in (2.29)-(2.30) stated for $x < 0$.*

The following result establishes the optimality of Theorem 2.2 and tells us that a strong non-trivial solution of (1.3) corresponding to data with fast decay at infinity will immediately behave asymptotically, in the x-variable at infinity.

Theorem 2.3. *Assume that for some $T > 0$, and $s > 3/2$, $u \in C([0,T]; H^s(\mathbb{R}))$ is a strong solution of the initial value problem associated to equation (1.3) and that $u_0(x) = u(x,0)$ satisfies that for some $\alpha \in (1/2, 1)$,*

$$|u_0(x)| = O(e^{-x}), \quad |\partial_x u_0(x)| = O(e^{-\alpha x}) \quad as \quad x \uparrow \infty.$$

Then

$$|u(x,t)| = O(e^{-x}) \quad as \quad x \uparrow \infty,$$

uniformly in the time interval $[0,T]$.

Proof: This proof is similar to the argument above, and therefore it will be omitted. □

From the result of Theorem 2.1, we will know that, as long as it exists, the solution $u(x,t)$ corresponding to compactly supported initial data $u_0(x)$ is positive at infinity and negative at minus infinity regardless of the profile of a fast-decaying data $u_0 \neq 0$. We would like to list it as follows (We can see its proof in [22]).

Theorem 2.4. *Assume that for some $T > 0$ and $s > 5/2$, $u \in C([0,T]; H^s(\mathbb{R}))$ is a strong solution of the initial value problem associated to equation (1.3).*

(a) If $u_0(x) = u(x,0)$ has compact support, then for any $t \in (0,T]$,

$$u(x,t) = \begin{cases} c_+(t)e^{-x}, & for\ x > \eta(b,t), \\ c_-(t)e^{x}, & for\ x < \eta(a,t) \end{cases}. \tag{2.40}$$

(b)If for some $\mu > 0$,

$$\partial_x^j u_0 \sim O(e^{-(1+\mu)|x|}) \quad as \quad |x| \uparrow \infty \quad j = 0, 1, 2, \tag{2.41}$$

then for any $t \in (0,T]$,

$$h(x,t) = u(x,t) - u_{xx}(x,t) \sim O(e^{-(1+\mu)|x|}) \quad as \quad |x| \uparrow \infty, \tag{2.42}$$

and

$$\lim_{x\to\pm\infty} e^{\pm x} u(x,t) = c_\pm(t), \tag{2.43}$$

where in (2.40), (2.43), $c_+(\cdot), c_-(\cdot)$ denote continuous non-vanishing functions, with $c_+(t) > 0$ and $c_-(t) < 0$ for $t \in (0,T)$. Furthermore, $c_+(\cdot)$ is a strictly increasing function, while $c_-(\cdot)$ is strictly decreasing.

Remark 2.2. *Here $\eta = \eta(x,t)$ is the flow of u, that is*

$$\begin{cases} \dfrac{d\eta(x,t)}{dt} = u(\eta(x,t),t), & x \in \mathbb{R}, \\ \eta(x, t=0) = x, & x \in \mathbb{R}. \end{cases}$$

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