MONTEL'S CRITERION AND SHARED SET

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ABSTRACT. We prove a normality criterion for a family of meromorphic functions involving set sharing, which improves a result of P. Montel.

1. Introduction, Definitions and results

Let $\mathfrak{F}$ be a family of meromorphic functions defined in a domain $\mathfrak{D}$ in the open complex plane $\mathbb{C}$. The family is said to be normal in $\mathfrak{D}$, in the sense of Montel, if for any sequence $\{f_n\} \subset \mathfrak{F}$ there exists a subsequence $\{f_{n_j}\}$, which converges locally spherically uniformly in $\mathfrak{D}$ to a meromorphic function or ∞ {see p.71, [2]}.

The most acclaimed result of the theory of normal family is the following theorem of P. Montel, which was called by J. L. Schiff {p.74, [2]} the fundamental normality test.

Theorem A. *Let $\mathfrak{F}$ be a family of meromorphic functions defined in a domain $\mathfrak{D}$, which omit three complex values a, b, c. Then $\mathfrak{F}$ is normal in $\mathfrak{D}$.*

Theorem A has been further improved in the following manner {Theorem 4.1.3, p.105, [2]}.

Theorem B. *Let $\mathfrak{F}$ be a family of meromorphic functions in a domain $\mathfrak{D}$. If for each $f \in \mathfrak{F}$, the poles are of multiplicities $\geq h$, the zeros are of multiplicities $\geq k$ and the 1-points are of multiplicities $\geq l$, with*

$$\frac{1}{h} + \frac{1}{k} + \frac{1}{l} < 1,$$

then $\mathfrak{F}$ is normal in $\mathfrak{D}$.

Let f be a meromorphic function in a domain $\mathfrak{D}$ in the complex plane $\mathbb{C}$. For a complex number a, finite or infinite, we denote by $\overline{E}(a; f)$ the set of distinct a-points of f. Two meromorphic functions f and g, defined in $\mathfrak{D}$, are said to share the value a if $\overline{E}(a; f) = \overline{E}(a; g)$. Let S be a nonempty subset of the extended complex plane. We say that f and g share the set S if $\cup_{a \in S} \overline{E}(a; f) = \cup_{a \in S} \overline{E}(a; g)$.

Considering the concept of value sharing, D. C. Sun [3] improved Theorem A and proved the following result.

Theorem C. *Let $\mathfrak{F}$ be a family of meromorphic functions defined in a domain $\mathfrak{D}$. If for each pair of functions f and g in $\mathfrak{F}$, f and g share $0, 1, \infty$ in $\mathfrak{D}$, then $\mathfrak{F}$ is normal in $\mathfrak{D}$.*

In 2010 Y. Xu [5] considered the case of sharing of two values and one value respectively and proved the following results.

Theorem D. *Let $\mathfrak{F}$ be a family of meromorphic functions in a domain $\mathfrak{D}$. Suppose that for each pair of functions $f, g \in \mathfrak{F}$, f and g share $0, \infty$ and all 1-points of each $f \in \mathfrak{F}$ are multiple. Then $\mathfrak{F}$ is normal in $\mathfrak{D}$.*

Theorem E. *Let $\mathfrak{F}$ be a family of meromorphic functions defined in a domain $\mathfrak{D}$. Suppose that*

(i) for each pair of functions $f, g \in \mathfrak{F}$, f and g share 0 in $\mathfrak{D}$;
(ii) all poles of f have multiplicity at least 2 (or 3) and all 1-points of f have multiplicity 3 (or 2) for each $f \in \mathfrak{F}$.

Then $\mathfrak{F}$ is normal in $\mathfrak{D}$.

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One may see [1] and [4] for some normality criteria concerning sharing values and sharing functions.

In this paper we improve the result of Montel by considering shared set instead of shared value and prove the following result.

Theorem 1. *Let $\mathfrak{F}$ be a family of meromorphic functions defined in a domain $\mathfrak{D}$, M be a positive number and $S=\{\alpha,\beta\}$, where α, β are distinct elements of $\mathbb{C}\cup\{\infty\}$. Suppose further that*

(i) each pair of functions $f,g\in\mathfrak{F}$ share the set S in $\mathfrak{D}$;
(ii) there exists a $\gamma\in\mathbb{C}\backslash\{\alpha,\beta\}$ such that for each $f\in\mathfrak{F}$, $\mid f'(z)\mid\leq M$ whenever $f(z)=\gamma$ in $\mathfrak{D}$;
(iii) each $f\in\mathfrak{F}$ has no simple β-points in $\mathfrak{D}$.

Then $\mathfrak{F}$ is normal in $\mathfrak{D}$.

The following example shows that there do exist a normal family of meromorphic functions which do not always share values but share a set.

Example 1. *Let $\mathfrak{F}=\{f_n:n=1,2,3,\ldots\}$ and $\mathfrak{D}=\{z:\mid z\mid<1\}$, where $f_{2n}(z)=\dfrac{2n}{z^k}$, $f_{2n+1}(z)=\dfrac{z}{2n+1}$ and $k(\geq 2)$ is an integer. Then every pair of members of $\mathfrak{F}$ share the set $S=\{0,\infty\}$. Now $f'_{2n}(z)=\dfrac{-2nk}{z^{k+1}}$ and $f'_{2n+1}(z)=\frac{1}{2n+1}$. Hence $f_{2n}(z)=1$ implies $\mid f'_{2n}(z)\mid=\dfrac{2nk}{\mid z\mid^{k+1}}=\dfrac{k}{\sqrt[k]{2n}}\leq k$. Also we see that $\mid f'_{2n+1}(z)\mid<1$. Therefore $f_n(z)-1=0$ implies $\mid f'_n(z)\mid\leq k$ for $n=1,2,3,\ldots$. Further we see that $f^{\#}_{2n}(z)\leq\frac{k}{2n}<k$ and $f^{\#}_{2n+1}(z)\leq\frac{1}{2n+1}<1$. So by Marty's criterion the family $\mathfrak{F}$ is normal in $\mathfrak{D}$.*

The following example shows that the hypothesis (ii) of Theorem 1 cannot be removed.

Example 2. *Let $\mathfrak{F}=\{f_n:n=1,2,\ldots\}$ and $\mathfrak{D}=\{z:\mid z\mid<1\}$, where $f_n(z)=nz$. Then every pair of members of $\mathfrak{F}$ share the set $S=\{0,\infty\}$. No member of $\mathfrak{F}$ has a simple pole. Also $\mid f'_n(z)\mid=n\to\infty$ as $n\to\infty$. Since $f^{\#}_n(0)=n\to\infty$ as $n\to\infty$, by Marty's criterion $\mathfrak{F}$ is not normal in $\mathfrak{D}$.*

To prove Theorem 1 we require the following lemma, known as Zalcman lemma.

Lemma 1. *{p.101, [2]} Let $\mathfrak{F}$ be a family of meromorphic functions in a domain $\mathfrak{D}$. If $\mathfrak{F}$ is not normal at $z_0\in\mathfrak{D}$, then there exist a sequence of points $\{z_n\}\subset\mathfrak{D}$ with $z_n\to z_0$, a sequence of positive numbers $\rho_n\to 0$ and a sequence of functions $f_n\in\mathfrak{F}$ such that*

$$g_n=f_n(z_n+\rho_n\zeta)\to g(\zeta)\quad as\quad n\to\infty$$

locally uniformly with respect to the spherical metric, where g is a nonconstant meromorphic function in $\mathbb{C}$.

2. Proof of Theorem 1

Proof of Theorem 1. Let $L(z)=\dfrac{(z-\alpha)(\gamma-\beta)}{(z-\beta)(\gamma-\alpha)}$ if $\beta\neq\infty$ and $L(z)=\dfrac{z-\alpha}{\gamma-\alpha}$ if $\beta=\infty$. Since $\mathfrak{F}$ is normal if and only if the family $\{L(f):f\in\mathfrak{F}\}$ is normal, without loss of generality we may choose $\alpha=0$, $\beta=\infty$ and $\gamma=1$.

Since normality is a local property, it is sufficient to show that $\mathfrak{F}$ is normal at $z_0\in\mathfrak{D}$. Suppose that $\mathfrak{F}$ is not normal at z_0. Then by Lemma 1 there exist functions $\{f_n\}\subset\mathfrak{F}$, points in $\mathfrak{D}$ $z_n\to z_0$ and positive numbers $\rho_n\to 0$ such that

$$g_n=f_n(z_n+\rho_n\zeta)\to g(\zeta)\quad\text{as}\quad n\to\infty$$

spherically uniformly on compact subsets of $\mathbb{C}$, where g is a nonconstant meromorphic function in $\mathbb{C}$. Also by Hurwitz's theorem and the hypothesis (iii) we know that g does not have a simple pole.

We now verify that $g-1$ has only multiple zeros. Let ζ_0 be a zero of $g-1$. Since $g-1$ is nonconstant, by Hurwitz's theorem, there exists a sequence $\zeta_n\to\zeta_0$ such that $g_n(\zeta_n)-1=0$ for $n=1,2,3,\ldots$ and so $f_n(z_n+\rho_n\zeta_n)=1$. Therefore by the hypothesis (ii) we get

$$\mid g'_n(\zeta_n)\mid=\rho_n\mid f'_n(z_n+\rho_n\zeta_n)\mid\leq\rho_n M\to 0\quad\text{as}\quad n\to\infty.$$

Hence $g'(\zeta_0)=\lim_{n\to\infty}g'_n(\zeta_n)=0$. This shows that $g-1$ can not have a simple zero. We now consider the following cases:

Case 1. Suppose that there exists $f\in\mathfrak{F}$ such that $f(z_0)\neq 0,\infty$. Then we can find $r>0$ such that

$D_r(z_0) \subset \mathfrak{D}$ and $f(z) \neq 0, \infty$ in $D_r(z_0)$, where $D_r(z_0)$ is a circular neighbourhood of z_0 with radius r. By the hypothesis we see that $h(z) \neq 0, \infty$ in $D_r(z_0)$ for all $h \in \mathfrak{F}$. Hence g is entire and does not have a zero. Since $g-1$ does not have a simple zero, by the second fundamental theorem we see that g is a constant, which is a contradiction.

Case 2. Suppose that there exists $f \in \mathfrak{F}$ such that $f(z_0) = 0$ or ∞. Then we can choose $r > 0$ such that $D_r(z_0) \subset \mathfrak{D}$ and $f(z) \neq 0, \infty$ in $D'_r(z_0)$, where $D'_r(z_0) = D_r(z_0) \backslash \{z_0\}$. By the hypothesis we see that $h(z) \neq 0, \infty$ in $D'_r(z_0)$ and $h(z_0) = 0$ or ∞ for all $h \in \mathfrak{F}$.

We now verify that g has at most one zero in $\mathbb{C}$. Suppose that $g(\eta_1) = g(\eta_2) = 0$ and $\eta_1 \neq \eta_2$. We put $r_0 = \frac{1}{4} \mid \eta_1 - \eta_2 \mid$. Since $g_n(\zeta)$ converges to $g(\zeta)$ locally uniformly, by Hurwitz's theorem we can find that there exists a subsequence of g_n, say itself, such that $g_n(\zeta_n) = g_n(\xi_n) = 0$ for all large positive integers n, where $\mid \zeta_n - \eta_1 \mid < r_0$ and $\mid \xi_n - \eta_2 \mid < r_0$. Therefore for all large positive integers n we get $f_n(z_n + \rho_n \zeta_n) = 0 = f_n(z_n + \rho_n \xi_n)$. Again $\mid z_n + \rho_n \zeta_n - z_0 \mid \leq \mid z_n - z_0 \mid + \rho_n \{\mid \eta_1 \mid + \mid \zeta_n - \eta_1 \mid\} < r$ and $\mid z_n + \rho_n \xi_n - z_0 \mid \leq \mid z_n - z_0 \mid + \rho_n \{\mid \eta_2 \mid + \mid \xi_n - \eta_2 \mid\} < r$ for all large positive integers n.

Therefore $z_n + \rho_n \zeta_n \in D_r(z_0)$ and $z_n + \rho_n \xi_n \in D_r(z_0)$ for all large positive integers n. Since $f_n(z) \neq 0, \infty$ in $D'_r(z_0)$, we get $z_n + \rho_n \zeta_n = z_n + \rho_n \xi_n = z_0$ and so $\zeta_n = \xi_n$ for all large positive integers n. Hence

$$\mid \eta_1 - \eta_2 \mid \leq \mid \eta_1 - \zeta_n \mid + \mid \eta_2 - \xi_n \mid < 2r_0 = \frac{1}{2} \mid \eta_1 - \eta_2 \mid,$$

a contradiction. Therefore g has at most one zero in $\mathbb{C}$.

Considering the reciprocals of g and f_n we can similarly show that g has at most one pole in $\mathbb{C}$.

Suppose that g is a transcendental meromorphic function. Then from the above properties of g and the second fundamental theorem we have

$$\begin{aligned} T(r,g) &\leq \overline{N}(r,\infty;g) + \overline{N}(r,0;g) + \overline{N}(r,1;g) + S(r,g) \\ &\leq \frac{1}{2}T(r,g) + O(\log r) + S(r,g) \\ &= \frac{1}{2}T(r,g) + S(r,g) \end{aligned}$$

and so $T(r,g) = S(r,g)$, a contradiction. Therefore g is a rational function. We now consider the following sub-cases.

Sub-case 2.1. Let g have one zero but no pole. Then g is a polynomial. So we can put $g = A(\zeta - a)^m$, where $a, A(\neq 0)$ are constants and m is a positive integer. Then $g - 1 = A(\zeta - a)^m - 1$ has only simple zeros, a contradiction.

Sub-case 2.2 Let g have one pole but no zero. So we can put $g = \frac{B}{(\zeta - b)^n}$, where $b, B(\neq 0)$ are constants and n is a positive integer. Then $g - 1 = \frac{B - (\zeta - b)^n}{(\zeta - b)^n}$ has only simple zeros, a contradiction.

Sub-case 2.3 Let g have one zero and one pole. So we can put $g = \frac{A(\zeta - a)^m}{(\zeta - b)^n}$, where $A(\neq 0)$, $a, b(\neq a)$ are constants and m, n are positive integers.

Now

$$g - 1 = \frac{A(\zeta - a)^m - (\zeta - b)^n}{(\zeta - b)^n} = \frac{H(\zeta)}{(\zeta - b)^n}, \quad \text{say.}$$

If $H(\zeta)$ is a constant, say α, then $g = \frac{\alpha + (\zeta - b)^n}{(\zeta - b)^n}$, which is impossible as g has at most one zero. Hence $H(\zeta)$ is nonconstant.

Since $g-1$ has only multiple zeros, we see that each zero of $H(\zeta)$ is a zero of $H'(\zeta)$. Therefore if c is a zero of $H(\zeta)$, we get $A(c-a)^m = (c-b)^n$ and $Am(c-a)^{m-1} = n(c-b)^{n-1}$. Since a, b, c are pairwise mutually distinct, this gives $m \neq n$ and $c = \frac{mb - na}{m - n}$. So $g - 1$ has only one zero in $\mathbb{C}$.

First we suppose that $n > m$. Then $g - 1 = \left(\frac{\zeta - c}{\zeta - b}\right)^n$. Differentiating g and $g-1$ we get

$$g' = A\left(\frac{m}{\zeta - a} - \frac{n}{\zeta - b}\right)\frac{(\zeta - a)^m}{(\zeta - b)^n} = \left(\frac{n}{\zeta - c} - \frac{n}{\zeta - b}\right)\frac{(\zeta - c)^n}{(\zeta - b)^n}.$$

So
$$Am(\zeta-a)^{m-1}-\frac{An(\zeta-a)^m}{\zeta-b}\equiv n(\zeta-c)^{n-1}-\frac{n(\zeta-c)^n}{\zeta-b}. \tag{2.1}$$
If $m>1$, putting $\zeta=a$ we get from above $b=c$, a contradiction. So $m=1$ and from (2.1) we get
$$A-\frac{An(\zeta-a)}{\zeta-b}\equiv n(\zeta-c)^{n-1}-\frac{n(\zeta-c)^n}{\zeta-b}.$$
Differentiating we obtain
$$\frac{An(b-a)}{(\zeta-b)^2}\equiv n(n-1)(\zeta-c)^{n-2}-\frac{n^2(\zeta-c)^{n-1}(\zeta-b)-n(\zeta-c)^n}{(\zeta-b)^2}.$$
If $n>2$, putting $\zeta=c$ we get $\dfrac{An(b-a)}{(c-b)^2}=0$, which is impossible. So $n=2$ and $g=\dfrac{A(\zeta-a)}{(\zeta-b)^2}$ and $g-1=\left(\dfrac{\zeta-c}{\zeta-b}\right)^2$. Therefore $\dfrac{A(\zeta-a)}{(\zeta-b)^2}\equiv\left(\dfrac{\zeta-c}{\zeta-b}\right)^2+1$ and so $2\zeta^2-2(b+c+\dfrac{A}{2})\zeta+b^2+c^2+Aa\equiv 0$, which is impossible. Hence $m>n$ so that $g=\dfrac{A(\zeta-a)^m}{(\zeta-b)^n}$ and $g-1=\dfrac{A(\zeta-c)^m}{(\zeta-b)^n}$.

Since $\overline{N}(r,0;g)=\overline{N}(r,\infty;g)=\overline{N}(r,1;g)=\log r+O(1)$ and $T(r,g)=m\log r+O(1)$, we get $m\le 3$ by the second fundamental theorem.

Differentiating g and $g-1$ we get
$$g'=\frac{A}{(\zeta-b)^n}\left\{m(\zeta-a)^{m-1}-\frac{n(\zeta-a)}{\zeta-b}\right\}=\frac{A}{(\zeta-b)^n}\left\{m(\zeta-c)^{m-1}-\frac{n(\zeta-c)^m}{\zeta-b}\right\}.$$
Therefore $m(\zeta-c)^{m-1}(\zeta-b)-n(\zeta-c)^m\equiv m(\zeta-a)^{m-1}(\zeta-b)-n(\zeta-a)^m$. Since $m>n\ge 1$, putting $\zeta=a$ we get from above $c=\dfrac{m(b-a)}{n}+a$. Since $c=\dfrac{mb-na}{m-n}$, we get $\dfrac{m(b-a)}{m-n}=\dfrac{m(b-a)}{n}$. Since $a\ne b$, we get $m=2n$. Now $1\le n<m\le 3$ and $m=2n$ imply $m=2$ and $n=1$. Hence $g=\dfrac{A(\zeta-a)^2}{\zeta-b}$, which is a contradiction as g has no simple poles. Therefore $\mathfrak{F}$ is normal at z_0. This proves the theorem. □

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